Approximation algorithms for some vehicle routing problems [*]

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Abstract

We study vehicle routing problems with constraints on the distance traveled by each vehicle or on the number of vehicles. The objective is either to minimize the total distance traveled by vehicles or to minimize the number of vehicles used. We design constant differential approximation algorithms for kVRP. Note that, using the differential bound for METRIC 3VRP, we obtain the randomized standard ratio $\frac{197}{99} + \varepsilon$, $\forall \varepsilon > 0$. This is an improvement of the best-known bound of 2 given by Haimovich et al. [12]. For natural generalizations of this problem, called EDGE COST VRP, VERTEX COST VRP, MIN VEHICLE and kTSP we obtain constant differential approximation algorithm and we show that these problems have no differential approximation scheme, unless $P=NP$.

Keywords: differential ratio, approximation algorithm, VRP, TSP

1   Introduction

Vehicle routing problems that involve the periodic collection and delivery of goods and services as mail delivery or trash collection are of great practical importance. Simple variants of these real problems can be modeled naturally with graphs. Unfortunately even simple variants of vehicle routing problems are NP-hard. In this paper we consider approximation algorithms, and measure their efficiencies in two ways. One is the *standard* measure giving the ratio $\frac{apx}{opt}$, where opt and apx are the values of an optimal and approximate solution, respectively. The other measure is the *differential* measure, that compares the worst ratio of, on the one hand, the difference between the cost of the solution generated by the algorithm and the worst cost, and on the other hand, the difference between the optimal cost and the worst cost. Formally, the differential measure gives the ratio $\alpha = \frac{wor-apx}{wor-opt}$, where wor is the value of the optimal solution for the complementary problem. In [15], the measure $1 - \alpha$ is considered and it is called there z-approximation. Justification for this measure can be found for example in [1, 6, 27, 15, 20].

The main subject of this paper is differential approximation of routing problems. In these problems n *customers* have to be served by *vehicles* of limited capacity from a common *depot*. A solution consists of a set of routes, where each starts at the depot and returns there after visiting a subset of customers, such that each customer is visited exactly once. We refer to a problem as a VEHICLE ROUTING PROBLEM (VRP) if there is a constraint on the (possibly weighted) number of customers visited by a vehicle. This constraint reflects

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the assumption that the vehicle has a finite capacity and that it *collects* from the customers (or *distributes* among them) a commodity. The goal is to find a solution such that the total length of the routes is as small as possible. In other cases, the vehicle is just supposed to *visit* the customers, for example, in order to serve them. In such cases we refer to the problem as a TSP problem. We will assume in such cases that the limitation is on the total distance traveled by a vehicle and not on the number of customers it visits, and in this case we search solution with a minimum number of vehicles used.

The problems that are considered here generalize the (undirected) TRAVELING SALESMAN PROBLEM (TSP). Differential approximation algorithms for the TSP are given by Hassin and Khuller [15] and Monnot [20]. We will sometimes use these algorithms to generate approximations for the problems of this paper. However, we note an important difference. In the TSP, adding a constant k to all of the edge length does not affect the set of optimal solutions or the value of the differential ratio. The reason is that every solution contains exactly n edges and therefore every solution value increases by exactly the same value, namely nk. In particular, this means that for the purpose of designing algorithms with bounded differential ratio, it doesn't matter whether d is a metric or not (it can be made a metric by adding a suitable constant to the edge lengths). In contrast, in some of problems dealt with here, the number of edges used by a solution is not the same for every solution and therefore it may turn out, as we will see, that in some cases the metric version is easier to approximate.

It is easy to see that 2VRP is polynomial time solvable. For $k \geq 3$, METRIC kVRP was proved *NP*-hard by Haimovich and Rinnooy Kan [11]. In [12], Haimovich, Rinnooy Kan and Stougie gave a $\frac{5}{2} - \frac{3}{2k}$ standard approximation for Metric kVRP. We study for the first time the differential approximability of kVRP. More exactly we give a $\frac{1}{2}$ differential approximation for the non-metric case for any $k \geq 3$. We improve this bound to $\frac{3}{5}$ for METRIC 4VRP and $\frac{2}{3}$ for METRIC kVRP with $5 \leq k \leq 8$. We also improve the cases $k = 3$ and $k \geq 9$ to $\frac{50}{99} - \varepsilon$, $\forall \varepsilon > 0$ and $\frac{25(k-1)}{33k} - \varepsilon$, $\forall \varepsilon > 0$ respectively by using a random algorithm. An approximation lower bound of $\frac{2219}{2220}$ is given here for METRIC nVRP with length 1 and 2 using a lower bound of TSP(1,2) [8].

We study a generalization of VRP, called EDGE COST VRP, where the maximum length traversed by each vehicle is bounded. We establish a $\frac{1}{3}$ differential approximation for this problem.

MIN-MAX kTSP is a generalization of TSP where we search to cover the customers by at most k vehicles such that the maximum length traversed by the vehicles is minimum. The metric case of the problem was studied by Fredrickson, Hecht and Kim [9] where they give a $\frac{5}{2} - \frac{1}{k}$ standard approximation algorithm by constructing a reduction from this problem to METRIC TSP and using Christofides' algorithm [4] . We establish a $\frac{1}{2}$ differential approximation for METRIC MIN-MAX kTSP and prove that it has no differential approximation scheme. We also give a standard lower bound of $\frac{p+1}{p}$ for MIN-MAX $\lfloor \frac{n}{p} \rfloor$TSP, for $p \geq 6$.

MIN-SUM EkTSP is another generalization of TSP where we search to cover the customers by *exactly* k vehicles such that the total length is minimum. We show that METRIC MIN-SUM EkTSP is $\frac{2}{3}$ differential approximable and it has no differential approximation scheme unless $P=NP$.

In MIN VEHICLE the goal is to minimize the number of vehicles subject to a constraint on the maximum length traversed by any single vehicle. In [19], Li, Simchi-Levi and Desrochers proved that MIN VEHICLE is not standard 2 approximable, unless $P=NP$ and it is $1 + \frac{\alpha}{\alpha-2}$

standard approximable where $\alpha = \frac{\lambda}{d_m}$ and $d_m = \max\{d_{0,1}, \ldots, d_{0,n}\}$. We first present a $\frac{2}{3}$ differential approximation algorithm and show how to improve the bound to $\frac{289}{360}$. We also show that even when λ is constant and the lengths are 1 and 2, MIN VEHICLE has no standard and differential approximation scheme, unless $P=NP$. We can improve the bound of $\frac{2}{3}$ to $\frac{289}{360}$.

The paper is organized as follows: In section 2 we give the necessary definitions. In section 3 we give a constant differential approximation algorithm for GENERAL kVRP, and a better constant differential approximation for the metric case. In section 4 the main result is a constant differential approximation for EDGE COST VRP. In the last three sections we show that MIN-MAX kTSP, MIN-SUM EkTSP and METRIC MIN VEHICLE are constant differential approximable and have no differential approximation scheme, if $P \neq NP$.

2   Terminology

Given an instance x of an optimization problem and a feasible solution y of x, we denote by $val(x, y)$ the value of the solution y, by $opt(x)$ the value of an optimal solution of x, and by $wor(x)$ the value of a worst solution of x. The *differential approximation ratio* of y is defined as $\delta(x, y) = \frac{|val(x,y) - wor(x)|}{|opt(x) - wor(x)|}$. This ratio measures how the value of an approximate solution $val(x, y)$ is located in the interval between $opt(x)$ and $wor(x)$. In particular, it is equivalent for a minimization problem to prove $\delta(x, y) \geq \varepsilon$ and $val(x, y) \leq \varepsilon opt(x) + (1 - \varepsilon)wor(x)$.

For a function f, $f(n) < 1$, an algorithm is a $f(n)$ *differential approximation algorithm* for a problem Q if, for any instance x of Q, it returns a solution y such that $\delta(x, y) \geq f(|x|)$. We say that an optimization problem is *constant differential approximable* if, for some constant $\delta < 1$, there exists a polynomial time δ differential approximation algorithm for it. An optimization problem has a *differential polynomial time approximation scheme* if it has a polynomial time $(1 - \varepsilon)$ differential approximation, for every constant $\varepsilon > 0$. We say that two optimization problems are standard (differential) *equivalent* if a δ differential approximation algorithm for one of them implies a δ standard (differential) approximation algorithm for the other one.

We consider in this paper several routing problems. The problems are defined on a complete undirected graph denoted $G = (V, E)$. The vertex set V consists of a *depot vertex* 0, and *customer vertices* $\{1, \ldots n\}$, and each edge $(i, j) \in E$ is endowed with a *weight* $d_{i,j} \geq 0$. We call a such graph a *complete valued graph*. We refer to the version of the problem in which d is assumed to satisfy the triangle inequality as the *metric case*. The output to the problems consists of a *p-tour*, that is, a set of simple cycles, $C_1, \ldots, C_p$, such that $V(C_i) \cap V(C_j) = \{0\}$, $\forall i \neq j$, and $\cup_{i=1}^{p} V(C_i) = V$. The sequence $(0, i, 0)$ with $i \neq 0$ is accepted as a cycle. We now describe the problems. For each one we specify the input, the problem's constraints, and the output.

kVRP
Input: A complete valued graph.
Constraint: $|C_j| \leq k + 1$, $j = 1, \ldots, p$.
Output: A p-tour minimizing the total weight of the cycles.

EDGE COST VRP
Input: A complete valued graph and a metric $\{\ell_{i,j} \ (i, j \in E)$, and $\lambda > 0$.

Constraint: $\sum_{e \in E(C_j)} \ell_e \leq \lambda$, $j = 1, \ldots, p$.
Output: A p-tour minimizing the total weight of the cycles.

VERTEX COST kVRP
Input: A complete valued graph and a function $\{c_i \geq 0 \ i \in V\}$, where c_i denotes the cost of the vertex i.
Constraint: $\sum_{i \in V(C_j)} c_i \leq k$, $j = 1, \ldots, p$.
Output: A p-tour minimizing the total weight of the cycles.

MIN-MAX kTSP
Input: A complete valued graph.
Constraint: $p \leq k$.
Output: A p-tour minimizing the maximum weight of the cycles.

MIN-SUM EkTSP
Input: A complete valued graph.
Constraint: $p = k$.
Output: A p-tour minimizing the total weight of the cycles.

MIN VEHICLE
Input: A complete valued graph and $\lambda > 0$.
Constraint: $\sum_{e \in E(C_q)} d_e \leq \lambda$, $j = 1, \ldots, p$.
Output: A p-tour minimizing p.

MIN DISTANCE
Input: A complete valued graph and $\lambda > 0$.
Constraint: $\sum_{e \in E(C_q)} d_e \leq \lambda$, $j = 1, \ldots, p$.
Output: A p-tour minimizing the total weight of the cycles.

For an optimization problem Q with edge lengths, we denote by Q(a, b) the version of Q where weights are between a and b and more specifically Q$[t]$, for $t > 1$, the variant where $b \leq ta$ for any $a > 0$. We will use the following problem:

MIN TSP PATH$(1,2)$ is the variant of MIN TSP$(1,2)$ problem where instead of a tour we ask for a Hamiltonian path of minimum weight. MIN TSP PATH$(1,2)$ has no differential approximation scheme [22] even if $opt = n - 1$ and $wor = 2(n - 1)$ where n is the number of vertices since it is proved in [2] that MIN TSP$(1,2)$, when the subgraph restricted to edges of length 1 is Hamiltonian and cubic, has no standard approximation scheme.

We will also use the following problems:

PARTITIONING INTO PATHS OF LENGTH k (kPP): Given a graph $G = (V, E)$ with $|V| = (k + 1)q$, is there a partition of V into q paths $P_1, \ldots, P_q$, each path with $k + 1$ vertices? 2PP have been proved NP-complete in [10] whereas, more generally, the NP-completeness of kPP is proved in [18] as a special of G-PARTITION PROBLEM. Thus $(n - 1)$PP is the decision version of HAMILTONIAN PATH.

MAXIMUM WEIGHTED PARTITIONING INTO PATHS WITH AT MOST k VERTICES (MAX WEIGHTED ATMOSTkPP): Given a weighted complete graph G where each edge $(i, j) \in E$ is endowed with a *weight $d_{i,j} \geq 0$*, we want to find a partition of vertices into paths $P_1, \ldots, P_q$, each path with at most k vertices (or indifferently $k - 1$ edges) such that $\sum_{i=1}^{q} d(P_i)$ is max-

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imum. There is an easy reduction proving the *NP*-hardness of this problem between kPP and MAX WEIGHTED ATMOST$(k+1)$PP that consist to complete the graph G instance of kPP by edges of weight 0.

A *binary 2-matching* (also called *2-factor* or *cycle cover*) is a subgraph in which each vertex in V has a degree of exactly 2. Since the graph is simple, each cycle has at least three vertices. A *minimum binary 2-matching* is one with minimum total edge weight. Hartvigsen [14] has shown how to compute a minimum binary 2-matching in $O(n^3)$ time (see [25] for another $O(n^2|E|)$ algorithm). More generally, a *binary f-matching*, where f is a vector of size $n+1$, is a subgraph in which each vertex i of V has a degree of exactly f_i. A *minimum binary f-matching* is one with minimum total edge weight and is computable in polynomial time [5].

3 kVRP

nVRP is standard equivalent to TSP. So, using the result of Sahni and Gonzalez [26] we deduce that nVRP is not $2^{p(n)}$ standard approximable for any polynomial p, unless $P{=}NP$. In fact for any $k \geq 5$ the problem is as hard to approximate as nVRP.

Theorem 3.1 *For all $k \geq 5$ (even if k is a function of n), kVRP, is not $2^{p(n)}$ standard approximable for any polynomial p, unless $P{=}NP$.*

Proof: We use a reduction from PARTITIONING INTO PATHS OF LENGTH k (kPP). Given the graph $G = (V, E)$ on $n = (k+1)q$ vertices we construct a graph G' instance of $(k+3)$VRP. We add a vertex 0 (the depot) to G and a set A of $2q$ vertices. We define the function d as follows: $d_{i,j} = 1$, if $i \in V \cup \{0\}$ and $j \in A$ or if $(i, j) \in E$ and $i, j \in V$. Finally, the remaining edges have weight $n2^{p(n)}$.

If G contains a decomposition into disjoint paths of $k+1$ vertices then $opt(G') = q(k+4)$, otherwise $opt(G') > n2^{p(n)}$. So, a $2^{p(n)}$ standard approximation for $(k+3)$VRP could decide kPP in polynomial time. The conclusion follows. ∎

3.1 General kVRP

When d is a metric, the reduction of TSP to nVRP is straightforward, and it easily follows that computing *opt* is NP-hard. On the other hand, this reduction between the corresponding maximization problems MAX TSP and MAX nVRP leading to the conclusion that computing *wor* is also NP-hard, does not work. We can easily prove this result by applying a reduction from kPP with weight 1 and 3.

In the following we give a $\frac{1}{2}$ differential approximation for non-metric kVRP.

We first compute a lower bound LB. Then we generate a feasible solution for G with value $good = LB + \delta_1$. Next, we generate another feasible solution of value $bad = LB + \delta_2$ where $\delta_2 \geq \delta_1$. This proves that the approximate solution with value *good* is an α-differential approximation where

$$\alpha = \frac{wor - good}{wor - opt} \geq \frac{bad - good}{bad - opt} \geq \frac{\delta_2 - \delta_1}{bad - LB} = \frac{\delta_2 - \delta_1}{\delta_2} = 1 - \frac{\delta_1}{\delta_2}, \tag{1}$$

since for a minimization problem $wor \geq bad \geq good \geq opt \geq LB$. To generate LB we replace 0 by a complete graph with a set V_0 of $2n$ vertices and zero length edges. The

distance between a vertex of V_0 and a vertex i of $V \setminus V_0$ is the same as the distance between 0 and i. Denote the resulting graph by G'. Compute in G' a minimum weight binary 2-matching M'.

Lemma 3.2 *Let LB denote the weight of M', and denote by opt the value of an optimal VRP solution. Then $opt \geq LB$.*

Proof: It is sufficient to show that for any VRP solution in G there exists a binary 2-matching in G' with the same value. Consider an optimal VRP solution in G and let C be a cycle in it. Generate in G' a cycle C' which is as C except for that 0 is replaced by two new adjacent vertices from V_0. Repeat this process for every cycle in the VRP solution, taking care that the subsets of vertices selected from V_0 are disjoint (an optimal solution may only contain cycles $(0, i, 0)$ for $i = 1, \ldots, n$ and in such a case, we need to use all vertices of V_0). In the last cycle insert all the remaining vertices of V_0. The result is a binary 2-matching since every cycle has at least three vertices and the cycles are disjoint and cover V. Since the value of cycle C' is the same as the value of C, the optimum of VRP is greater than or equal to the minimum binary 2-matching. ■

Lemma 3.3 *A binary 2-matching M' of G' can be transformed in polynomial time into a set M of cycles covering vertices of G with the same weight.*

Proof: If a cycle of M' does not contain a vertex of V_0 then this cycle is considered in M. If a cycle of M' contains more than one consecutive vertices from V_0 then replace these vertices by one vertex of V_0. Consider in the following a cycle C' of M' containing at least one vertex from V_0 and one from $V(G') \setminus V_0$. Suppose that $C' = (v_0^1, \mu_1, v_0^2, \mu_2, \ldots, v_0^t, \mu_t, v_0^1)$ where paths $\mu_1, \ldots, \mu_t$ contain only vertices from $V(G') \setminus V_0$. Then M will contain t cycles $(0, \mu_1, 0), (0, \mu_2, 0), \ldots, (0, \mu_t, 0)$ that have the same weight as C'. ■

We suggest the following algorithm. W.l.o.g. we suppose that the current cycle is $(0, 1, \ldots, m, 0)$.

`Algo_Differential VRP`

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1 Compute LB the weight of a minimum weight binary 2-matching M' in G';

2 Transform M' into $M = \{C_1, \ldots, C_p\}$, using Lemma 3.3;

3 For every cycle $C_i = (1, \ldots, m_i, 1)$ of M do

    3.1 If $m_i \equiv 0 \mod 2$ then

        3.1.1 $sol_{i,1} := \{(0, 1, 2, 0), (0, 3, 4, 0), \ldots, (0, m_i - 1, m_i, 0)\}$;

        3.1.2 $sol_{i,2} := \{(0, m_i, 1, 0), (0, 2, 3, 0), \ldots, (0, m_i - 2, m_i - 1, 0)\}$;

    3.2 If $m_i \equiv 1 \mod 2$ then

        3.2.1 $sol_{i,1} := \{(0, 1, 2, 0), (0, 3, 4, 0), \ldots, (0, m_i - 4, m_i - 3, 0)\}$
            $\cup \{(0, m_i - 2, m_i - 1, m_i, 0)\}$;

        3.2.2 $sol_{i,2} := \{(0, m_i, 1, 0), (0, 2, 3, 0), \ldots, (0, m_i - 3, m_i - 2, 0)\} \cup \{(0, m_i - 1, 0)\}$;
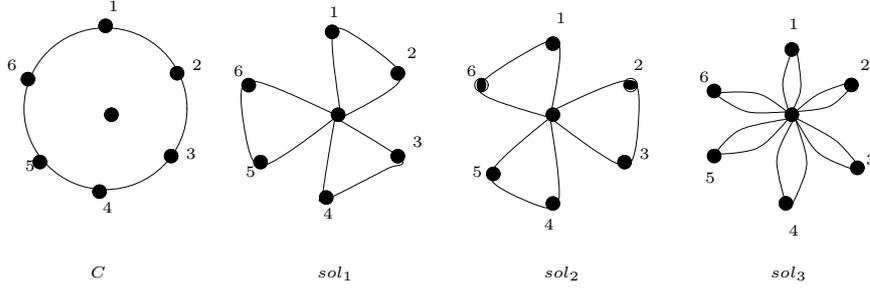
Figure 1: $m = 6$

4 For every cycle $C_i = (0, 1, \ldots, m_i, 0)$ of M with $m_i > k$ do

    4.1 If $m_i \equiv 0 \bmod 2$ then

        4.1.1 Construct $sol_{i,1} = \{(0, 2, 3, 0), \ldots, (0, m_i - 2, m_i - 1, 0)\} \cup \{(0, 1, 0), (0, m_i, 0)\}$;

        4.1.2 Construct $sol_{i,2} = \{(0, 1, 2, 0), \ldots, (0, m_i - 1, m_i, 0)\}$;

    4.2 If $m_i \equiv 1 \bmod 2$ then

        4.2.1 Construct $sol_{i,1} = \{(0, 2, 3, 0), \ldots, (0, m_i - 1, m_i, 0)\} \cup \{(0, 1, 0)\}$;

        4.2.2 Construct $sol_{i,2} = \{(0, 1, 2, 0), \ldots, (0, m_i - 2, m_i - 1, 0)\} \cup \{(0, m_i, 0)\}$;

5 For every cycle $C_i = (0, 1, \ldots, m_i, 0)$ of M with $m_i \leq k$ do $sol_{i,1} = sol_{i,2} = C_i$;

6 Output $APX = \cup_{i=1}^{p} argmin\{d(sol_{i,1}), d(sol_{i,2})\}$;

Theorem 3.4 `Algo_Differential VRP` *is a* $\frac{1}{2}$ *differential approximation algorithm for* kVRP.

Proof: Consider an arbitrary cycle C_i of M and let $add_{i,j}$ denote the added weight of $sol_{i,j}$ for $j = 1, 2$ with respect to the length of C_i. Note that since M was computed to have a minimum weight, $add_{i,j} \geq 0$ and we have $d(sol_{i,j}) = d(C_i) + add_{i,j}$ for $j = 1, 2$.

On the other hand, let bad_i be the weight of the feasible solution $sol_{i,3}$ defined by C_i if $0 \in C_i$ and $|C_i| \leq k + 1$ and by $\{(0, 1, 0), \ldots, (0, m_i, 0)\}$ otherwise; in any case, we have $bad_i = d(C_i) + add_{i,1} + add_{i,2}$

Figure 1 and 2 give an illustration of these solutions when $C_i = (1, \ldots, m_i, 1)$ and $m_i = 6$ and respectively $m_i = 3$. Sum these inequality over i and let $\delta_1 = \sum_{i=1}^{p} \min\{add_{i,1}, add_{i,2}\}$ and $\delta_2 = \sum_{i=1}^{p}(add_{i,1} + add_{i,2})$. We have $\delta_2 \geq 2\delta_1$, $LB = d(M) = \sum_{i=1}^{p} d(C_i)$ and $wor \geq \sum_{i=1}^{p} bad_i$. So, the theorem is proved by (1). ∎

When we use *bounded metrics* (i.e., when the maximum weight d_{max} is not very far from the minimum weight d_{min}), we are able to give some relations between differential and standard ratios. Bounded metric variants of TSP were studied by Papadimitriou and Yannakakis [24] and more recently by Papadimitriou and Vempala [23], and Engebretsen and Karpinski [8]. In the following, we denote by kVRP[t] the version of kVRP satisfying $\frac{d_{max}}{d_{min}} \leq t$ for some $t > 1$.

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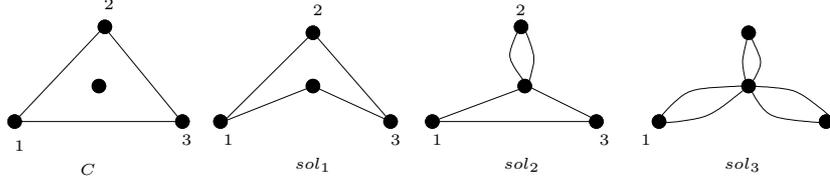
Figure 2: $m = 3$

Theorem 3.5 *A δ differential approximation algorithm for kVRP$[t]$ is also a $\delta + (1-\delta)\frac{2tk}{k+1}$ standard approximation algorithm for kVRP$[t]$.*

Proof: Let $G = (V, E)$ be a graph where $V = \{0, \ldots, n\}$ and $\frac{d_{max}}{d_{min}} \leq t$ for some $t > 1$. An optimal solution for G contains at least $n + \lceil \frac{n}{k} \rceil$ edges since it has at least $\lceil \frac{n}{k} \rceil$ cycles, and then we have:

$$opt \geq \frac{nd_{min}(1 + k)}{k}. \tag{2}$$

On the other hand, any solution of G contains at most $2n$ edges and then, we deduce the following upper bound for the worst solution:

$$wor \leq 2d_{max}n. \tag{3}$$

Finally, regrouping inequalities (2) and (3) and since we have $d_{max} \leq td_{min}$, we obtain the inequality: $wor \leq 2t\frac{k}{k+1}opt$.

Let apx be a δ differential approximation for kVRP$[t]$. Using the previous inequality we deduce:

$$apx \leq \delta opt + (1 - \delta)wor \leq \delta opt + (1 - \delta)2t\frac{k}{k+1}opt. \tag{4}$$

∎

Using the previous theorems we deduce some new standard results for kVRP$[t]$. More exactly, we obtain a $\frac{7}{2} - \frac{3}{k+1}$ standard approximation for kVRP$[3]$ and a $\frac{9}{2} - \frac{4}{k+1}$ standard approximation for kVRP$[4]$.

3.2 METRIC kVRP

The first part of this section starts with some positive differential approximation results and ends with a negative result. In the second part, we present an improvement of the best known approximation algorithm for 3VRP.

3.2.1 Differential approximation results

When d is a metric, computing a worst solution becomes easy as shown by the next lemma:

Lemma 3.6 $wor = 2\sum_{i=1}^{n} d_{0,i}$

Proof: Let sol be a feasible solution and denote by $(0, 1, \ldots, m_i, 0)$ one of these cycles. We replace it by $(0, 1, 0), \ldots, (0, m_i, 0)$ and by the triangle inequality, this change does not decrease the value of the solution. So, we can repeat it on each cycle and finally obtain the solution $(0, 1, 0), \ldots, (0, n, 0)$. ∎

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In Theorem 3.4 we have shown that kVRP is $\frac{1}{2}$ differential approximable. We now show that in the metric case, the same bound can be achieved by a simpler algorithm.

We compute a minimum weight perfect matching M on the subgraph induced by $\{1, \ldots, n\}$, if n is even, or by $\{0, 1, \ldots, n\}$ if n is odd. We link each endpoint different of 0 of M to the depot. We claim that

$$opt \geq 2d(M). \tag{5}$$

Indeed, consider an optimum solution for kVRP. Walk around it and shortcut in order to obtain a Hamiltonian cycle C on $\{0, 1, \ldots, n\}$ if n is odd and a Hamiltonian cycle C on $\{1, \ldots, n\}$ if n is even. We have $opt \geq d(C)$ by the triangle inequality and this cycle is the sum of two perfect matchings which are greater than or equal to M.

Using (5), Lemma 3.6 and the construction of the approximate solution, we obtain:

$$apx = d(M) + \sum_{i=1}^{n} d_{0,i} \leq \frac{1}{2}opt + \frac{1}{2}wor , \tag{6}$$

proving that the result is a $\frac{1}{2}$ differential approximation.

Theorem 3.7 METRIC kVRP *is* $\delta \cdot \frac{k-1}{k}$ *differential approximable, where* δ *is the differential approximation ratio for* METRIC TSP.

Proof: Our algorithm modifies the *Optimal Tour Partitioning* heuristic of Haimovich, Rinnooy Kan and Stougie [12]: first construct a tour T of value $val(T)$ on V using the δ differential approximation algorithm for TSP. W.l.o.g., assume that this tour is described by the sequence $(0, 1, \ldots, n, 0)$. We produce k solutions sol_i for $i = 1, \ldots, k$ and we select the best solution. The first cycle of sol_i is formed by the sequence $(0, 1, \ldots, i, 0)$ and then each other cycle (except possibly the last) of sol_i has exactly k consecutive vertices (for instance, the second cycle is $(0, i+1, \ldots, i+k, 0)$) and finally, the last cycle is formed by the unvisited vertices (connecting n to the depot 0). Denote by apx_i for $i = 1, \ldots, k$ the values of the solution sol_i and by apx the value of the best one.

In the union of solutions $sol_1, \ldots, sol_k$ each edge of $T \setminus \{(0,1), (0,n)\}$ appear exactly $(k-1)$ times and each edge $(0, j)$ for $j \neq 1, n$ appears exactly twice. Finally, edges $(0,1)$ and $(0,n)$ appear exactly $(k+1)$ times. Since $wor_{VRP} = 2\sum_{i=1}^{n} d_{0,i}$ by Lemma 3.6, we deduce:

$$apx \leq \frac{1}{k}\sum_{i=1}^{k} apx_i \leq \frac{(k-1)}{k}val(T) + \frac{1}{k}wor_{VRP}. \tag{7}$$

Since T is a δ differential approximation then

$$val(T) \leq (1-\delta)wor_{TSP} + \delta opt_{TSP}. \tag{8}$$

Since it is possible to construct from an optimum solution of VRP a solution of TSP with a smaller value (using the triangle inequality), it follows that

$$opt_{TSP} \leq opt_{VRP} \tag{9}$$

Also, by connecting the depot twice with each customer, we can construct from a solution of TSP a solution of VRP with a greater value, and therefore

$$wor_{TSP} \leq wor_{VRP} \tag{10}$$

Using (7)-(10) we obtain that

$$apx \leq \delta \frac{k-1}{k} opt_{VRP} + \left(1 - \delta \frac{k-1}{k}\right) wor_{VRP}.$$

∎

Since the best known differential approximation algorithm for TSP is $\frac{2}{3}$ [15, 20] then the algorithm of Theorem 3.7 is an $\frac{2}{3} \cdot \frac{k-1}{k}$ differential approximation algorithm for metric kVRP. For $k > 4$ this is an improvement over the bound of $\frac{1}{2}$ given by Theorem 3.4 for the general (non-metric) kVRP.

We will proceed now to improve the bound given in Theorem 3.7 by using a generic algorithm. When we deal with a cycle of size m we consider the vertices modulo m.

Algo_Differential Metric$k$VRP

---

1 Find a partition of $V \setminus \{0\}$ by cycles $M = \{C_1, \ldots, C_p\}$ using a Preprocessing algorithm;

2 For every cycle $C_i = (1, \ldots, m_i, 1)$ of M with $m_i = kq + r$, $0 \leq r < k$ do

    2.1 For $j = 1$ to m_i do

        2.1.1 Let $(\mu_1, \ldots, \mu_{\lceil \frac{m_i}{k} \rceil}) = C_i \setminus [\{(j, j+1)\} \cup \{(j+r+\ell k, j+r+1+\ell k) : 0 \leq \ell < q\}]$;

        2.1.2 Construct $sol_{i,j} = \cup_{\ell=1}^{\lceil \frac{m_i}{k} \rceil} \{(0, \mu_\ell, 0)\}$;

    2.2 Let $sol_i = argmin\{d(sol_{i,1}), \ldots, d(sol_{i,m_i})\}$

3 Output $APX = \cup_{i=1}^{p} sol_i$;

By using the construction of solutions $sol_{i,1}, \ldots, sol_{i,m_i}$, we easily deduce the following lemma:

Lemma 3.8 *Consider a cycle $C_i = (1, \ldots, m_i, 1)$ of M with $m_i = kq + r$, $0 \leq r < k$. We have:*

  (i) $\sum_{j=1}^{m_i} d(sol_{i,j}) = (m_i - q)d(C_i) + 2q \sum_{j=1}^{m_i} d(0, j)$ *if $r = 0$.*

  (ii) $\sum_{j=1}^{m_i} d(sol_{i,j}) = (m_i - q - 1)d(C_i) + 2(q+1) \sum_{j=1}^{m_i} d(0, j)$ *if $r \neq 0$.*

Proof:(*i*): $sol_{i,j}$ contains $\lceil \frac{m_i}{k} \rceil = q$ cycles for every $j = 1, \ldots, m_i$. Thus, in $\cup_{j=1}^{m_i} sol_{i,j}$, each edge of C_i appears exactly $m_i - q$ times and each edge $(0, j)$ appears exactly $2q$ times.

(*ii*): $sol_{i,j}$ contains $\lceil \frac{m_i}{k} \rceil = q + 1$ cycles for every $j = 1, \ldots, m_i$. So, the same argument as previously shows that each edge of C_i appears exactly $m_i - (q+1)$ times and each edge $(0, j)$ appears exactly $2(q+1)$ times in $\cup_{j=1}^{m_i} sol_{i,j}$. ∎

Theorem 3.9 METRIC 4VRP *is $\frac{3}{5}$ differential approximable and* METRIC kVRP *is $\frac{2}{3}$ differential approximable with $5 \leq k \leq 8$.*

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Proof: Our preprocessing algorithm works as follows: we compute a minimum weight binary 2-matching $M = (C_1, \ldots, C_p)$ on the subgraph induced by $V \setminus \{0\}$. Consider a cycle $C_i = (1, \ldots, m_i, 1)$ of M with $m_i = kq + r$ and let $wor_i = 2 \sum_{j=1}^{m_i} d_{0,j}$.

Assume $q = 0$. Since the best solution (i.e., sol_i) is better than the average one, we obtain using Lemma 3.8:

$$d(sol_i) \leq \frac{r-1}{r} d(C_i) + \frac{1}{r} wor_i = \frac{1}{r}(wor_i - d(C_i)) + d(C_i) \ . \tag{11}$$

Since $wor_i \geq d(C_i)$ by the triangle inequality and $r \geq 3$ (C_i contains at least 3 vertices), we deduce:

$$d(sol_i) \leq \frac{2}{3} d(C_i) + \frac{1}{3} wor_i \ . \tag{12}$$

Now, assume $q \geq 1$. If $r = 0$, then we deduce:

$$d(sol_i) \leq \frac{k-1}{k} d(C_i) + \frac{1}{k} wor_i \leq \frac{2}{3} d(C_i) + \frac{1}{3} wor_i \ . \tag{13}$$

since $k \geq 3$. Otherwise, we have $r \geq 1$ and we obtain:

$$d(sol_i) \leq \frac{q+1}{kq+r}(wor_i - d(C_i)) + d(C_i)$$

and we deduce since $r, q \geq 1$:

$$d(sol_i) \leq \frac{k-1}{k+1} d(C_i) + \frac{2}{k+1} wor_i \tag{14}$$

On the one hand, it is possible to construct from an optimum solution of METRIC VRP a feasible solution of TSP on the subgraph induced by $V \setminus \{0\}$ (by shortcutting) with a smaller value and we deduce $d(M) = \sum_{i=1}^{p} d(C_i) \leq opt_{TSP} \leq opt_{VRP}$. On the other hand $wor = \sum_{i=1}^{q} wor_i$. Finally, by summing over i the inequalities (12), (13) and (14) and by distinguishing the case $k = 4$ and $k > 4$ we obtain the expected result. ∎

The algorithm of Theorem 3.9 works for any $k \geq 3$ and it gives the ratio $\frac{1}{2}$ for METRIC 3VRP and $\frac{2}{3}$ for $k \geq 9$. We now improve the previous bound for $k = 3$ and $k \geq 9$ using another preprocessing algorithm. But surprisingly, this algorithm compute an approximate TSP with maximum weight.

Observation 3.10 *The differential and standard approximation ratios for* MAX WEIGHTED ATMOSTkPP *coincide. Indeed, we have $wor = 0$ since $\{P_i\}_{i \in V}$ where $P_i = \{i\}$ is a feasible solution.*

This problem is very close to METRIC kVRP when we deal with differential ratio:

Theorem 3.11 *For any $k \geq 3$,* MAX WEIGHTED ATMOSTkPP *and* METRIC kVRP *are differential equivalent.*

Proof: In order to reduce METRIC kVRP to MAX WEIGHTED ATMOSTkPP, consider an instance G of METRIC kVRP with n customers. We construct an instance I' of MAX WEIGHTED ATMOSTkPP as follows: we delete the depot 0 and consider the graph K_n and set $d'_{x,y} = d_{0,x} + d_{0,y} - d_{x,y}$ for any vertices $x, y \in V \setminus \{0\}$. By the triangle inequality,

$d'_{x,y} \geq 0$. $d'_{x,y}$ denotes the saving gained with respect to the worst solution, by joining x and y in a cycle rather then reaching each of them from the depot. We have an one to one correspondence between a path $P = (1, \ldots, j)$ using at most k vertices in I' and the cycle $C = (0, 1, \ldots, j, 0)$ with at most k customers in G. Moreover, $d'(P) = 2\sum_{i=1}^{j} d_{0,i} - d(C)$. Finally, we also have an one to one correspondence between feasible solutions of these two problems, and since $wor = 2\sum_{i=1}^{n} d_{0,i}$, for any solution of G of value val we have

$$val' = wor_{VRP} - val. \tag{15}$$

Conversely we reduce MAX WEIGHTED ATMOSTkPP to METRIC kVRP. Let G and d be an instance of MAX WEIGHTED ATMOSTkPP. We add a depot 0 and we set: $d'_{0,i} = \max_{e\in E} d_e, \forall i \in V$ and $d'_{i,j} = 2\max_{e\in E} d_e - d_{i,j}, \forall i, j \in V$. The rest of the proof is similar. ∎

Let ρ be the standard approximation ratio for MAX TSP. The current best value for ρ is $\frac{25}{33}$ obtained by a randomized algorithm in [17].

Theorem 3.12 METRIC kVRP *is* $(\frac{25}{33}\frac{k-1}{k} - \varepsilon)$ *differential randomized approximable for* $k \geq 3$ *and any* $\varepsilon > 0$.

Proof: Let G be an instance of METRIC kVRP with n customers and let $\varepsilon > 0$. In order to obtain a good solution for G, we apply algorithm `Algo_Differential Metric`k`VRP` where the preprocessing is a tour $T = C_1$. This tour is produced by the algorithm from [17] applied on the instance $I' = (K_n, d')$ with $n = kq + r$ obtained from G as in Theorem 3.11, algorithm that is a $\frac{25}{33}$ randomized approximation. Using the definition of weight d' and the Lemma 3.8, we obtain:

$$wor_{VRP} - apx = \max_{1\leq j\leq n} d'(sol_{1,j}) \geq \frac{\sum_{j=1}^{n} d'(sol_{1,j})}{n} \geq (\frac{k-1}{k} - \varepsilon)d'(C_1).$$

when $q \geq \frac{k-1}{\varepsilon k^2} - \frac{1}{k}$. Otherwise, we exhaustively solve the problem.

On the other hand, an optimal solution of MAX WEIGHTED ATMOSTkPP on I' can be used to construct a feasible solution of MAX TSP on I' by joining the endpoints of the paths. Hence $opt_{MaxTSP} \geq opt_{\text{MAX WEIGHTED ATMOST}k\text{PP}}$. Finally, by using the $\frac{25}{33}$ standard approximation algorithm for MAX TSP for obtaining the tour T, we have $d'(C_1) \geq \frac{25}{33} opt_{MaxTSP}$ and $opt_{\text{MAX WEIGHTED ATMOST}k\text{PP}} = wor_{VRP} - opt_{VRP}$ since (15). ∎

In particular, we obtain a $(\frac{50}{99} - \varepsilon)$ differential randomized approximation for METRIC 3VRP, that is better than the $\frac{1}{2}$ differential approximation given in Theorem 3.4. It also improves the result of Theorem 3.9 for $k \geq 9$ since we obtain the differential ratio $\delta = \frac{25(k-1)}{33k} - \varepsilon > \frac{2}{3}$ for METRIC kVRP. For instance, this ratio is $\frac{200}{297} \simeq 0.67$ for $k = 9$.

We summarize in the following the differential results that we obtain for METRIC kVRP:

- METRIC 3VRP is $(\frac{50}{99} - \varepsilon)$ differential randomized approximable for any $\varepsilon > 0$.

- METRIC 4VRP is $\frac{3}{5}$ differential approximable.

- METRIC kVRP is $\frac{2}{3}$ differential approximable for $5 \leq k \leq 8$.

- METRIC kVRP is $(\frac{25}{33}\frac{k-1}{k} - \varepsilon)$ differential randomized approximable for any $k \geq 9$ and for any $\varepsilon > 0$.

Finally, note the similarity between the results given in Theorem 3.7 and the one given in Theorem 3.12. They both deal with the reduction in approximation from METRIC kVRP to MAX TSP (MAX TSP and MIN METRIC TSP are equivalent with respect to the differential ratio [20]) and the expansion is very similar $\delta\frac{k-1}{k}$ for Theorem 3.7 and $\rho\frac{k-1}{k} - \varepsilon$ for Theorem 3.12. The only difference is on the measure used: The first reduction considers the differential ratio for the two problems whereas the second one considers the standard ratio for MAX TSP. Actually, the standard ratio $\rho = \frac{25}{33}$ is better than differential ratio $\delta = \frac{2}{3}$ for MAX TSP and more generally the best standard ratio ρ_{best} for MAX TSP will be always better than the best differential ratio δ_{best} (i.e., $\rho_{best} \geq \delta_{best}$) since we have a trivial reduction from any maximization problem to itself transforming a differential result into a standard result (see Lemma 1.3 in Monnot [20]), leading to the conclusion that the reduction of Theorem 3.12 is better. Nevertheless, if the optimal result is $\rho_{best} = \delta_{best}$ then the reduction of Theorem 3.7 will be better.

Since nVRP and TSP are standard equivalent, from the result of Papadimitriou and Yannakakis [24] we deduce immediately that nVRP(1,2) has no standard approximation scheme unless $P = NP$. Also TSP(1,2) has no differential approximation scheme [21] but we cannot deduce immediately that nVRP(1,2) has no differential approximation scheme since wor_{nVRP} and wor_{TSP} may be very far. However, we prove in the following a lower bound for the differential approximation of nVRP(1,2).

Theorem 3.13 nVRP$(1,2)$ *is not* $(\frac{2219}{2220} - \varepsilon)$ *differential approximable, for any constant* $\varepsilon > 0$, *unless P=NP.*

Proof: Since $wor_{nVRP} \leq 4n \leq 4opt_{nVRP}$, a δ differential approximation for nVRP$(1,2)$ gives a $\delta + 4(1-\delta)$ standard approximation for nVRP$(1,2)$. Using the negative result given in [8] that TSP(1,2) is not $\frac{741}{740} - \varepsilon$ standard approximable, we obtain the expected result.

∎

3.2.2 Some standard approximation results

Despite these observations, by using Theorem 3.9 for METRIC kVRP and Theorem 3.5 we establish better standard approximation ratio than Haimovich, Rinnooy Kan and Stougie (i.e., $(\frac{5}{2} - \frac{3}{2k})$ standard approximation) when we deal with bounded metrics, i.e., $d_{max} \leq td_{min}$. More exactly, METRIC 4VRP[2] is $\frac{47}{25}$ standard approximable and METRIC kVRP[2] is $(2 - \frac{4}{3(k+1)})$ standard approximable for $k \geq 5$.

We now describe some results concerning the standard approximability of METRIC kVRP. In [12], a $(\frac{5}{2} - \frac{3}{2k})$ standard approximation for METRIC kVRP is obtained by reduction to METRIC TSP and using Christofides' algorithm.

The following theorem gives a reduction preserving the approximation scheme between standard and differential ratio even if we deal with unbounded metrics ($\frac{d_{max}}{d_{min}}$ is not upper bounded).

Theorem 3.14 *A* δ *differential approximation algorithm for* METRIC kVRP *is also a* $k - \delta(k-1)$ *standard approximation algorithm.*

Proof: Consider an optimal solution for an instance G of METRIC kVRP and w.l.o.g. denote by $(0, 1, \ldots, m_i, 0)$ one of its cycles. Using the triangle inequality, the length of this cycle is at least $2max\{d_{0,i} : i = 1, \ldots, m_i\} \geq \frac{2}{k} \sum_{i=1}^{m_i} d_{0,i}$. Summing over each cycle, we obtain using Lemma 3.6:

$$opt \geq \frac{2}{k} \sum_{i=1}^{n} d_{0,i} = \frac{wor}{k}. \tag{16}$$

Let apx be a δ differential approximation for G. Using the inequality (16) we deduce:

$$apx \leq \delta opt + (1 - \delta)wor \leq \delta opt + k(1 - \delta)opt. \tag{17}$$

∎

Using Theorem 3.14, Observation 3.10 and Theorem 3.12 we obtain:

Corollary 3.15 METRIC 3VRP *is* $(3 - \frac{4}{3}\rho + \varepsilon)$ *standard approximable for all* $\varepsilon > 0$ *where* δ *is the standard approximation ratio for* MAX TSP.

More exactly, since $\rho = \frac{25}{33}$ [17] we obtain the bound $\frac{197}{99} \simeq 1.99$ that is an improvement of the 2 standard approximation of Haimovich et al. [12].

4   EDGE COST VRP

We assume now that each edge is associated with a cost ℓ satisfying the triangle inequality, and the solution must satisfy that the total cost on each cycle does not exceed λ.

Note that if we do not assume that ℓ is a metric then even deciding whether the problem has any feasible solution is NP-complete. For a proof see Theorem 7.1 below. Therefore, we assume that ℓ satisfies the triangle inequality, and to ensure feasibility we also assume that $2\ell_{0,i} \leq \lambda$ for $i = 1, \ldots, n$.

Theorem 4.1 EDGE COST VRP *is* $\frac{1}{3}$ *differential approximable.*

Proof: We start with a binary 2-matching as described in Lemma 3.2 except that the initial graph is not a complete undirected graph G but a partial graph G' of it built by deleting the edges (i, j) for $i \neq 0$ and $j \neq 0$ such that $\ell_{0,i} + \ell_{i,j} + \ell_{j,0} > \lambda$. Observe that M is still a lower bound of an optimal solution of EDGE COST VRP. Then, we apply the algorithm `Algo_Differential VRP` except that we change steps 3.2, 5 and 6. The step 3.2 becomes the following: we produce m_i solutions $sol_{i,1}, \ldots, sol_{i,m_i}$ where $sol_{i,j} = \{(0, j+1, j+2, 0), \ldots, (0, j-2, j-1, 0)\} \cup \{(0, j, 0)\}$ for $j = 1, \ldots, m_i$.

The step 5 becomes: for every cycle $C_i = (0, 1, \ldots, m_i, 0)$ of M with $\sum_{e \in E(C_i)} \ell_e \leq \lambda$ do $sol_{i,1} = sol_{i,2} = C_i$ whereas the step 6 becomes: the solution APX is the solution obtained by concatenating the shortest of $sol_{i,j}$ for each cycle C_i.

Observe that in step 3.2, each edge of C_i appears exactly $\lfloor \frac{m_i}{2} \rfloor$ times in $(\cup_{j \leq m_i} sol_{i,j})$ and each edge $(0, j)$ appears exactly $m_i + 1$ times. Thus, since $m_i \geq 2$, the same arguments as in Theorem 3.4 proved that APX is a $\frac{1}{3}$ differential approximation. ∎

In [12], the authors consider two versions of kVRP with additional constraint on the length of each cycles. In the first problem that we will call here VERTEX COST kVRP, each customer has a cost and we want to find a solution such that the total customer cost on

each cycle does not exceed k. In the second, called in [19] MIN METRIC DISTANCE, we want to find a solution such that the total cost on each cycle does not exceed a given bound λ. For each of these two problems, we give a differential reduction preserving approximation scheme from EDGE COST VRP.

Lemma 4.2 *A δ differential approximation solution for* EDGE COST VRP *(respectively, metric case) is also a δ differential approximation for* VERTEX COST kVRP *(respectively, metric case).*

Proof: Let $G = (V, E)$ with d and c be an instance of VERTEX COST kVRP. We construct an instance of EDGE COST VRP as follows. First we set $\lambda = k$. The graph and the function d are the same whereas the function ℓ is defined by: $\ell_{i,j} = \frac{c_i + c_j}{2}$ where we assume that $c_0 = 0$. This function satisfies the triangle inequality. Moreover, let C be a cycle linking the depot to a subset of customers. We have $\sum_{i \in V(C)} c_i \leq \lambda$ iff $\sum_{e \in E(C)} \ell_e \leq \lambda$. ∎

Corollary 4.3 VERTEX COST kVRP *is $\frac{1}{3}$ differential approximable.*

MIN METRIC DISTANCE is a particular case of EDGE COST VRP where the function ℓ is exactly the function d. Thus, from Theorem 4.1 we deduce the corollary:

Corollary 4.4 MIN METRIC DISTANCE *is $\frac{1}{3}$ differential approximable.*

For EDGE COST VRP and VERTEX COST kVRP there have neither standard nor differential approximation scheme unless $P = NP$ since these two problems contain nVRP.

5 MIN-MAX kTSP

The metric case of the problem was studied by Fredrickson, Hecht and Kim [9] where they give a $\frac{5}{2} - \frac{1}{k}$ standard approximation algorithm by constructing a reduction from this problem to METRIC TSP and using Christofides' algorithm [4].

Theorem 5.1 MIN-MAX rTSP *is not $2^{p(n)}$ standard approximable for any polynomial p and $r \geq 1$, unless P=NP.*

Proof: We reduce HAMILTONIAN PATH problem to MIN-MAX rTSP. We start with the reduction described in Theorem 3.1 with $k = n - 1$ and $q = 1$ and the weight $n2^{p(n)}$ is replaced by $(n + 3)2^{p(n)}$ (recall that the $(n - 1)$PP problem is the HAMILTONIAN PATH problem) and we apply r times this reduction (so, the final graph consists of depot and r copies of G and set A of $2r$ vertices). Thus, a $2^{p(n)}$ standard approximation for MIN-MAX rTSP could decide HAMILTONIAN PATH, that is proved NP-hard in [10]. ∎

We now turn to the metric case. We give a $\frac{1}{2}$ differential approximation algorithm for METRIC MIN-MAX kTSP, $k \geq 2$ and we show that the problem has neither standard nor differential approximation scheme unless P=NP.

Theorem 5.2 METRIC MIN-MAX 2TSP *is $\frac{1}{2}$ differential approximable.*

Proof: Consider a tour $T = (0, \ldots, n, 0)$ of G. Let i be the smallest index such that $\sum_{j=0}^{i} d_{j,j+1} \geq \frac{d(T)}{2}$. We consider the solution $C_1 = (0, 1, \ldots, i, 0)$ and $C_2 = (0, i+1, \ldots, n, 0)$.

Note that

$$d(C_1) - d_{0,i} = \sum_{j=0}^{i-1} d_{j,j+1} \leq \frac{d(T)}{2}$$

and

$$d(C_2) - d_{0,i+1} = d(T) - \sum_{j=0}^{i} d_{j,j+1} \leq d(T) - \frac{d(T)}{2} = \frac{d(T)}{2}.$$

So, $\max\{d(C_1), d(C_2)\} \leq \frac{d(T)}{2} + \max\{d_{0,i}, d_{0,i+1}\} \leq \frac{wor_{TSP}}{2} + \frac{opt_{2TSP}}{2}$. Since a worst tour on V with the value wor_{TSP} is a feasible solution for 2TSP then $wor_{2TSP} \geq wor_{TSP}$. Thus, $\max\{d(C_1), d(C_2)\} \leq \frac{wor_{2TSP}}{2} + \frac{opt_{2TSP}}{2}$. ∎

Corollary 5.3 METRIC MIN-MAX kTSP *is* $\frac{1}{2}$ *differential approximable.*

Proof: The previous algorithm is a $\frac{1}{2}$ differential approximationfor general $k \geq 3$ since we have also $wor_{kTSP} \geq wor_{TSP}$ and $\max\{d_{0,i}, d_{0,i+1}\} \leq \frac{opt_{kTSP}}{2}$. ∎

Theorem 5.4 MIN-MAX kTSP(1,2), $k \geq 2$ *has neither standard nor differential polynomial time approximation scheme, unless P=NP.*

Proof: We construct a L-reduction from MIN TSP(1,2), when the subgraph restricted to the edges of length 1 is Hamiltonian, to METRIC MIN-MAX kTSP.

Let G be a complete graph on n vertices, with edges of length 1 and 2. We construct an instance G' of MIN-MAX kTSP adding to G a depot, the vertex 0, and we set the distance between 0 and a vertex i of G to 2. Suppose that $n = q \cdot k + r$, $0 < r \leq k$. It is easy to see that $opt(G) = opt_{TSP}(G) = n$ and $opt(G') = opt_{Min-Max\ kTSP}(G') = q + 4$ since the optimum of G' is obtained when the Hamiltonian cycle is divided in k paths where the difference of sizes is at most 1. So $opt(G') = \frac{opt(G)-r}{k} + 4$, or $opt(G) = k \cdot opt(G') - 4k + r$. From a solution of G' with value val' we construct a solution in G putting together the paths induced by the solution in G and linking these paths by edges of length at most 2. This solution has the value $val \leq k(val' - 4) + 2k$. So,

$$val - opt(G) \leq k \cdot val' - 2k - k \cdot opt(G') - 4k + r \leq k[val' - opt(G')] - 5k \leq k[val' - opt(G')].$$

Since MIN TSP(1,2), when the subgraph restricted to edges of length 1 is Hamiltonian, has no standard polynomial time approximation scheme [2], then MIN-MAX kTSP also has no standard approximation scheme.

In order to see that MIN-MAX kTSP has no differential approximation scheme, we show that if it was the case then MIN-MAX kTSP on the particular instances that we consider above would have a standard approximation scheme. Suppose that MIN-MAX kTSP has a differential approximation scheme A_δ, $\forall \delta, 0 < \delta < 1$. So, A_δ gives a solution for G' with a value $val \leq \delta opt(G') + (1 - \delta)wor(G')$. For the above instances G' of MIN-MAX kTSP, $opt(G') = \frac{n-r}{k} + 4$ and $wor(G') = 2(n-1) + 4 \leq 2k opt(G')$. Thus, $val \leq [\delta + 2k(1-\delta)]opt(G')$, and for an $(1 + \varepsilon)$ standard approximation solution for an instance of MIN-MAX kTSP, $\forall \varepsilon > 0$, we apply A_δ with $\delta = 1 - \frac{\varepsilon}{2k-1}$. ∎

For certain cases we can give inapproximability gaps, for examples, when we have $\lfloor \frac{n}{6} \rfloor$ vehicles we can prove that the problem is not $\frac{7}{6}$ approximable and more generally we obtain:

Theorem 5.5 MIN-MAX $\lfloor \frac{n}{k} \rfloor$TSP(1,2), $k \geq 6$ *is not* $\frac{k+1}{k} - \varepsilon$ *standard approximable for any* $\varepsilon > 0$, *unless P=NP*.

Proof: We use a reduction from $(k-4)$PP with $k \geq 6$. We use the reduction described in Theorem 3.1 except that we replace the distances $n2^{p(n)}$ by distances 2. Then, if G contains a decomposition in paths of length $k-4$ then $opt(G') = k$, otherwise $opt(G') \geq k+1$. So, a $\frac{k+1}{k} - \varepsilon$ standard approximation for MIN-MAX $\lfloor \frac{n}{k} \rfloor$TSP(1,2) could decide $(k-4)$PP in polynomial time. ∎

6 MIN-SUM EkTSP

Bellmore and Hong [3] showed that when the constraint $p = k$ is replaced by $p \leq k$, then MIN-SUM kTSP is standard equivalent to TSP on an extended graph. This is true even for the directed version of the problem and when there is a cost associated with activating a salesman. For our case the transformation simply involves replacing the depot vertex 0 by k vertices of zero distance. Also, the metric case of the $p \leq k$ version is not of interest since the solution is just a single cycle (thus, we deal with the case $p = k$ and MIN-SUM EkTSP denote this problem).

MIN-SUM EkTSP is differential equivalent to METRIC MIN-SUM EkTSP. This observation follows since the number of edges in every solution is the same (like in the TSP case). Hence, we add a constant to all the edge lengths and achieve the triangle inequality without affecting the best and worst solutions.

Similarly, MIN-SUM EkTSP is differential equivalent to MAX-SUM EkTSP.

Theorem 5.1 can be adapted in order to prove that MIN-SUM EkTSP is not $2^{p(n)}$ standard approximable, for any polynomial p, unless $P=NP$.

We now give the main results of this section.

Theorem 6.1 METRIC MIN-SUM EkTSP *is* $\frac{2}{3}$ *differential approximable,* $\forall k \geq 1$.

Proof: Let G and d be an instance of METRIC MIN-SUM EkTSP. Add to every edge incident with the depot a parallel copy. Compute a minimum binary f-matching $M = \{C_1, \ldots, C_p\}$ ($C_1, \ldots, C_k$ denote the cycles of M containing the depot 0) on G where $f(0) = 2k$ and $f(v) = 2$ for $v \in V \setminus \{0\}$. Compute by using a $\frac{2}{3}$ differential approximation algorithm of [15] or [20] a solution C' for TSP on the subgraph G' of G induced by $V' = V \setminus (\cup_{i=1}^{k-1} V(C_i)) \cup \{0\}$. The approximate solution sol for METRIC MIN-SUM EkTSP is composed of C' and the cycles $C_1, \ldots, C_{k-1}$. See Figure 3. Since M is a minimum binary f-matching M on G then $M' = M \setminus (\cup_{i=1}^{k-1} C_i)$ is an optimum binary 2-matching on G'. Let $r = \sum_{i=1}^{k-1} d(C_i)$. It is proved in [15] or [20] that the TSP algorithm gives a solution satisfying $val \leq \frac{2}{3} d(M') + \frac{1}{3} wor_{TSP}(G')$. Since $wor_{kTSP}(G) \geq wor_{TSP}(G') + r$ and $opt_{kTSP}(G) \geq d(M') + r$, we deduce that the value of sol satisfies:

$$apx = val + r \leq \frac{2}{3}[d(M') + r] + \frac{1}{3}[wor_{TSP}(G') + r] \leq \frac{2}{3} opt_{kTSP}(G) + \frac{1}{3} wor_{kTSP}(G)$$

∎

Theorem 6.2 *Unless P=NP,* MIN-SUM EkTSP(1,2) *has no standard and differential approximation scheme for any* $k \geq 2$.

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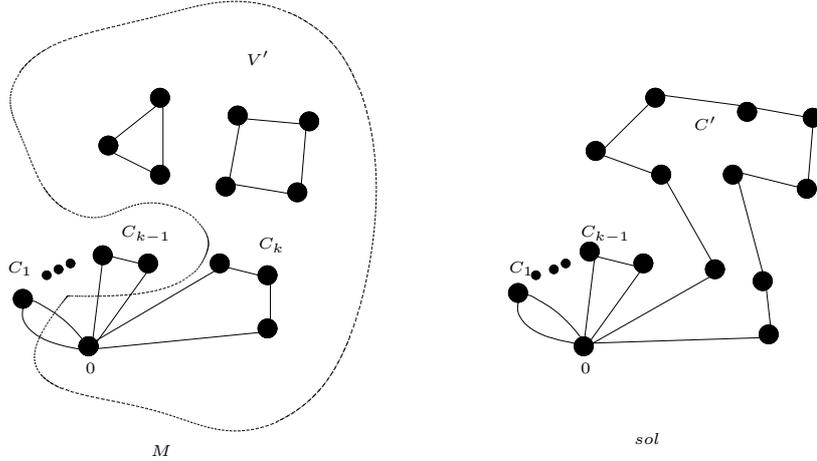
Figure 3: M and sol

Proof: We reduce MIN TSP PATH (1,2) on Hamiltonian cubic graphs to MIN-SUM E2TSP(1,2). From a graph $G = (V, E)$ on n vertices, we construct a graph G' instance of MIN-SUM E2TSP(1,2). G' consists of two copies of G and a vertex 0 (the depot). Within a copy, the edges have the same distance as in G; $d_{0,i} = 1$, for each vertex i in one of the two copies; $d_{i,j} = 2$ if i and j are vertices in different copies. Using the equalities $opt(G) = n - 1 = \frac{wor(G)}{2}$ and $opt(G') = 2n + 2$, $wor(G') = 4n$, we have $opt(G') = 2opt(G) + 4$ and $wor(G') = 2wor(G) + 4$. Given a solution S of G' with two cycles, we can transform it in another one S' that contains exactly two cycles $(0, P_1, 0)$, $(0, P_2, 0)$, each of these two paths are contained in a copy of G and with a better value. The idea for doing this is to remove the edges between the two copies in the solution S and in each copy, we arbitrarily connect the resulting paths. We consider as solution for G the path with the smallest value among the two. So, $val = \min\{val(P_1), val(P_2)\} \leq \frac{val(P_1) + val(P_2)}{2} = \frac{val(S') - 4}{2} \leq \frac{val(S) - 4}{2}$. Since $opt(G) = \frac{opt(G')}{2} - 2$ and $wor(G) = \frac{wor(G')}{2} - 2$ then a δ differential approximation of MIN-SUM E2TSP(1,2) gives a δ differential approximation for MIN TSP PATH (1,2) on Hamiltonian and cubic graphs. The conclusion follows for MIN-SUM E2TSP(1,2) since MIN TSP PATH (1,2) on Hamiltonian and cubic graphs has no differential approximation scheme ([2, 22]). It is easy to see that if S is a $(1 + \frac{\varepsilon}{2})$ standard approximation of MIN-SUM E2TSP(1,2) then the same solution as above with value val is a $(1 + \varepsilon)$ standard approximation of MIN TSP PATH (1,2). The proof for $k \geq 3$ is similar. ∎

7 MIN VEHICLE

In this problem, the goal is to visit the customers by a minimum number of vehicles, under a constraint on the total distance traveled by a vehicle.

In [19], it is proved that METRIC MIN VEHICLE is not standard 2 approximable, unless $P = NP$. Indeed even deciding whether the problem has a feasible solution is NP-complete:

Theorem 7.1 *Deciding the feasibility of* MIN VEHICLE *is NP-complete.*

Proof: In order to prove the NP-hardness, we reduce HAMILTONIAN PATH problem to

Min Vehicle. We again apply the reduction described in Theorem 3.1 with $k = n - 1$ and $q = 1$, except that the distances $n2^{p(n)}$ are replaced by the distances λ. Trivially there is a feasible solution for G' only if $\lambda \geq n + 3$. It is easy to see that Min Vehicle has a feasible solution iff G contains a Hamiltonian path. ∎

In contrast, deciding the feasibility of Metric Min Vehicle is trivial, and the condition simply amounts to $d(0, i) \leq \frac{\lambda}{2}$ for $i = 1, \ldots, n$. The following theorem gives a positive result for this problem:

Theorem 7.2 Metric Min Vehicle *is* $\frac{2}{3}$ *differential approximable.*

Proof: Consider the collection $\mathcal{C}$ of sets of vertices of feasible cycles (cycles that include the depot and whose length is at most λ). Since we assume that d is a metric, $\mathcal{C}$ is a *monotone collection*, that is, if $C' \subset C$ and $C \in \mathcal{C}$ then also $C' \in \mathcal{C}$. This means that if G' is a subgraph of G that includes the depot, then the optimal solution on G' is at most that of G. This allows us to apply the following "greedy" approach:

Construct feasible cycles with the depot and three vertices, as long as this is possible. Let G' be the graph G except the vertices of these cycles (the depot is preserved in G'). For an edge (i, j), if $d_{0,i} + d_{0,j} + d_{i,j} > \lambda$ then we remove this edge from G'. Denote the resulting graph also by G'. Find a maximum size matching in G'. We will show below that a such maximum size matching in G' is an optimum solution on G'. We now show that the union of these cycles is a $\frac{2}{3}$ differential approximation.

Denote by k_3 the number of cycles on three vertices and the depot, constructed in the first step of the algorithm. Denote by k_2 (and k_1) the number of edges (and isolated vertices) obtained in G' when we search a maximum size matching. So, $val(G) = k_1 + k_2 + k_3$. The value of the solution obtained in G' in this way is $val' = k_1 + k_2 = |V(G')| - k_2$ since $k_1 + 2k_2 = |V(G')|$. Since we want to minimize val' a maximum size matching gives an optimum solution. Since $opt(G) \geq opt(G')$ and $wor = n = |V(G)|$, we obtain that $val(G) = k_1 + k_2 + k_3 = k_1 + k_2 + \frac{n - k_1 - 2k_2}{3} \leq \frac{2}{3}opt(G) + \frac{1}{3}wor(G)$. ∎

The algorithm of Theorem 7.2 is similar to the approach in [16] for obtaining differential approximation for Graph Coloring. By applying approximation algorithms for 3-Set Cover and following the ideas of Halldórsson [13] for obtaining better differential approximation for Graph Coloring (see also [15]), the bound can be improved. Consider the following algorithm: Construct feasible cycles with four vertices as long as this is possible. Let G' be the graph G except the vertices of these cycles. List all the feasible cycles in G'. Note that such cycles include the depot and at most three other vertices, and therefore their number is polynomial. Apply an approximation algorithm for Min 3-Set Exact Cover of a Monotone Collection, such as the algorithm of Halldórsson [13] or Duh and Fürer [7]. This former result is a $\frac{3}{4}$-differential approximation (see Theorem 5.2 in [13]), and the latter gives a bound of $\frac{289}{360}$ (see Theorem 4.2 in [7]). Note that the mentioned results were developed to give differential approximations for Graph Coloring, but they apply as well to any problem of exact covering by sets that correspond to a monotone collection (see Section 4 of [15]).

In [19], it is proved that unless $P{=}NP$, Min Vehicle is not standard 2 approximable and thus without standard approximation scheme when $\lambda \to \infty$. In the following we establish the same result for λ constant and for the differential case.

Theorem 7.3 MIN VEHICLE(1,2) *has no standard and differential approximation scheme even if* λ *is constant, unless P=NP.*

Proof: We prove firstly that MIN VEHICLE(1,2) has no standard approximation scheme, if $P \neq NP$ by reducing MIN TSP(1,2) problem on Hamiltonian graphs to MIN VEHICLE(1,2). MIN TSP(1,2) problem on cubic Hamiltonian graphs has no standard approximation scheme [2], thus there is a constant $\varepsilon_0, 0 < \varepsilon_0 < 1$, such that it is not $1 + \varepsilon$ standard approximable for $\varepsilon \leq \varepsilon_0$, if $P \neq NP$.

Given a graph $G = (V, E)$ on n vertices, we construct a graph G' instance of MIN VEHICLE. G' consists of one copy of G and a vertex 0 (the depot) and we define the function d' as follows: $d'_{0,i} = 1$, for $i \in \{1, \ldots, n\}$ and $d'_{i,j} = d_{i,j}$ if $i, j \in \{1, \ldots, n\}$. It is easy to see that $opt_1 = opt(G) = n$ and $opt_2 = opt(G') = \lceil \frac{n}{\lambda-1} \rceil \leq \frac{n}{\lambda-1} + 1 \leq \frac{n}{\lambda-2}$ when $n \geq (\lambda - 1)(\lambda - 2)$. Given a solution S' of G' with val_2 vehicles, $S' = C_1, \ldots, C_{val_2}$, we consider as solution S for G the restriction of this solution to the vertices of G. The value of S is $val_1 \leq \sum_{i=1}^{val_2} d(C_i) \leq \lambda val_2$ by the triangle inequality.

Suppose that MIN VEHICLE(1,2) would have a standard approximation scheme A_δ. We prove that this assumption imply that MIN TSP(1,2) has an approximation scheme, contradiction. In order to obtain an $(1 + \varepsilon)$ approximation for G, we apply $A_{\frac{\varepsilon}{3}}$ on G' with $\lambda = 3 + \lceil \frac{3}{\varepsilon} \rceil$. Thus

$$val_1 \leq \lambda(1 + \frac{\varepsilon}{3})\frac{n}{\lambda-2} \leq (1 + \varepsilon)n$$

since $\lambda \geq 3 + \frac{3}{\varepsilon}$.

Using this last result we prove that this problem has no differential approximation scheme if $P=NP$. Suppose that MIN VEHICLE(1,2) when the graph restricted to edges of weight 1 is Hamiltonian would have a differential δ approximation scheme $A_\delta, \forall \delta, 0 < \delta < 1$. Therefore, for each instance G of the problem on n vertices, with $\lambda = 3 + \lceil \frac{3}{\varepsilon_0} \rceil$, this algorithm gives a solution for G with a value $val(G) \leq \delta opt(G) + (1 - \delta) wor(G)$. Since on these instances $wor(G) = n$ and $opt(G) = \lceil \frac{n}{\lambda-1} \rceil \geq \frac{n}{\lambda-1}$ then $wor(G) \leq (2 + \frac{3}{\varepsilon_0}) opt(G)$ and so $val(G) \leq [\delta + (2 + \frac{3}{\varepsilon_0})(1 - \delta)] opt(G)$. Thus, in order to obtain a standard $(1 + \varepsilon)$ approximation algorithm, $0 < \varepsilon < 1$, we have to take the solution given by A_δ with $\delta = 1 - \varepsilon \frac{\varepsilon_0}{3+\varepsilon_0}$. The result follows since as we prove above MIN VEHICLE(1,2) on these instances has no standard approximation scheme, unless $P=NP$. ∎

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