Coloring tournaments: from local to global *

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Abstract

The *chromatic number* of a directed graph D is the minimum number of colors needed to color the vertices of D such that each color class of D induces an acyclic subdigraph. Thus, the chromatic number of a tournament T is the minimum number of transitive subtournaments which cover the vertex set of T. We show in this note that tournaments are significantly simpler than graphs with respect to coloring. Indeed, while undirected graphs can be altogether "locally simple" (every neighborhood is a stable set) and have large chromatic number, we show that locally simple tournaments are indeed simple. In particular, there is a function f such that if the out-neighborhood of every vertex in a tournament T has chromatic number at most c, then T has chromatic number at most $f(c)$. This answers a question of Berger et al.

Keywords: chromatic number of tournaments, Erdős-Hajnal conjecture, digraph coloring

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1 Introduction

A directed graph is said to be *acyclic* if it does not contain any directed cycles. Given a loopless digraph D, a k-*coloring* of D is a coloring of each of the vertices of D with one of the colors from the set $\{1, ..., k\}$ such that each color class induces an acyclic subdigraph. The *chromatic number* $\vec{\chi}(D)$ of D is the smallest number k for which D admits a k-coloring. This digraph invariant was introduced by Neumann-Lara [14], and naturally generalizes many results on the graph chromatic number (see, for example, [4], [10] [11], [12], [13]). As shown by Chen, Hu, and Zang [5], it is NP-hard to determine the chromatic number of a digraph D, even when D is restricted to a tournament. In this note, we study the chromatic number of a class of tournaments where the out-neighborhood of every vertex has bounded chromatic number.

A *tournament* is a loopless digraph such that for every pair of distinct vertices u, v, exactly one of uv, vu is an arc. Given a tournament T, a subset X of $V(T)$ is *transitive* if the subtournament of T induced by X contains no directed cycle. Thus, $\vec{\chi}(T)$ is the minimum k such that $V(T)$ can be colored with k colors where each color class is a transitive set. The coloring of tournaments has close relationship with the celebrated Erdős–Hajnal conjecture (cf. [1, 9]) and has been studied in [3, 6, 7, 2, 8].

Given $t \geq 1$, a tournament T is t-*local* if for every vertex v, the subtournament of T induced by the set of out-neighbors of v has chromatic number at most t. The following conjecture was raised in [3] (Conjecture 2.6) and settled for $t = 2$ in [8].

Conjecture 1. *There is a function f such that every t-local tournament T satisfies $\vec{\chi}(T) \leq f(t)$.*

The goal of this note is to provide a proof of Conjecture 1 for all t.

Given a set $S \subset V(T)$, we say that S is a *dominating set* of T if every vertex in $V \setminus S$ has an in-neighbor in S. The *dominating number* $\gamma(T)$ of a tournament T is the smallest number k such that T has a dominating set of size k. The main tool to prove Conjecture 1 is the following theorem, which seems more interesting than our original goal.

Theorem 2. *For every integer $k \geq 1$, there exist integers K and ℓ such that every tournament T with dominating number at least K contains a subtournament on ℓ vertices having chromatic number at least k.*

Roughly speaking, Theorem 2 asserts that if the dominating number of a tournament is sufficiently large, then it contains a bounded-size subtournament with large chromatic number. One may ask whether high dominating number is enough to force an induced copy of a specific (high chromatic number) subtournament. The following tournaments may be potential candidates. Let S_1 be the tournament with a single vertex. For every $i > 1$, let S_i be the tournament (with $2^i - 1$ vertices) obtained by blowing up two vertices of an oriented triangle into two copies of S_{i-1}. It is easy to check that $\vec{\chi}(S_i) \geq i$. The following problem is trivial for $i \leq 2$ and verified for $i = 3$ in [8], while still open for all $i \geq 4$.

Problem 3. *For every integer $i \geq 1$, there exist $f(i)$ such that every tournament T with dominating number at least $f(i)$ contains an isomorphic copy of S_i.*

On another note, it is natural to ask whether Theorem 2 still holds with a weaker hypothesis. In particular, is it true that for every k, if the chromatic number of a tournament is huge, then it contains a bounded-size subtournament with chromatic number at least k? Unfortunately, the answer is negative for any $k \geq 3$. It is well known that for any ℓ, there is an undirected simple graph G with arbitrarily high chromatic number and girth at least $\ell + 1$. We fix an arbitrary enumeration of vertices of G and create a tournament T as follows: If ij with $i < j$ is an edge of G then ij is an arc of T; otherwise, ji is an arc of T. Then T has arbitrarily high chromatic number while every subtournament of T of size ℓ has chromatic number at most 2. However, a similar question for dominating number is still open.

Problem 4. *For every integer $k \geq 1$, there exist integers K and ℓ such that every tournament T with dominating number at least K contains a subtournament with ℓ vertices and dominating number at least k.*

2 Proof of Conjecture 1

For every vertex v in a tournament T, we denote by $N_T^+(v)$ the set of out-neighbors of v in T. Given a subset X of $V(T)$, let $N_T^+(X)$ denote the union of all $N_T^+(v)$, for $v \in X$, and let $N_T^+[X] := X \cup N_T^+(X)$. For every subset X of $V(T)$, let $\vec{\chi}_T(X)$ denote the chromatic number of the subtournament of T induced by X.

Given a tournament T and a subset X of $V(T)$, we say a set $R \subseteq V(T)$ (not necessary disjoint from X) is a dominating set of X in T if every vertex in $X \backslash R$ has an in-neighbor in R. The *dominating number* $\gamma_T(X)$ of X in T is the smallest number k such that X has a dominating set of size k. When it is clear in the context, we omit the subscript T in the notation.

Let T be a tournament and $X, Y \subseteq V(T)$. The following inequalities are straightforward:

$$\gamma_T(N^+[X]) \leq |X|, \tag{1}$$

and

$$\gamma_T(Y) \leq \gamma_T(X) + \gamma_T(Y \backslash X). \tag{2}$$

Let us restate Theorem 2.

Theorem 5. *For every integer $k \geq 1$, there exist integers K and ℓ such that every tournament T with $\gamma(T) \geq K$ contains a subtournament A on ℓ vertices with $\vec{\chi}(A) \geq k$.*

Proof. We proceed by induction on k. The claim is trivial for $k = 1$. For $k = 2$, we can choose $K = 2$ and $\ell = 3$. Indeed, if a tournament T satisfies $\gamma(T) \geq K = 2$, then T is not transitive and thus it contains an oriented triangle A of size $\ell = 3$ and $\vec{\chi}(A) \geq k = 2$.

Assuming now that (K, ℓ) exists for k, we want to find (K', ℓ') for $k+1$. For this, we set $K' := k(K+\ell+1)+K$, and fix ℓ' later. Let T be a tournament such that $\gamma(T) \geq K'$. Let D be a dominating set of T of minimum size. Consider a subset W of D of size $k(K + \ell + 1)$. From (1) and (2) we have

$$\gamma(V \setminus N^+[W]) \geq \gamma(T) - \gamma(N^+[W]) \geq K' - |W| \geq K,$$

where V is the vertex set of T. Thus by induction hypothesis on k, one can find a set $A \subseteq V \setminus N^+[W]$ such that A has ℓ vertices and $\vec{\chi}(A) \geq k$. Note that by construction, $A \cap W = \emptyset$ and all arcs between A and W are directed from A to W.

Consider now a subset S of W of size $K + \ell + 1$. We claim that $\gamma(N^+(S)) \geq K + \ell$. If not, we can choose a dominating set S' of $N^+(S)$ of size at most $K + \ell - 1$. Note that x dominates S for any $x \in A$, and so $S' \cup \{x\}$ dominates $N^+[S]$. Hence $(D \setminus S) \cup S' \cup \{x\}$ would be a dominating set of T of size less than $|D|$, which contradicts the minimality of $|D|$. Therefore $\gamma(N^+(S)) \geq K + \ell$.

Let N' be the set of vertices $N^+(S) \setminus N^+(A)$. From (1) and (2) we have

$$\gamma(N') \geq \gamma(N^+(S)) - \gamma(N^+(A)) \geq K + \ell - |A| = K.$$

Thus by induction hypothesis on k, there is a subset A_S of N' such that $|A_S| = \ell$ and $\vec{\chi}(A_S) \geq k$. Note that by construction, $A_S \cap A = \emptyset$ and all arcs between A_S and A are directed from A_S to A.

We now construct our subtournament of T with chromatic number at least $k + 1$. For this we consider the set of vertices $A \cup W$ to which we add the collection of A_S, for all subsets $S \subseteq W$ of size $K + \ell + 1$. Let A' denote this new tournament and observe that its number of vertices is at most

$$\ell' := \ell + k(K + \ell + 1) + \ell \binom{k(K + \ell + 1)}{K + \ell + 1}.$$

To conclude, it is sufficient to show that $\vec{\chi}(A') \geq k+1$. Suppose not, and for contradiction, take a k-coloring of A'. Since $|W| = k(K+\ell+1)$ there is a monochromatic set S in W of size $K+\ell+1$ (say, colored 1). Recall that we have all arcs from A_S to A and all arcs from A to S, and note that since $\vec{\chi}(A) \geq k$ and $\vec{\chi}(A_S) \geq k$, both A and A_S have a vertex of each of the k colors. Hence there are $u \in A$ and $w \in A_S$ colored 1. Since $A_S \subseteq N^+(S)$, there is $v \in S$ such that vw is an arc. We then obtain the monochromatic cycle uvw of color 1, a contradiction. Thus, $\vec{\chi}(A') \geq k+1$, completing the proof. □

We now show that Conjecture 1 is true.

Theorem 6. *There is a function f such that every t-local tournament T satisfies $\vec{\chi}(T) \leq f(t)$.*

Proof. Let (K, ℓ) satisfy Theorem 5 for $k := t+1$. Let T be a t-local tournament. Thus, if $\gamma(T) \geq K$ then T contains a set A of ℓ vertices and $\vec{\chi}(A) \geq t+1$. If a vertex $v \in V(T) \backslash A$ does not have an in-neighbor in A, then $A \subseteq N^+(v)$, and so $t+1 \leq \vec{\chi}(A) \leq \vec{\chi}(N^+(v)) \leq t$, a contradiction. Hence, A is a dominating set of T. Note that

$$\vec{\chi}(N^+[v]) \leq \vec{\chi}(N^+(v)) + \vec{\chi}(\{v\}) \leq t+1$$

for every $v \in V(T)$. Thus

$$\vec{\chi}(T) = \vec{\chi}(N^+[A]) \leq \sum_{v \in A} \vec{\chi}(N^+[v]) \leq (t+1)|A| = (t+1)\ell.$$

Otherwise, $\gamma(T) < K$. Let D be a dominating set of T with minimum size. Then

$$\vec{\chi}(T) = \vec{\chi}(N^+[D]) \leq \sum_{v \in D} \vec{\chi}(N^+[v]) \leq (t+1)|D| < (t+1)K.$$

Consequently, t-local tournaments have chromatic number at most $f(t) := \max\big((t+1)K, (t+1)\ell\big)$. □

The implication of our result is that we are possibly missing a key-definition of what is a "large" (or "dense") hypergraph (i.e., a set of subsets). It could be that for a suitable definition of "large" (for which "large" intersecting "large" would be "large"), we would obtain that for any tournament T on vertex set V, the set of out-neighborhoods of vertices of T is "large", and in addition the set of subsets of vertices of a K-chromatic tournament inducing at least chromatic number k is also "large". Hence, if two large sets are intersecting in a non-empty way, one could find an out-neighborhood with chromatic number k.

If such a notion would exist, it should decorrelate the two large sets (out-neighborhoods and k-chromatic), and thus imply the following: If T_1, T_2 are tournaments on the same set of vertices and $\vec{\chi}(T_1)$ is huge, then there is a vertex v such that T_1 induces on $N^+_{T_2}(v)$ a subtournament of large chromatic number. A very similar conjecture was proposed by Alex Scott and Paul Seymour.

Conjecture 7. *[15] For every k, there exists K such that if T and G are respectively a tournament and a graph on the same set of vertices with G of chromatic number at least K, then there is a vertex v such that G induces on $N^+_T(v)$ a subgraph of G of chromatic number at least k.*

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