Gallai's Theorem for List Coloring of Digraphs

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Abstract

A classical theorem of Gallai states that in every graph that is critical for k-colorings, the vertices of degree $k-1$ induce a tree-like graph whose blocks are either complete graphs or cycles of odd length. We provide a generalization to colorings and list colorings of digraphs, where some new phenomena arise. In particular, the problem of list coloring digraphs with the lists at each vertex v having $\min\{d^+(v), d^-(v)\}$ colors turns out to be NP-hard.

Keywords: Digraph coloring, dichromatic number, list coloring, Gallai tree, algorithmic complexity, NP-complete, critical digraph.

1 Introduction

A theorem of Gallai [8] describes the structure of low degree vertices in graphs that are critical for the chromatic number. It states that the induced subgraph on the vertices of degree $k-1$ in a k-critical graph is composed of blocks that are either complete graphs or odd cycles. In this paper, we consider the chromatic number of digraphs and show that Gallai theorem can

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be extended to this setting. It is interesting to note that another structure appears in addition to cliques and odd cycles. These are directed cycles of any length. For a parallel, we observe that this kind of graphs also occur in the version of Brooks' Theorem for digraphs, see Theorem 1.3 below.

The Gallai theorem has a natural setting in terms of list colorings. For undirected graphs, it can be viewed as a list coloring problem where the list at each vertex has the same number of available colors as the degree of that vertex. The coloring problem for this type of lists is easily solvable for undirected graphs. However, as we show in Section 3, the colorability of this type of list coloring problems on digraphs is NP-hard.

List colorings and Gallai trees

A graph G is *k-color-critical* or *k-critical* if $\chi(G) = k$ and $\chi(H) < \chi(G)$, for every proper subgraph $H \subset G$. The minimum degree of a k-critical graph is at least $k-1$. A classical theorem of Gallai [8] states that in every k-critical graph, the vertices of degree $k-1$ induce a graph whose blocks are either odd cycles or complete graphs. Because of this result, a graph all of whose blocks are either odd cycles or complete graphs is called a *Gallai tree*.

A natural setting of applying Gallai's theorem is that of list colorings. Given a graph G and a list $L(v)$ of colors for each vertex v, we say G is *L-colorable* if there is a proper coloring of G (i.e. each color class is an independent set) such that each vertex v is assigned a color from $L(v)$. Having a k-critical graph G, one may assume that we have (somehow) colored vertices of degree larger than $k-1$ with $k-1$ colors and that only vertices whose degree in G is $k-1$ are left to be colored. Denote the subgraph induced by the vertices of degree $k-1$ by S. Now, each vertex $v \in V(S)$ has a list $L(v)$ of available colors, and $|L(v)| = \deg_S(v)$. This setting is used to formulate Gallai's theorem for list colorings. It was obtained independently by Borodin [3] and Erdős et al. [5]. Kostochka et al. [9] generalized it to hypergraphs.

Theorem 1.1 ([3],[5]). *Let G be a connected graph, and L a list-assignment for G. Suppose that $|L(x)| \geq \deg(x)$ for each $x \in V(G)$, and G is not L-colorable. Then G is a Gallai tree.*

The following theorem has been proved by Thomassen [13], while the generalization to hypergraphs can be found in [9].

Theorem 1.2. *Let L be an arbitrary list-assignment for a graph G. Let X be a subset of vertices such that $G[X]$ is connected and $|L(x)| \geq \deg_G(x)$ for* *each $x \in X$. Assume that $G - X$ is L-colorable. If G is not L-colorable, then $G[X]$ is a Gallai tree and $|L(x)| = \deg_G(x)$ for every $x \in X$.*

Digraph colorings and the Brooks Theorem

Let D be a digraph. A vertex set $A \subset V(D)$ is called *acyclic* if the induced subdigraph $D[A]$ has no directed cycles. A *k-coloring* of D is a partition of $V(D)$ into k acyclic sets. The minimum integer k for which there exists a k-coloring of D is the *chromatic number* $\chi(D)$ of the digraph D. The above definition of the chromatic number of a digraph was first introduced by Neumann-Lara [12]. The same notion was independently introduced much later by the second author when considering the circular chromatic number of weighted (directed or undirected) graphs [10]. The chromatic number of digraphs was further investigated by Bokal et al. [2]. The notion of chromatic number of a digraph shares many properties with the notion of the chromatic number of undirected graphs. Note that if G is an undirected graph, and D is the digraph obtained from G by replacing each edge with the pair of oppositely directed arcs joining the same pair of vertices, then $\chi(D) = \chi(G)$ since any two adjacent vertices in D induce a directed cycle of length two. The second author [11] provides some further evidence for the close relationship between the chromatic number of a digraph and the usual chromatic number.

Note that the blocks in Gallai's theorem for undirected graphs are precisely complete graphs and odd cycles, which also appear in Brooks' theorem. For digraphs, a version of Brooks' theorem was proved in [11].

Theorem 1.3 ([11])**.** *Suppose that D is a k-critical digraph in which for every vertex $v \in V(D)$, $d^+(v) = d^-(v) = k - 1$. Then one of the following cases occurs:*

1. *$k = 2$ and D is a directed cycle of length $n \geq 2$.*
2. *$k = 3$ and D is a bidirected cycle of odd length $n \geq 3$.*
3. *D is bidirected complete graph of order $k \geq 4$.*

Note that the last two cases of Theorem 1.3 are the analogues of odd cycles and complete graphs in the undirected version of Brooks' and Gallai's theorems. Thus, it is expected that the first case of Theorem 1.3 will appear in the Gallai's theorem for digraphs, which is proved in the sequel.

Basic definitions and notation

We end this section by introducing some terminology that we will be using throughout the paper. The notation is standard and we refer the reader to [1] for an extensive treatment of digraphs. We use xy to denote the arc joining vertices x and y, where x is the *initial vertex* and y is the *terminal vertex* of the arc xy. We denote by $A(D)$ the set of arcs of the digraph D. Digraphs discussed in the paper will not have parallel arcs. We do allow, however, the existence of two arcs between two vertices going in opposite directions. For $v \in V(D)$ and $e \in A(D)$, we denote by $D - v$ and $D - e$ the subdigraph of D obtained by deleting v and the subdigraph obtained by removing e, respectively. We let $d_D^+(v)$ and $d_D^-(v)$ denote the *out-degree* (the number of arcs whose initial vertex is v) and the *in-degree* (the number of arcs whose terminal vertex is v) of v in D, respectively. A vertex $v \in V(D)$ is said to be *Eulerian* if $d^+(v) = d^-(v)$. The digraph D is *Eulerian* if every v in D is Eulerian. We say that u is an *out-neighbor (in-neighbor)* of v if vu (uv) is an arc. We denote by $N^+(v)$ and $N^-(v)$ the set of out-neighbors and in-neighbors of v, respectively. Every undirected graph G determines a *bidirected* digraph $D(G)$ that is obtained from G by replacing each edge with two oppositely directed edges joining the same pair of vertices. If D is a digraph, we let $G(D)$ be the *underlying undirected graph* obtained from D by "forgetting" all orientations. A digraph D is said to be *(weakly) connected* if $G(D)$ is connected. The *blocks* of a digraph D are the maximal subdigraphs D' of D whose underlying undirected graph $G(D')$ is 2-connected. A *cycle* in a digraph D is a cycle in $G(D)$ that does not use parallel edges. A *directed cycle* in D is a subdigraph forming a directed closed walk in D whose vertices are all distinct. A directed cycle consisting of exactly two vertices is called a *digon*.

The rest of the paper is organized as follows. In Section 2, we derive an analogue of Gallai's theorem for directed graphs. In Section 3, we consider algorithmic questions for list coloring a digraph.

2 List coloring and Gallai Theorem

We define list colorings of digraphs in an analogous way as for undirected graphs. Let $\mathcal{C}$ be finite set of colors. Given a digraph D, let $L : v \mapsto L(v) \subseteq \mathcal{C}$ be a *list-assignment* for D, which assigns to each vertex $v \in V(D)$ a set of colors. The set $L(v)$ is called the *list* (or the set of *admissible colors*) for v. We say D is *L-colorable* if there is an *L-coloring* of D, i.e., each vertex v is assigned a color from $L(v)$ such that every color class induces an acyclic subdigraph in D. We say that D is *L-critical* if D is not L-colorable but every proper subdigraph of D is L-colorable. Clearly, by saying that a subdigraph H is L-colorable, we use the restriction of the list-assignment L to $V(H)$. The main result of this section is the following digraph analogue of Gallai Theorem.

Theorem 2.1. *Let D be a connected digraph, and L an assignment of colors to the vertices of D such that $|L(v)| \geq \max\{d^+(v), d^-(v)\}$. Suppose that D is not L-colorable. Then D is Eulerian and every block of D is one of the following:*

(a) *directed cycle (possibly a digon),*

(b) *an odd bidirected cycle, or*

(c) *a bidirected complete digraph.*

Moreover, for each block B of D, whose vertices have in-degree and out-degree d_B, there is a set C_B of colors so that for each vertex $v \in V(D)$, we have

$$L(v) = \{C_B \mid B \text{ is a block of } D \text{ and } v \in V(B)\}.$$

Furthermore, $|L(v)| = d^+(v)$, implying that the blocks B containing v have pairwise disjoint color sets C_B.

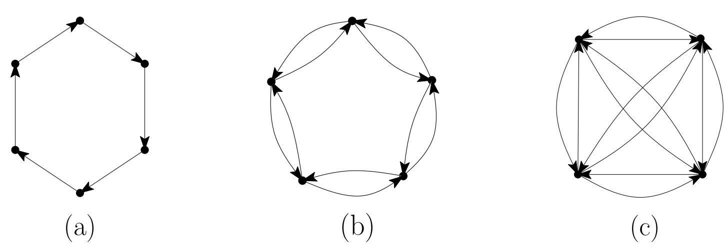

Figure 1: Possible blocks in Gallai trees: (a) a directed cycle, (b) a bidirected odd cycle, and (c) a bidirected complete graph.

The proof of Theorem 2.1 relies on several lemmas. The first of these gives information about the lists of L-critical Eulerian digraphs.

Lemma 2.2. *Let D be an Eulerian digraph, and let L be an assignment of colors to the vertices of D. Suppose that $|L(v)| = d^+(v)$ $(v \in V(D))$ and that D is L-critical. Given a vertex $v \in V(D)$, let f be an L-coloring of $D - v$. Then the following holds:*

1. *$L(v) = \{f(u) \mid u \in N^-(v)\} = \{f(w) \mid w \in N^+(v)\}$, and so each color in $L(v)$ appears exactly once in $N^-(v)$ and once in $N^+(v)$.*

2. *If u is a neighbor of v with $f(u) = c$, then uncoloring u and coloring v with c gives an L-coloring of $D - u$.*

Proof. If a color $c \in L(v)$ would not appear on the out-neighborhood of v, we could color v by c and obtain an L-coloring of D. Similarly, each color $c \in L(v)$ also appears on the in-neighborhood of v. This establishes the first claim.

To prove the second claim, remove color c from u and color v with c. Suppose, without loss of generality, that u is an out-neighbor of v. Since c appeared on the out-neighbors of v only once, we get an L-coloring of $D - u$. □

Lemma 2.3. *Let D be a connected digraph. Let L be an assignment of colors to the vertices of D with $|L(v)| \geq \max\{d^+(v), d^-(v)\}$ for each $v \in V(D)$. Suppose that D is not L-colorable. Then*

1. *D is Eulerian and $|L(v)| = d^+(v) = d^-(v)$ for every $v \in V(D)$.*

2. *D is L-critical.*

Proof. To prove 1), we will use induction on $|V(D)|$. The claim is clear if $|V(D)| = 1$. If $|V(D)| = 2$, then D is either a directed edge (and hence L-colorable since $L(v) \neq \emptyset$ for $v \in V(D)$) or a digon, in which case 1) holds. So, assume now that $|V(D)| \geq 3$.

Suppose there exists a vertex $v \in V(D)$ such that $d_D^+(v) \neq d_D^-(v)$. Let $D' = D - v$. If D' was L-colorable so would be D, since one of the colors in $L(v)$ would not appear among the in-neighbors or out-neighbors of v. Thus, D' is not L-colorable. Then D' has a connected component D'' that is not L-colorable. Applying the induction hypothesis to D'', we conclude that D'' is Eulerian and $|L(u)| = d_{D''}^+(u) = d_{D''}^-(u)$ for every $u \in V(D'')$. Now, choosing a vertex $u \in V(D'')$ which is a neighbor of v we obtain that $d_{D''}^+(u) = |L(u)| \geq \max\{d_D^+(u), d_D^-(u)\} \geq d_{D''}^+(u) + 1$, a contradiction. Therefore, $|L(v)| = \min\{d_D^+(v), d_D^-(v)\} = \max\{d_D^+(v), d_D^-(v)\}$, and the result follows.

To prove 2), we use induction on $|A(D)|$. The claim is clearly true when $|A(D)| \leq 2$. So, suppose $|A(D)| \geq 3$. Now, let $e = uv$ be any arc, and let $D' = D - e$. Let D'' be any component of D'. By part 1), D is Eulerian which implies that D'' is not Eulerian. Therefore, by the induction hypothesis, D'' is L-colorable. Similarly, if there exists a second component of D', it is also L-colorable. Therefore, D' is L-colorable, and thus D is L-critical. □

Let $C = v_1v_2...v_k$ be a cycle (not necessarily directed) in a digraph D. Let f be a coloring of $D - v_1$. A *shift of colors around* C is a color assignment g for $D - v_1$, where $g(v_2) = f(v_3)$, $g(v_3) = f(v_4)$, ..., $g(v_k) = f(v_2)$ and $g(v) = f(v)$ for $v \in V(D) \backslash V(C)$. Let us observe that in the case of Eulerian L-critical graphs, Lemma 2.2 guarantees that g is a (proper) L-coloring of $D - v_1$ since g can be obtained by repeatedly using part (2) of Lemma 2.2: first we uncolor v_2 and color v_1, then uncolor v_3 and color v_2, etc. until the last step when we uncolor v_1 and color v_k. This fact will be used throughout this section.

Lemma 2.4. *Let D be a connected digraph, and L an assignment of colors to the vertices of D such that $|L(v)| = \max\{d^+(v), d^-(v)\}$ for each $v \in V(D)$. Suppose that D is not L-colorable. Let C be a cycle of length 3 or 4 in the underlying graph $G(D)$. If the orientation of the edges of C in D is not cyclic, then $V(C)$ induces a complete bidirected graph in D.*

Proof. By Lemma 2.3, D is Eulerian and L-critical. First, assume that $C = v_1v_2v_3$ has length three. We may assume that the edges of C are directed as follows: v_3v_1, v_1v_2 and v_3v_2. We will show that the arcs v_1v_3, v_2v_3 and v_2v_1 are also present in D. Consider a coloring f of $D - v_1$. Let $f(v_2) = a$. If $f(v_3) = a$, then uncoloring v_3 and coloring v_1 with a would give an L-coloring of $D - v_3$ where v_3 has two out-neighbors colored a, a contradiction by Lemma 2.2. Therefore, $f(v_3) = b \neq a$. Now, the out-neighbor of v_1 that is colored b must be on the cycle C since otherwise doing a shift of colors around C we would get a new L-coloring of $D - v_1$ with v_1 having two out-neighbors colored b, so we could complete the coloring. The only way the out-neighbor of v_1 colored b is on C is when $v_1v_3 \in A(D)$. By a similar reasoning, $v_2v_1 \in A(D)$. To show the existence of the arc v_2v_3, consider an L-coloring of $D - v_3$ and the cycle C' consisting of the arcs v_1v_2, v_1v_3, and v_3v_2. The same proof as above shows that $v_2v_3 \in A(D)$. This settles the case when C is a cycle of length 3.

Suppose now that $C = v_1v_2v_3v_4v_1$ is a 4-cycle, and assume that the arcs of C are not cyclic. We may assume that the vertex v_1 has both vertices, v_2 and v_4, as its out-neighbors. Now, by criticality, $D - v_1$ is L-colorable. Moreover, every coloring f assigns different colors to v_2 and v_4 by Lemma 2.2. So suppose $f(v_2) = a$ and $f(v_4) = b$, $a \neq b$. Now, $f(v_3) \neq a$, since otherwise making the counter-clockwise shift of colors around C we would get two out-neighbors of v_1 colored a. Similarly, if we do a clockwise shift of colors around C we deduce that $f(v_3) \neq b$. Therefore, assume $f(v_3) = c$, $c \neq a, b$. Now, if we do a clockwise shift of colors around C we get that the color a disappears in the out-neighborhood of v, unless the vertex v_3 is an out-neighbor of v_1. Thus, by Lemma 2.2, $v_1v_3 \in A(D)$.

Now, regardless of the orientation of edges v_2v_3 and v_3v_4, the two triangles $v_1v_2v_3$ and $v_1v_3v_4$ have acyclic orientations and therefore by the first part of the proof, these sets induce bidirected cycles in D. Therefore, we have that C is a bidirected cycle that also has the chords v_1v_3 and v_3v_1. Now we apply the same proof to the cycle C' with arcs $v_2v_3, v_3v_4, v_4v_1, v_2v_1$ in which v_2 has two out-neighbors. We conclude that also the chords v_2v_4 and v_4v_2 are in D. This completes the proof of the lemma. □

Using Lemma 2.4, we now obtain the following.

Lemma 2.5. *Let D be a connected digraph, and L an assignment of colors to the vertices of D such that $|L(v)| = \max\{d^+(v), d^-(v)\}$ for each $v \in V(D)$. Suppose that D is not L-colorable. Let $C = v_1v_2...v_kv_1$, $k \geq 3$, be a cycle of length k in the underlying graph. Suppose that the orientation of the edges of C is not cyclic. Then the following holds:*

1. *If k is even, then $V(C)$ induces a complete bidirected subdigraph in D,*
2. *If k is odd, then $V(C)$ either induces a complete bidirected cycle or a complete bidirected subdigraph in D.*

Proof. By Lemma 2.3, D is Eulerian and L-critical. We proceed by induction on k. The cases $k = 3$ and $k = 4$ are established by Lemma 2.4. So we assume that $k \geq 5$. First, suppose that k is odd. We may assume that the two neighbors of v_1 on the cycle C, v_2 and v_k, are an out-neighbor and an in-neighbor, respectively. Such a vertex must exist by parity. We consider two cases. First, suppose there is a chord incident to v_1, say v_1v_i, $2 < i < k$. Then regardless of the orientation of the edge v_1v_i, one of the two cycles $v_1v_2...v_iv_1$ and $v_1v_iv_{i+1}...v_kv_1$ has acyclic orientation. By induction, we must have the arcs v_1v_i and v_iv_1 present in D. The arcs v_1v_i and v_iv_1 divide the cycle C into an odd cycle and an even cycle. Suppose $C_1 = v_1v_2...v_i$ is the even cycle. We can make sure that C_1 has its edges oriented acyclically by appropriately picking either the arc v_1v_i or v_iv_1. Thus, by induction, C_1 induces a complete bidirected digraph. Similarly, $C_2 = v_1v_iv_{i+1}...v_kv_1$ induces either a bidirected cycle or a bidirected clique. Now, consider the cycle $C_3 = v_2v_iv_{i+1}...v_kv_1v_2$. We can choose the appropriate bidirected arcs to ensure that C_3 has acyclic orientation. Since C_3 is an even cycle and it is shorter than C, it follows that C_3, and hence also C_2, induces a complete bidirected digraph. It remains to show that every vertex on C_1 has bidirected arcs to every vertex on C_2. But this is clear, since for any v_j on

C_2, $v_1v_jv_iv_{i+1}...v_kv_1$ is an even cycle and thus induces a complete bidirected graph by the same argument as used above.

Now, suppose there is no chord incident to v_1. Let f be an L-coloring of $D - v_1$. First, we claim that $f(v_k) \neq f(v_2)$. Suppose, for a contradiction, that $f(v_k) = f(v_2) = a$. By repeatedly making a shift of colors around C, we conclude that all the original colors on C were equal to a. Let v_i be a vertex on C that has both of its neighbors on C as in-neighbors. Passing the color of v_2 to v_1 (by using Lemma 2.2(2)), the color of v_3 to $v_2, \cdots$, the color of v_i to v_{i-1}, we get a proper L-coloring of $D - v_i$. But now v_i has two in-neighbors colored a, so we can complete the coloring to a coloring of D, a contradiction. So we may assume that $f(v_2) = a$ and $f(v_k) = b$, $a \neq b$. Now, the out-neighbor of v_1 that has color b must be v_k for otherwise doing a shift of colors we would get a coloring of $D - v_1$ with two out-neighbors colored b. So, $v_1v_k \in A(D)$. By a similar argument, $v_2v_1 \in A(D)$. Now, consider the vertex v_2 and a coloring of $D - v_2$. Since the edges $v_1v_2, v_2v_1, v_1v_k, v_kv_1$ exist, we can change C to a non-directed cycle C' in which v_2 has an in-neighbor and an out-neighbor. As above, we either get a bidirected clique or both arcs v_2v_3 and v_3v_2. Repeating this argument, we deduce that $V(C)$ induces a bidirected cycle or a bidirected clique.

Next, suppose k is even. We may assume that v_1's neighbors on C, v_2 and v_k, are both in-neighbors. We claim that there is a chord of C incident to v_1 and directed inwards (i.e., v_1 has another in-neighbor on C). Suppose not. Consider a coloring of $D - v_1$ and let $f(v_2) = a$ and $f(v_k) = b$. Now if we do a shift of colors around C we deduce that $f(v_3) = f(v_5) = f(v_7) = \cdots = f(v_{k-1}) = b$. But this is impossible since after performing a shift of colors in the opposite direction, we will obtain a valid coloring of $D - v_1$ with v_k and v_2 both colored b. Therefore, there is an arc $v_iv_1 \in A(D)$. If this arc divides C into two even cycles, then by an inductive argument similar to the case when k is odd we can deduce that C is a complete bidirected digraph. Therefore, assume that i is odd so that v_iv_1 splits the cycle C into two odd cycles $C_1 = v_1v_2...v_iv_1$ and $C_2 = v_1v_iv_{i+1}...v_kv_1$. By induction, we have that all the edges of C are actually bidirected arcs. Also, we know that $v_iv_1, v_1v_i \in A(D)$. Next, we show that there must be further chords incident to v_1 in addition to those coming from v_i. Suppose not. Consider a coloring g of $D - v_1$, and suppose $g(v_2) = a$, $g(v_k) = b$ and $g(v_i) = c$. Now, if we do shift of colors around C_1, we conclude that $g(v_2) = g(v_4) = \cdots = g(v_{i-1}) = a$ and $g(v_3) = g(v_5)... = g(v_i) = c$. Similarly, doing shift of colors around C_2 we conclude that $g(v_i) = g(v_{i+2}) = g(v_{k-1}) = c$ and $g(v_{i+1}) = g(v_{i+3}) = \cdots = g(v_k) = b$. Since $k \geq 6$, if we now do two shifts of colors around C, we will get a coloring of $D - v_1$ where there is the same color appearing twice in the neighborhood of v_1, contradicting Lemma 2.2. Therefore, there are other chords incident to v_1 except the ones coming from v_i. This implies that one of the cycles C_1 or C_2 is divided into an even cycle and an odd cycle and we are done by a similar argument as in the case when k is odd. □

Now, we can prove the main result of this section.

Proof of Theorem 2.1. By Lemma 2.3, D is Eulerian and L-critical. Let H be a block of D, for which none of (a)-(c) applies. Note that H cannot be a single arc by L-criticality. The theorem is clear if $|V(H)| \leq 3$. Note that H cannot be a non-directed cycle or a cycle with some but not all edges bidirected, since every such cycle induces new arcs by Lemma 2.5. So we may assume that $|V(H)| \geq 4$ and that H (as an undirected graph) is not a cycle. Then there are two vertices in H with three internally vertex-disjoint paths between them, say P_1, P_2, P_3. Two of these paths, say P_1 and P_2, create a cycle C of even length. We claim that the cycle C induces a complete bidirected graph. Suppose not. Then C is a directed cycle by Lemma 2.5. This implies that at least one of the cycles $P_1 \cup P_3$ or $P_1 \cup P_2$ is not directed. By applying Lemma 2.5 again, this new cycle induces at least a bidirected cycle and therefore some of the arcs of C are bidirected. But this is a contradiction, which shows that C induces a complete bidirected digraph.

Let v be any vertex of H that is not on C. Since H is a block, there are two paths P and Q from v to C whose only common vertex is v. Now, simply take an even cycle C' that contains the path $P \cup Q$ and one or two additional arcs of C. We may choose the arcs of C' so that C' is a non-directed cycle. Now, Lemma 2.5 shows that C' induces a complete bidirected digraph. By using different vertices of C when making C' (by possibly including more than two arcs of C), we conclude that every vertex of $P \cup Q$ is adjacent to each other and to every vertex on C. Therefore, if we take any maximal bidirected clique K in H we conclude that all the vertices of H are on K. Hence, H is a complete bidirected digraph.

It remains to prove the last part of the theorem. Let us consider a block B of D. Note that B satisfies one of (a)–(c). If $B = D$, then it is an easy exercise to show that the only list assignment L, for which D is not L-colorable, has all lists $L(v)$, $v \in V(D)$, equal to each other. So, we may assume that $B \neq D$. Next, we L-color $D' = D - V(B)$. After this, each vertex $v \in V(B)$ is left with at least $d_B^+(v)$ colors that do not appear on $N(v)$. Let $L'(v) \subseteq L(v)$ denote these colors. Now, every L'-coloring of B gives rise to an L-coloring of D, so B is not L'-colorable. But since $|L'(v)| \geq d_B^+(v)$ for all $v \in V(B)$, we conclude, by the same arguments as above, that $|L'(v)| = d_B^+(v)$ for each $v \in V(B)$ and that all lists $L'(v)$ are the same. By denoting this common color set by C_B, we obtain the last part of the theorem. Since $|L(v)| = d^+(v)$, it is easy to see that the color sets C_B of all blocks B containing v are pairwise disjoint. $\square$

Note that the condition $|L(v)| \geq \max\{d^+(v), d^-(v)\}$ in Theorem 2.1 cannot be strengthened to, say, $|L(v)| \geq d^+(v)$, since we could take any digraph which has a vertex with no out-neighbors and an empty list of colors. However, this becomes possible if we know that the digraph is L-critical.

Corollary 2.6. *Let D be a connected digraph and L an assignment of colors to the vertices of D such that $|L(v)| \geq d^+(v)$, for every $v \in V(D)$. Suppose that D is L-critical. Then D is Eulerian, and hence the conclusions of Theorem 2.1 hold.*

Proof. If D is not Eulerian, then there exists a vertex $v \in V$ with $d^+(v) > d^-(v)$. Consider an L-coloring of $D - v$. Now, since $|L(v)| \geq d^+(v) > d^-(v)$, there is a color $c \in L(v)$ that does not appear on the in-neighborhood of v. Coloring v with color c gives an L-coloring of D, a contradiction. $\square$

The next corollary obtains a similar result when the criticality conditioned is dropped, but we insist that vertices whose out-degree is larger than their in-degree have an extra admissible color.

Corollary 2.7. *Let D be a connected digraph, and L an assignment of colors to the vertices of D such that $|L(v)| \geq d^-(v)$ if $d^+(v) \leq d^-(v)$ and $|L(v)| \geq d^-(v) + 1$ otherwise. Suppose that D is not L-colorable. Then D is Eulerian, and hence the conclusions of Theorem 2.1 hold.*

Proof. We use induction on $|A(D)|$. If $|A(D)| \leq 3$ and D is not Eulerian, then D is L-colorable for any choice of L. So, we may assume from now on that $|A(D)| \geq 4$.

We first show that D is L-critical. Let $e = uv$ be an arc of D and suppose for a contradiction that $D - uv$ is not L-colorable. Consider a component C of $D - uv$ that is not L-colorable. By the induction hypothesis, we have that C is Eulerian and that conclusions of Theorem 2.1 hold. If $u \in V(C)$ (say), then u is not an Eulerian vertex in D, so $|L(u)| > d_C^+(u)$, which contradicts the conclusions of Theorem 2.1 for C.

Now, suppose that D is not Eulerian. Since $\sum_v d^+(v) = \sum_v d^-(v) = |A(D)|$, there exists a vertex v such that $d^+(v) > d^-(v)$. Then $|L(v)| \geq$

$d^-(v)+1$. Remove an arc e incident to v from D, and choose an L-coloring of $D - e$. Now, putting the edge e back, we see that we still have a color in $L(v)$ not appearing on the in-neighborhood of v, allowing us to complete the coloring to an L-coloring of D, a contradiction. □

The reader may wonder why do we request an additional color for non-Eulerian vertices. As we shall see in the next section, the situation changes drastically if this were not the case.

3 Complexity of list coloring of digraphs with Brooks' condition

It is natural to ask whether the condition of Corollary 2.7 can be relaxed to $|L(v)| \geq \min\{d^+(v), d^-(v)\}$. It turns out that the answer is negative even if the digraph is L-critical. There is an example on four vertices; see Figure 2, where the numbers at the vertices indicate the corresponding lists of colors. Further examples of digraphs that are L-critical with $|L(v)| \geq \min\{d^+(v), d^-(v)\}$ for every $v \in V(D)$, and yet do not admit a block decomposition described by Theorem 2.1, are not hard to construct.

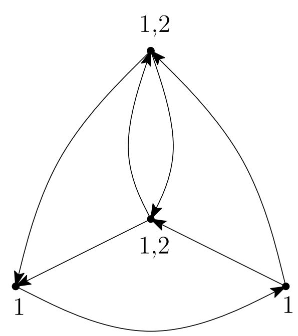

Figure 2: An L-critical digraph with $|L(v)| \geq \min\{d^+(v), d^-(v)\}$ that is not Eulerian

Not only that there are many such examples, it turns out that the list coloring problem restricted to such a restricted class of instances is NP-hard. This (surprising) fact and its proof is the subject of the remainder of this section.

Computational complexity of digraph colorings has been studied by several authors. We have the following complexity theorem for digraphs proven in Bokal et al. [2].

Theorem 3.1 ([2])**.** *Let D be a digraph. It is NP-complete to decide whether $\chi(D) \leq 2$.*

Stronger results were obtained by Feder, Hell and Mohar [6].
We study the following problem.

> Problem: List Coloring with Brooks' Condition
> Instance: A digraph D, a list-assignment L such that for every vertex $v \in V(D)$, $|L(v)| = \min\{d^+(v), d^-(v)\}$.
> Question: Is the digraph D L-colorable?

If we restrict the instances to planar graphs, we get the Planar List Coloring Problem with Brooks' Condition.

Theorem 3.2. *The* Planar List Coloring Problem with Brooks' Condition *is NP-complete.*

For a polynomial time reduction, we shall use the following problem, which was proved to be NP-complete in [7].

> Problem: Planar $(\leq 3, 3)$-Satisfiability
> Instance: A formula Φ in conjunctive normal form with a set C of clauses over a set X of boolean variables such that
>
> (1) each clause involves at most three distinct variables,
>
> (2) every variable occurs in exactly three clauses, once positive and twice negative, and
>
> (3) the graph $G_\Phi = (X \cup C, \{xc \mid x \in X, x \in c \in C$ or $\neg x \in c \in C\})$ is planar.
> Question: Is Φ satisfiable?

Proof. Clearly, every list coloring problem is in NP since after guessing an L-coloring, one can check in polynomial time whether each color class induces an acyclic subdigraph using Breadth-First-Search.

For the polynomial-time reduction we use Planar $(\leq 3, 3)$-Satisfiability. Let the formula Φ be an instance of Planar $(\leq 3, 3)$-Satisfiability. Note that $G = G_\Phi$ is a bipartite graph with bipartition $\{X, C\}$. We create an instance of list coloring for digraphs as follows.

- Direct all the edges of G from X to C.
- For each $x \in X$, we create a new vertex x' and add the arcs $x'x$ and c_1x', c_2x', where c_1, c_2 are the two clauses that contain $\neg x$.

- Add the arc c_3x, where c_3 is the clause containing the literal x.
- For every variable $x \in X$, we define two colors, x and $\bar{x}$. For each $x \in X$, set $L(x) = \{x, \bar{x}\}$. For each $c \in C$, we set $L(c) = \{\bar{x} \mid x \in c\} \cup \{x \mid \neg x \in c\}$. Finally, let $L(x') = \{x\}$ for every x'.

Let D be the resulting digraph. Clearly, every $x \in X$ has out-degree 3 and in-degree 2 because x appears in three clauses, twice negative and once positive. Therefore, $|L(x)| = \min\{d^+(v), d^-(v)\}$. For a given clause $c \in C$, for every arc xc we have exactly one of the two arcs cx or cx. Therefore, $d^+(c) = d^-(c) = |L(c)|$. Now, every x' has in-degree 2 and out-degree 1, which implies that $|L(x')| = \min\{d^+(x'), d^-(x')\}$. Therefore, all the list sizes match with minimum degree. Now, we claim that Φ is satisfiable if and only if D is L-colorable.

Suppose first that f is an L-coloring of D. Define a truth assignment ϕ as follows: $\phi(x) = \texttt{true}$ if $f(x) = x$ and $\phi(x) = \texttt{false}$ if $f(x) = \bar{x}$. We need to show that every clause c is satisfied. If $f(c) = x$ for some variable x, then $\neg x \in c$. Also, $f(x) \neq x$ for otherwise we would have a monochromatic triangle $cx'x$ of color x. Therefore, $f(x) = \bar{x}$, thus $\phi(x) = \texttt{false}$, and hence c is satisfied. Similarly, if $f(c) = \bar{x}$, then $x \in c$. Further, $f(x) = x$ for otherwise we would have a monochromatic digon. Therefore, $\phi(x) = \texttt{true}$ and c is satisfied.

Conversely, let ϕ be a satisfying truth assignment. Define the following L-coloring f: $f(x) = x$ if $\phi(x) = \texttt{true}$, and $f(x) = \bar{x}$ if $\phi(x) = \texttt{false}$. For each clause c, choose a variable x which satisfies c and set $f(c) = x$ if $\neg x \in c$, and $f(c) = \bar{x}$, if $x \in c$. Clearly, $f(x') = x$ for all x'. To see that f is a coloring, consider an arc xc. We claim that $f(x) \neq f(c)$. Suppose $f(x) = x$ (the other case is similar) and that $\neg x \in c$. Since $f(x) = x$, $\phi(x) = \texttt{true}$ which implies that $\neg x = \texttt{false}$. Therefore, $f(c) \neq x$. Thus, no arc from X to C is monochromatic, so f is a coloring. This completes the proof. $\square$

We note that the above proof implies the following obvious corollary.

Corollary 3.3. *List coloring of digraphs is NP-complete even if restricted to planar digraphs where each vertex v has $d_0(v) = \min\{d^+(v), d^-(v)\} \leq 3$ and the list size for v is equal to $d_0(v)$.*

Next, we consider the problem where the list sizes of vertices with $d^+(v) > d^-(v)$ have an additional color.

PROBLEM: LIST COLORING WITH RELAXED BROOKS' CONDITION

INSTANCE: A digraph D, a list-assignment L such that for every vertex $v \in V(D)$ with $d^+(v) \leq d^-(v)$, $|L(v)| \geq d^+(v)$, and for every vertex v with $d^+(v) > d^-(v)$, we have $|L(v)| \geq d^-(v)+1$.
QUESTION: Is the digraph D L-colorable?

Theorem 3.4. *The problem* LIST COLORING WITH RELAXED BROOKS' CONDITION *can be solved in linear time* $O(|V(D)|+|A(D)|)$.

Proof. Note that it is sufficient to provide an algorithm for connected digraphs because we can then apply it to all the components. We first give an algorithm for the Eulerian instances of D, and then show that the general case can be reduced to the Eulerian case.

So suppose D is Eulerian. We will apply Theorem 2.1. If there exists a vertex $v \in V(D)$ such that $|L(v)| > d^+(v)$, then D is L-colorable by Theorem 2.1. So we may assume that $|L(v)| = d^+(v)$ for all $v \in V(D)$. We first find the blocks of D; this can be done in time $O(|V(D)|+|A(D)|)$ using Depth-First-Search, see for example [4]. By Theorem 2.1, if there exists a block of D that is not of type (a)–(c), then D is L-colorable. So we may assume that all blocks of D are of type (a), (b) or (c). Let B be a leaf block in the block-cutpoint tree of D. If $B = D$, then as mentioned in the proof of Theorem 2.1, D is not L-colorable if and only if all the lists of D are the same. This can be checked in linear time. Otherwise, let $v \in V(B)$ be the single cut-vertex in B. If there are two vertices in $u, w \in V(B)\backslash\{v\}$ with $L(u) \neq L(w)$ or there exists a vertex $x \in V(B)\backslash\{v\}$ such that $L(x) \nsubseteq L(v)$, then D is L-colorable by Theorem 2.1. Therefore, we may assume that for all $u, w \in V(B)\backslash\{v\}$, $L(u) = L(w)$ and $L(u) \subseteq L(v)$. In this case, it is easy to see that D is L-colorable if and only if $D - (V(B)\backslash\{v\})$ is L'-colorable, where $L'(v) = L(v)\backslash L(u)$, for some $u \in V(B)\backslash\{v\}$, and $L'(x) = L(x)$ for all $x \in V(D)\backslash V(B)$. Thus, we can reduce the problem by deleting a leaf block B at each step by using at most $O(|V(B)|+|A(B)|)$ time, which results in a $O(|V(D)|+|A(D)|)$ overall time.

Next, suppose that D is not Eulerian. We give a linear time reduction to the Eulerian case. Since $\sum_v d^+(v) = \sum_v d^-(v) = |A(D)|$, there exists a vertex u such that $d^+(u) > d^-(u)$. Consider $D-u$. We claim that D is L-colorable if and only if $D-u$ is L-colorable. Clearly, if D is L-colorable then $D-u$ is L-colorable. Now, suppose $D-u$ is L-colorable, and let f be such a coloring. Since $d^+(u) > d^-(u)$, we have that there is a color in $L(u)$ that does not appear in the in-neighborhood of u. By using such a color, we can complete the coloring of $D-u$ to an L-coloring of D.

Repeating this reduction we will obtain a (possibly empty) digraph D^* such that $d^+_{D^*}(v) = d^-_{D^*}(v)$ for every $v \in V(D^*)$. Since $d^+(v) \geq d^+_{D^*}(v)$, it follows that $|L(v)| \geq d^+_{D^*}(v) = d^-_{D^*}(v)$. Now, using the algorithm for the Eulerian case, we can decide whether each component of D^* is L-colorable. Then clearly D is L-colorable if and only if each component of D^* is L-colorable.

To keep the list of vertices v with $d^+(v) > d^-(v)$, and updating this list after every vertex-removal takes overall linear time. We only need to consider at most $\min\{d^+(v), d^-(v)\} + 1$ colors at v, so when comparing the lists in the blocks we only need $O(|V(D)| + |A(D)|)$ time. Thus, it takes $O(|V(D) + |A(D)|)$ time to reduce D to the Eulerian digraph D^*. Since we need linear time to decide whether an Eulerian digraph is L-colorable, we have an $O(|V(D)| + |A(D)|)$ algorithm. □

The algorithm of Theorem 3.4 can be extended to a linear-time algorithm which also returns an L-coloring if D is L-colorable. The additional steps for doing this follow the reductions made in the proof of Theorem 2.1.

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