The satisfactory partition problem

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Abstract

The SATISFACTORY PARTITION problem consists in deciding if a given graph has a partition of its vertex set into two nonempty parts such that each vertex has at least as many neighbors in its part as in the other part. This problem was introduced by Gerber and Kobler [GK98, GK00] and further studied by other authors but its complexity remained open until now. We prove in this paper that SATISFACTORY PARTITION, as well as a variant where the parts are required to be of the same cardinality, are *NP*-complete. However, for graphs with maximum degree at most 4 the problem is polynomially solvable. We also study generalizations and variants of this problem where a partition into k nonempty parts ($k \geq 3$) is requested.

Keywords: Satisfactory partition, graph, complexity, polynomial algorithm, *NP*-complete.

1 Introduction

Gerber and Kobler introduced in [GK98, GK00] the problem of deciding if a given graph has a vertex partition into two nonempty parts such that each vertex has at least as many neighbors in its part as in the other part. A graph satisfying this property is called *partitionable*. As remarked by Gerber and Kobler, SATISFACTORY PARTITION may have no solution. In particular, the following graphs are not partitionable: complete graphs, stars, and complete bipartite graphs with at least one of the two vertex sets having odd size. Some other graphs are easily partitionable: cycles of length at least 4, trees which are not stars, and disconnected graphs. After [GK98, GK00] this problem was further studied in [SD02], [GK03], [BTV03] and [GK04] but its complexity remained open until now, while some generalizations were studied and proved to be *NP*-complete.

Gerber and Kobler showed in [GK98, GK00] the strong *NP*-hardness of a first generalization of this problem where vertices are weighted and we ask for a vertex partition into two nonempty parts such that for each vertex the sum of weights of the neighbors in the same part is at least as large as the sum of weights of the neighbors in the other part. Another generalization where the *edges* are weighted was also proved to be *NP*-hard in the strong sense.

An “unweighted” generalization of SATISFACTORY PARTITION was studied in [BTV03] where each vertex v is required to have at least $s(v)$ neighbors in its own part, for a given

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function s representing the degree of satisfiability. Obviously, when $s = \lceil \frac{d}{2} \rceil$, where d is the degree function, we obtain Satisfactory Partition. Stiebitz proved in [Sti96] that if $s \leq \lceil \frac{d}{2} \rceil - 1$ then such a partition always exists; and we gave in [BTV03] a polynomial-time algorithm that finds one such partition. We also proved in [BTV03] that for $\lceil \frac{d}{2} \rceil + 1 \leq s \leq d-1$ the problem is *NP*-complete. Only the complexity for $s = \lceil \frac{d}{2} \rceil$ remained open in [BTV03].

We define in this paper another variant of Satisfactory Partition, called Balanced Satisfactory Partition, where the parts are required to have the same cardinality. A graph admitting such a partition is said to be *balanced partitionable*. One can easily see that graphs like cycles of even length and complete bipartite graphs with both vertex classes of even size are trivially balanced partitionable. A graph of even order formed by two non-partitionable connected components of unequal size is an example of a graph partitionable but not balanced partitionable. We show in this paper that Satisfactory Partition and Balanced Satisfactory Partition are polynomially equivalent and *NP*-complete.

The paper is structured as follows. Section 2 contains some notation, definitions of problems and a preliminary result. In Section 3, we prove that for graphs with maximum degree at most 4, Satisfactory Partition is polynomially solvable and a satisfactory partition can be found in polynomial time if it exists. In Section 4 we show the polynomial equivalence of Satisfactory Partition and Balanced Satisfactory Partition, and the *NP*-completeness of these problems. In Section 5 we study the complexity of some extensions where partitions into k nonempty parts ($k \geq 3$) are requested.

2 Preliminaries

The following notation will be used in the rest of the paper. For a graph $G = (V, E)$, a vertex $v \in V$, and a subset $Y \subseteq V$ we denote by $d_Y(v)$ the number of vertices in Y that are adjacent to v; and, as usual, we write $d(v)$ for the degree $d_V(v)$ of v in V. The minimum and maximum degree of G will be denoted by $\delta(G)$ and $\Delta(G)$, respectively. For any subgraph G' of G, $V(G')$ and $E(G')$ denote respectively the set of vertices and edges of G'. A partition (V_1, V_2) of V is said to be *nontrivial* if both V_1 and V_2 are nonempty.

The problems we are interested in are defined as follows.

Satisfactory Partition
Input: A graph $G = (V, E)$.
Question: Is there a nontrivial partition (V_1, V_2) of V such that for every $v \in V$, if $v \in V_i$ then $d_{V_i}(v) \geq \lceil \frac{d(v)}{2} \rceil$?

Balanced Satisfactory Partition
Input: A graph $G = (V, E)$ on an even number of vertices.
Question: Is there a nontrivial partition (V_1, V_2) of V such that $|V_1| = |V_2|$ and for every $v \in V$, if $v \in V_i$ then $d_{V_i}(v) \geq \lceil \frac{d(v)}{2} \rceil$?

Considering $A \subseteq V$, a vertex $v \in A$ is said to be *satisfied* in A if $d_A(v) \geq \lceil \frac{d(v)}{2} \rceil$. Moreover A is a *satisfactory subset* if all of its vertices are satisfied in A. If $A, B \subseteq V$ are two disjoint, nonempty, satisfactory subsets, we say that (A, B) is a *satisfactory pair*. If, in addition, (A, B) is a partition of V, then it will be called a *satisfactory partition* and if the partition has the property $|A| = |B|$ then it will be called a *balanced satisfactory partition.* Given a *partition* (V_1, V_2) of V, a vertex $v \in V_i$ is *satisfied* if $d_{V_i}(v) \geq \lceil \frac{d(v)}{2} \rceil$.

We establish now a necessary and sufficient condition to obtain a satisfactory partition that will be useful afterwards. In [GK00] and [SD02] some sufficient as well as necessary and sufficient conditions are also given for the existence of a satisfactory partition in a graph.

Proposition 1 *A graph $G = (V, E)$ is partitionable if and only if it contains a satisfactory pair (A, B). Moreover, if a satisfactory pair (A, B) is given, then a satisfactory partition of G can be determined in polynomial time.*

Proof: The necessary part is obvious. The sufficient part is proved as follows. Let $V_1 = A$ and $V_2 = B$. While there is a vertex v in $V \setminus (V_1 \cup V_2)$ such that $d_{V_1}(v) \geq \lceil \frac{d(v)}{2} \rceil$, insert v into V_1. While there is a vertex v in $V \setminus (V_1 \cup V_2)$ such that $d_{V_2}(v) \geq \lceil \frac{d(v)}{2} \rceil$, insert v into V_2. At the end, if $C = V \setminus (V_1 \cup V_2) \neq \emptyset$, then $d_{V_1}(v) < \lceil \frac{d(v)}{2} \rceil$ and $d_{V_2}(v) < \lceil \frac{d(v)}{2} \rceil$ for any $v \in C$. For any $v \in C$ we have $d_{V_1 \cup C}(v) \geq \lceil \frac{d(v)}{2} \rceil$ and $d_{V_2 \cup C}(v) \geq \lceil \frac{d(v)}{2} \rceil$. Thus we can insert all vertices of C either into V_1 or into V_2, forming a satisfactory partition. $\square$

3 Graphs with degrees bounded by 4

Graphs G with $\Delta(G) \leq 4$ are such that any subgraph induced by a cycle corresponds to a satisfactory subset. This property seems to make the problem easier, which is indeed the case since we can decide in polynomial time if G with $\Delta(G) \leq 4$ is partitionable and find a partition when it exists. In particular, all cubic graphs except K_4 and $K_{3,3}$ are partitionable and all 4-regular graphs except K_5 are partitionable.

We first establish results on regular graphs.

Proposition 2 *Each cubic graph except K_4 and $K_{3,3}$ is partitionable.*

Proof: Let G be a cubic graph other than K_4 and $K_{3,3}$. If G is disconnected it is trivially partitionable. Hence, we assume that G is connected.

Suppose first that G contains a triangle and let C be a triangle of G with vertices v_1, v_2, v_3. Remark that a vertex outside C cannot have all its neighbors on C since $G \neq K_4$.

If each vertex of $V \setminus V(C)$ has at most one neighbor on C then $V_1 = V(C)$ and $V_2 = V \setminus V_1$ form a satisfactory partition.

Suppose that there is a vertex v_4 with two neighbors v_1, v_2 on C. If v_3 and v_4 have a common neighbor v_5, then $V_1 = \{v_1, v_2, v_3, v_4, v_5\}$ and $V_2 = V \setminus V_1 \neq \emptyset$ form a satisfactory partition of G. Otherwise $V_1 = \{v_1, v_2, v_3, v_4\}$ and $V_2 = V \setminus V_1 \neq \emptyset$ form a satisfactory partition of G.

Suppose now that G contains a cycle of length 4 and does not contain a triangle. Let $C = v_1v_2v_3v_4$ be a cycle of length 4. A vertex outside C cannot have more than two neighbors on C since otherwise G would contain a triangle.

If each vertex of $V \setminus V(C)$ has at most one neighbor on C, then $V_1 = V(C)$ and $V_2 = V \setminus V_1$ form a satisfactory partition.

Otherwise, suppose that a vertex v_5 has neighbors v_1 and v_3. Since $G \neq K_{3,3}$ there is no vertex of G with the three neighbors v_2, v_4, v_5. Thus, a vertex v_i with $i \geq 6$ has at most two neighbors among $\{v_2, v_4, v_5\}$. If all vertices v_i with $i \geq 6$ have at most one neighbor among $\{v_2, v_4, v_5\}$ then $V_1 = \{v_1, v_2, v_3, v_4, v_5\}$ and $V_2 = V \setminus V_1 \neq \emptyset$ form a satisfactory partition of

G. Otherwise, let v_6 be a vertex that has v_2, v_4 as neighbors. If all vertices v_i with $i \geq 7$ have at most one neighbor among $\{v_5, v_6\}$, then $V_1 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $V_2 = V \setminus V_1 \neq \emptyset$ form a satisfactory partition of G. Otherwise, there is another vertex v_7 with neighbors v_5, v_6. In this case $V_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $V_2 = V \setminus V_1 \neq \emptyset$ form a satisfactory partition of G.

Finally, suppose that G has no cycle of length at most 4. Let C be a shortest cycle in G. Since C has length $k \geq 5$, then no external vertex can have more than one neighbor in C, for otherwise G would contain a cycle of length at most $\lfloor k/2 \rfloor + 2 < k$, contradicting the choice of C. Thus, (V_1, V_2) with $V_1 = V(C)$ and $V_2 = V \setminus V_1$ is a satisfactory partition. □

Proposition 3 *Each 4-regular graph except K_5 is partitionable.*

Proof: Let G be a 4-regular graph other than K_5. If G is disconnected it is trivially partitionable. Hence, we assume that G is connected.

Suppose that G contains a triangle and let C be a triangle of G with vertices v_1, v_2, v_3. If each vertex of $V \setminus V(C)$ has at most two neighbors on C, then G is partitionable, and $V_1 = V(C)$ and $V_2 = V \setminus V_1$ form a satisfactory partition. Otherwise let v_4 be a vertex with neighbors v_1, v_2, v_3. If each vertex $v_i, i \geq 5$ has at most two neighbors among v_1, v_2, v_3, v_4 then G is partitionable with $V_1 = \{v_1, v_2, v_3, v_4\}$ and $V_2 = V \setminus V_1$. Otherwise, since $G \neq K_5$, there is a vertex v_5 with exactly three neighbors among v_1, v_2, v_3, v_4. Then $V_1 = \{v_1, v_2, v_3, v_4, v_5\}$ and $V_2 = V \setminus V_1 \neq \emptyset$ form a satisfactory partition of G.

Suppose now that G is triangle free. Let C be a shortest cycle in G. Since C has length $k \geq 4$, then there are no three vertices on C with a common neighbor outside C, since otherwise there exists in G a cycle of length at most $\lfloor k/3 \rfloor + 2$. For $k \geq 4$ this would be a cycle shorter than C. Thus, (V_1, V_2) with $V_1 = V(C)$ and $V_2 = V \setminus V_1$ is a satisfactory partition. □

Clearly Proposition 2 and Proposition 3 show that a satisfactory partition for cubic graphs except K_4 and $K_{3,3}$ and 4-regular graphs except K_5 can be found in polynomial time. We can even show that, for these graphs, a satisfactory partition can be found in linear time using the following algorithm.

Algorithm 1 Determination of a satisfactory partition for 3 and 4-regular graphs, $|V| > 10$

Let $G = (V, E)$ be a d-regular graph ($d = 3$ or 4) of order $n > 10$.
Search a cycle C of length less than $\frac{n}{2}$.
$V_1 \leftarrow V(C)$
while there exists a vertex $v \in V \setminus V_1$ with at least $d - 1$ neighbors in V_1 **do**
  $V_1 \leftarrow V_1 \cup \{v\}$
end while
$V_2 \leftarrow V \setminus V_1$

Remark The condition $n > 10$ is purely technical; it is imposed to ensure that for $d = 3$ the input graph do have a cycle of length less than $n/2$. On the other hand, for $d = 4$, a cycle shorter than $n/2$ exists whenever $n > 8$. What is more, in the 4-regular case we would just need a cycle of length at most $n/2$ (as shown later), and for this we should only assume $n > 5$. The small cases, however, can be settled in constant time, therefore we have put a uniform bound on n that works for both $d = 3$ and 4.

It is immediately seen that, for both $d = 3$ and $d = 4$, the algorithm terminates with a partition in which every vertex is satisfied, provided that the initial cycle C has been found. We will prove that the partition obtained is always nontrivial, and that the algorithm runs really fast. We begin with the latter, while the former will be split into two parts depending on the value of d.

Lemma 4 *Algorithm 1 can be implemented to run in linear time.*

Proof: We apply the Breadth-First Search algorithm that finds a spanning tree T in the input graph in linear time. If e is the first edge during BFS which does not become an edge of T, then this e together with a subpath of T defines a sufficiently short cycle. (If this cycle is longer than $2s$, for some $s \geq 3$, then up to distance s from the root of T we have d complete $(d-1)$-ary trees attached to the root; and if the cycle is also longer than $2s+1$, then this subtree of height s is an *induced* subgraph of G. Thus, $n > d\left(\sum_{i=0}^{s-1}(d-1)^i\right)$ holds, and if the length exceeds $2s+1$, then also $n > d\left(\sum_{i=0}^{s-1}(d-1)^i\right) + (d-1)^s$. From these bounds we obtain $n > 4s+4$, as needed.)

As regards the **while** loop, one solution for an efficient implementation is to create a counter for each $v \in V \setminus V_1$, whose value is set to 0 at the beginning. We also define a queue Q that initially contains the vertices of $V_1 = V(C)$. In each step, we remove the head element w from Q and increase the counters of all neighbors of w in $V \setminus V_1$; and if the counter of some v reaches the value $d-1$, we move v into V_1 and put it at the end of Q. The algorithm terminates when Q has become empty after some step. Since each edge is considered at most two times during the procedure, and $|E(G)| = \frac{d}{2}|V| \leq 2n$ for $3 \leq d \leq 4$, the **while** loop takes just $O(n)$ time. $\square$

Theorem 5 *All cubic graphs except K_4 and $K_{3,3}$ are partitionable in linear time.*

Proof: Let G be a cubic graph of order n. The cases $n \leq 10$ can be handled in constant time. For $n > 10$, let us verify that Algorithm 1 with $d = 3$ (running in linear time) is correct.

We have seen that $\ell = |V(C)| < \frac{n}{2}$. The key observation now is that the algorithm can move at most ℓ vertices from $V \setminus V_1$ to V_1. Indeed, moving m vertices yields $|V_1| = \ell + m$, and this V_1 induces at least $\ell + 2m$ edges. Thus, the average degree in the subgraph induced by V_1 is not smaller than $3 + \frac{m-\ell}{m+\ell}$, which implies $m \leq \ell$ since G is cubic. Consequently, if $n > 10$, Algorithm 1 stops with a satisfactory partition (V_1, V_2) where $|V_1| \leq 2\ell < n$, which implies $V_2 \neq \emptyset$, i.e. the partition is nontrivial. $\square$

Theorem 6 *All 4-regular graphs except K_5 are partitionable in linear time.*

Proof: Assuming that G is a 4-regular graph of order $n > 10$, we are going to prove that Algorithm 1 with $d = 4$ is correct.

We have seen that $\ell = |V(C)| < \frac{n}{2}$. Each vertex of C has at most two neighbors in $G - C$, therefore the degree sum in the induced subgraph $G - C$ is at least $4(n-\ell) - 2\ell = 2(n-\ell) + 2(n-2\ell) > 2(n-\ell)$. That is, the average degree in $G - C$ is at least two, and therefore $G - C$ contains some cycle C'. Clearly, Algorithm 1 stops before moving any vertex of C' into the set V_1. Thus, $V_2 \neq \emptyset$ and the partition (V_1, V_2) obtained is satisfactory. $\square$

Thus, all cubic graphs except K_4 and $K_{3,3}$ are partitionable and all 4-regular graphs except K_5 are partitionable. Moreover, as stated in the introduction, all 2-regular graphs (cycles) except K_3 are partitionable. These results cannot be extended for regular graphs with degree greater than 4 since there are 5-regular graphs, different from K_6 and $K_{5,5}$ that are not partitionable, and there are 6-regular graphs different from K_7 that are not partitionable (see Figure 1).

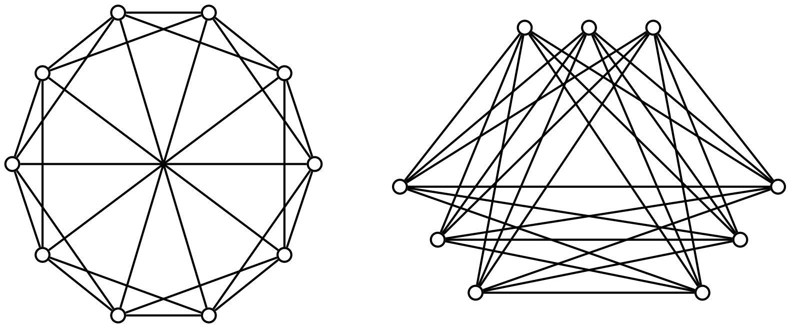

Figure 1: Non-partitionable 5-regular and 6-regular graphs

We consider now graphs with maximum degree at most 4. In [GK04], it is indicated that such graphs with at least 13 vertices are always partitionable. We give necessary and sufficient conditions for the existence of satisfactory partitions, and show how to generate such a partition in polynomial time when it exists.

Proposition 7 *A graph G with $\delta(G) = 3$ and $\Delta(G) \leq 4$ is partitionable if and only if it contains two vertex-disjoint cycles.*

Proof: (If) Immediate from Proposition 1. (Only if) If G is partitionable then each vertex has at least two neighbors in its part, so each part contains a cycle. $\square$

Proposition 8 *Let G be a graph with $\Delta(G) \leq 4$ and with no isolated vertex. Graph G is partitionable if and only if there exists at most two disjoint edges that can be inserted between vertices of degrees 1 or 2, such that the resulting multigraph contains two vertex-disjoint cycles.*

Proof: (If) If G contains two disjoint cycles C_1, C_2 then $V(C_1)$ and $V(C_2)$ can be completed to form a satisfactory partition, using Proposition 1.

If G has no two disjoint cycles but adding one edge (v_i, v_j), with $d(v_i), d(v_j) \leq 2$, the graph $G' = (V, E \cup \{(v_i, v_j)\})$ has two disjoint cycles C_1, C_2 then (v_i, v_j) belongs to one of these cycles. Then $V(C_1)$ and $V(C_2)$ form a satisfactory pair once we remove (v_i, v_j) since v_i and v_j have degree at most two.

Assume now that the addition of two non-adjacent edges $(v_i, v_j), (v_k, v_\ell)$, with $d(v_i), d(v_j)$, $d(v_k), d(v_\ell) \leq 2$, is such that the new graph contains two disjoint cycles. Since these two edges are not adjacent, as above, the two disjoint cycles can be completed to a satisfactory partition.

(Only if) Let (V_1, V_2) be a satisfactory partition of G. If V_i $(i = 1, 2)$ contains no cycle, then we add one edge between two degree-1 vertices of a tree component inside V_i. If the tree in question is just an edge, then we add a parallel edge creating a cycle of length 2 in the resulting multigraph. $\square$

Theorem 9 *Let G be a graph with $\Delta(G) \leq 4$. We can decide in polynomial time if G is partitionable, and find a satisfactory partition of G if it exists.*

Proof: If G is disconnected a satisfactory partition is trivially obtained. We consider now that G is connected so that we can apply Proposition 8. There is a polynomial number of choices to add at most two non-adjacent edges in G. For a fixed choice, we first verify if there are multiple edges. If there are two non-adjacent multiple edges, then we have found two disjoint cycles; if there is one multiple edge, then we search a cycle in the graph obtained by removing the two vertices incident to this edge. If the graph has no multiple edges, then we apply a polynomial algorithm that finds two disjoint cycles in a graph if they exist ([Bod94]).$\square$

4 The *NP*-completeness of (Balanced) Satisfactory Partition

In this section we establish the *NP*-completeness of SATISFACTORY PARTITION and BALANCED SATISFACTORY PARTITION. We first show that these two problems are polynomial equivalent.

Proposition 10 SATISFACTORY PARTITION *is polynomial reducible to* BALANCED SATISFACTORY PARTITION.

Proof: Let G be a graph, instance of the first problem on n vertices $v_1, \ldots, v_n$. The graph G', instance of BALANCED SATISFACTORY PARTITION, is obtained from G by adding an independent set of $n-2$ vertices $u_1, \ldots, u_{n-2}$. If G is partitionable and (V_1, V_2) is a satisfactory partition then (V_1', V_2') where $V_1' = V_1 \cup \{u_1, \ldots, u_{n-|V_1|-1}\}$ and $V_2' = V_2 \cup \{u_{n-|V_1|}, \ldots, u_{n-2}\}$ is a balanced satisfactory partition of G'.

If G' is balanced partitionable and (V_1', V_2') is a balanced satisfactory partition then both V_1' and V_2' contain at least one vertex from V and the restriction of this partition to V is a satisfactory partition of G. $\square$

Proposition 11 BALANCED SATISFACTORY PARTITION *is polynomial reducible to* SATISFACTORY PARTITION.

Proof: Let $G = (V, E)$ be a graph, instance of the first problem on n vertices. The graph $G' = (V', E')$, instance of SATISFACTORY PARTITION, is obtained from G by adding two cliques of size $\frac{n}{2}$, $A = \{a_1, \ldots, a_{\frac{n}{2}}\}$ and $B = \{b_1, \ldots, b_{\frac{n}{2}}\}$. In G', in addition to the edges of G, all vertices of V are adjacent with all vertices of A and B. Also each vertex $a_i \in A$ is linked to all vertices of B except b_i, $i = 1, \ldots, \frac{n}{2}$.

Let (V_1, V_2) be a balanced satisfactory partition of G. Then (V_1', V_2') where $V_1' = V_1 \cup A$ and $V_2' = V_2 \cup B$ is a satisfactory partition of G'. Indeed, a vertex from $A \cup B$ is satisfied, for example if $v \in A$, $d_{V_1'}(v) = |A| + \frac{n}{2} - 1 = d_{V_2'}(v)$. Also it is easy to see that a vertex from V is satisfied in G' since it is satisfied in G.

Let (V_1', V_2') be a satisfactory partition of G', where $V_1' = V_1 \cup A_1 \cup B_1$ and $V_2' = V_2 \cup A_2 \cup B_2$ with $V_i \subseteq V, A_i \subseteq A, B_i \subseteq B, i = 1, 2$. We claim that (V_1, V_2) is a balanced satisfactory partition of G.

We first show that $A_1 \cup B_1 \neq \emptyset$ and $A_2 \cup B_2 \neq \emptyset$, which means that no satisfactory partition can contain $A \cup B$ in one of its parts. Indeed, by contradiction, suppose we have

$V_1' = V_1 \cup A \cup B$ and $V_2' = V_2$. Then, the inequality specifying that $v \in V_2$ is satisfied is $d_{V_2}(v) \geq d_{V_1}(v) + n$ which is impossible. So, two cases are possible: either each part of the partition contains one clique, say $V_1' = V_1 \cup A$ and $V_2' = V_2 \cup B$ (case 1) or at least one of the cliques is cut by the partition (case 2).

In case 1, in order that a vertex of A be satisfied, we have $\frac{n}{2} - 1 + |V_1| \geq |V_2| + \frac{n}{2} - 1$ and in order that a vertex of B be satisfied, we have $\frac{n}{2} - 1 + |V_2| \geq |V_1| + \frac{n}{2} - 1$. These two inequalities imply $|V_1| = |V_2|$. Moreover, since $v \in V_1 \cup V_2$ is satisfied in G' where it is linked to $\frac{n}{2}$ vertices in A and $\frac{n}{2}$ vertices in B, v is also satisfied in G.

In case 2, suppose that clique A is cut by the partition into non-empty sets A_1 and A_2 while B_1 or B_2 may be empty. We show now that if $a_i \in A_1$ for some i, then also $b_i \in B_2$ for the same i. Assume by contradiction that $b_i \in B_1$. Since a_i is satisfied we have

$$(|A_1| - 1) + (|B_1| - 1) + |V_1| \geq |A_2| + |B_2| + |V_2| \tag{1}$$

This implies $|V_1'| > |V_2'|$.

Let $a_j \in A_2$. We may have $b_j \in B_1$ or $b_j \in B_2$. If $b_j \in B_2$ then the condition that a_j is satisfied is

$$(|A_2| - 1) + (|B_2| - 1) + |V_2| \geq |A_1| + |B_1| + |V_1| \tag{2}$$

If $b_j \in B_1$ then the condition that a_j is satisfied is

$$(|A_2| - 1) + |B_2| + |V_2| \geq |A_1| + (|B_1| - 1) + |V_1| \tag{3}$$

Each of (2) and (3) implies that $|V_2'| \geq |V_1'|$, contradicting (1). Thus $|A_1| = |B_2|$ and $|A_2| = |B_1|$, that means that both cliques are cut by the partition.

For $a_i \in A_1$ and $b_i \in B_2$ the inequalities specifying that a_i and b_i are satisfied are respectively:

$$(|A_1| - 1) + |B_1| + |V_1| \geq |A_2| + (|B_2| - 1) + |V_2|$$

and

$$|A_2| + (|B_2| - 1) + |V_2| \geq (|A_1| - 1) + |B_1| + |V_1|$$

from which we obtain $|A_1| + |B_1| + |V_1| = |A_2| + |B_2| + |V_2|$. Since $|A_1| = |B_2|$ and $|A_2| = |B_1|$, we get $|V_1| = |V_2|$.

Moreover, since $v \in V_1 \cup V_2$ is satisfied in G' where it is linked to $|A_1| + |B_1| = \frac{n}{2}$ vertices in V_1' among the vertices of the two cliques and $|A_2| + |B_2| = \frac{n}{2}$ vertices in V_2', v is also satisfied in G. □

We state now the main result of our paper.

Theorem 12 Satisfactory Partition *and* Balanced Satisfactory Partition *are NP-complete.*

Proof: Clearly, these two problems are in *NP*. We prove that Balanced Satisfactory Partition is *NP*-complete, which implies by Proposition 11 that Satisfactory Partition is *NP*-complete too.

We construct a polynomial reduction from a variant of Clique, the problem of deciding if a non-complete graph with n vertices contains a clique of size at least $\frac{n}{2}$, a problem proved to be *NP*-hard in [GJ79], to Balanced Satisfactory Partition. Let $G = (V, E)$ be a non-complete graph with n vertices $v_1, \dots, v_n$ and m edges, an input of Clique problem. We consider that n is even, since otherwise we can add an isolated vertex without changing the problem. Let $p = \frac{n(n-1)}{2} - m \geq 1$ corresponding to the number of non-edges in G. These non-edges are labelled $ne_1, \dots, ne_p$. We construct a graph $G'' = (V'', E'')$, instance of Balanced Satisfactory Partition as follows: the vertex set V'' consists of six sets F, F', T, T', V and V' where $F = \{f_1, \dots, f_{2p+1}\}$, $F' = \{f'_1, \dots, f'_{2p+1}\}$, $T = \{t_1, \dots, t_{2p+1}\}$, $T' = \{t'_1, \dots, t'_{2p+1}\}$ and $V' = \{v'_1, \dots, v'_n\}$. Vertices $f_{2\ell}$, $f_{2\ell+1}$ correspond to non-edge ne_ℓ ($\ell = 1, \dots, p$) and f_1 is an additional vertex. Vertices of F', T and T' are similarly defined. F and T are two cliques of size $2p+1$. Vertices f'_i (t'_i and v'_j) are only linked with f_i (t_i and v_j), $i = 1, \dots, 2p+1$, $j = 1, \dots, n$. In addition to these edges and E, the edge set E'' contains all edges between T and V and all edges between F and V except edges $(f_{2\ell}, v_i)$ and $(f_{2\ell+1}, v_j)$ for each non-edge $ne_\ell = (v_i, v_j), \ell = 1, \dots, p$.

It is easy to see that this construction can be accomplished in polynomial time. All that remains to show is that G has a clique of size at least $\frac{n}{2}$ if and only if G'' is balanced partitionable.

Suppose firstly that G has a clique of size at least $\frac{n}{2}$. Let C be a clique of size exactly $\frac{n}{2}$ of G. Let $V''_1 = F \cup F' \cup C \cup C'$ where $C' = \{v'_i : v_i \in C\}$ and $V''_2 = T \cup T' \cup \bar{C} \cup \bar{C}'$, where $\bar{C} = V \setminus C$ and $\bar{C}' = \{v'_i : v_i \in \bar{C}\}$. Let us check in the following that (V''_1, V''_2) is a balanced satisfactory partition. It is easy to see that all vertices of F, F', T, T' and V' are satisfied. Let $v \in C$. Since C is a clique, when v is not linked to a vertex of F, it is also not linked to a vertex of $\bar{C}$. Thus, $d_{V''_1}(v) = 2p+1 - (\frac{n}{2} - d_{\bar{C}}(v)) + \frac{n}{2} = 2p+1+d_{\bar{C}}(v) = d_{V''_2}(v)$ and so the vertices of C are satisfied. Given a vertex $v \in \bar{C}$, $d_{V''_1}(v) = 2p+1 - (n-1-d_{\bar{C}}(v) - d_C(v)) + d_C(v)$ $= 2p+2+d_{\bar{C}}(v) - (n - 2d_C(v)) \leq 2p+2+d_{\bar{C}}(v)$, while $d_{V''_2}(v) = 2p+2+d_{\bar{C}}(v)$, thus also the vertices of $\bar{C}$ are satisfied in G''.

Suppose now that G'' is balanced partitionable and let (V''_1, V''_2) be a balanced satisfactory partition. Observe that in any satisfactory partition vertices f'_i (respectively t'_i, v'_j) must be in the same part of the partition as f_i (respectively t_i, v_j), $i = 1, \dots, 2p+1$, $j = 1, \dots, n$.

We justify firstly that T and F cannot be cut by the partition. Assume for a contradiction that T is cut in (T_1, T_2) with $T_1 \subset V''_1$ and $T_2 \subset V''_2$. Consider now that V is cut in (V_1, V_2) with $V_1 \subset V''_1$ and $V_2 \subset V''_2$, where V_1 or V_2 could be empty. Since any vertex from T_1 must be satisfied, we have $|T_1| + |V_1| \geq |T_2| + |V_2|$. Since any vertex from T_2 must be satisfied, we have $|T_2| + |V_2| \geq |T_1| + |V_1|$. These inequalities imply $|T_1| + |V_1| = |T_2| + |V_2|$ which is impossible since $|T_1| + |V_1| + |T_2| + |V_2| = n + 2p + 1$ is odd. Therefore T cannot be cut by any satisfactory partition. We suppose in the following that $T \subset V''_2$.

Assume now by contradiction that F is cut in (F_1, F_2) with $F_1 \subset V''_1$ and $F_2 \subset V''_2$. Consider that V is cut in (V_1, V_2) with $V_1 \subset V''_1$ and $V_2 \subset V''_2$. Since (V''_1, V''_2) is balanced, we have

$$|F_1| + |V_1| = |F_2| + |T| + |V_2| \tag{4}$$

Consider $v \in F_2$. We have three cases:

- If $v = f_1$ the condition stating that v must be satisfied is $|F_2| + |V_2| \geq |F_1| + |V_1|$ which contradicts (4).

- If v is not linked to a vertex of V_2, the condition stating that v must be satisfied is $|F_2| + (|V_2| - 1) \geq |F_1| + |V_1|$ which also contradicts (4).

- If v is not linked to a vertex of V_1, the condition stating that v must be satisfied is $|F_2| + |V_2| \geq |F_1| + (|V_1| - 1)$ which also contradicts (4) since $|T| \geq 3$.

We show now that a balanced satisfactory partition (V_1'', V_2'') cannot contain F and T in the same part. Assume by contradiction that V is cut in (V_1, V_2) and $V_1'' = V_1 \cup V_1'$ and $V_2'' = V_2 \cup V_2' \cup F \cup F' \cup T \cup T'$. Since (V_1'', V_2'') is balanced we have $2|V_1| = 2|F| + 2|T| + 2|V_2|$ that is $|V_1| - |V_2| = 2(2p+1)$. For a vertex in T to be satisfied we must have $2p+1+|V_2| \geq |V_1|$ which contradicts the previous equality.

Thus (V_1'', V_2'') cuts the set V into two balanced sets V_1, V_2, where $V_1'' = F \cup F' \cup V_1 \cup V_1'$ and $V_2'' = T \cup T' \cup V_2 \cup V_2'$. We show that V_1 is a clique. A vertex $v \in V_1$ has $d_{V_1''}(v) = 2p + 1 - x + d_{V_1}(v) + 1$ where $x = n - 1 - d_{V_1}(v) - d_{V_2}(v)$ is the number of non-edges of G incident to v and $d_{V_2''}(v) = 2p+1+d_{V_2}(v)$. Since v is satisfied in G'' we have $d_{V_1''}(v) \geq d_{V_2''}(v)$ and we obtain that $d_{V_1}(v) \geq \frac{n}{2} - 1$. Thus V_1 is a clique of size $\frac{n}{2}$. □

5 Satisfactory k-Partitions

In this section we study the complexity of three generalizations of Satisfactory Partition where a partition into k nonempty parts is requested, for $k \geq 3$.

- Sum Satisfactory k-Partition where each vertex is required to have at least as many neighbors in its part as in all the other parts together. This condition is equivalent to asking that a vertex has at least the majority of its neighbors in its own part.

- Average Satisfactory k-Partition where each vertex is required to have at least $1/k$ proportion of its neighbors in its own part.

- Max Satisfactory k-Partition where each vertex is required to have at least as many neighbors in its own part as in each of the other parts.

Sum and Max versions were introduced by Gerber and Kobler in [GK98] where they proved the strong *NP*-hardness of generalizations of these problems where there are weights on the vertices or edges, and left as an open question the complexity of the unweigthed case.

Formally these problems can be stated as follows.

Sum Satisfactory k-Partition
Input: A graph $G = (V, E)$.
Question: Is there a partition into k nonempty parts $(V_1, \ldots, V_k)$ of V such that, for all $v \in V$, if $v \in V_i$ then $d_{V_i}(v) \geq \lceil \frac{d(v)}{2} \rceil$?

Average Satisfactory k-Partition
Input: A graph $G = (V, E)$.
Question: Is there a partition into k nonempty parts $(V_1, \ldots, V_k)$ of V such that, for all $v \in V$, if $v \in V_i$ then $d_{V_i}(v) \geq \lceil \frac{d(v)}{k} \rceil$?

Max Satisfactory k-Partition
Input: A graph $G = (V, E)$.
Question: Is there a partition into k nonempty parts $(V_1, \ldots, V_k)$ of V such that, for all $v \in V$, if $v \in V_i$ then $d_{V_i}(v) = \max_{j=1,\ldots,k} d_{V_j}(v)$?

We also consider the balanced version of these three problems. Observe that for $k = 2$ all these generalizations boil down to (BALANCED) SATISFACTORY PARTITION. As could be expected, all these problems are also *NP*-complete for every value of k.

We give now the proofs of *NP*-completeness, observing that all these problems are clearly in *NP*.

Proposition 13 SUM SATISFACTORY k-PARTITION *and* BALANCED SUM SATISFACTORY k-PARTITION *are NP-complete for every* $k \geq 3$.

Proof: We reduce SATISFACTORY PARTITION to SUM SATISFACTORY k-PARTITION as follows. Given a graph G, instance of SATISFACTORY PARTITION, we construct an instance of SUM SATISFACTORY k-PARTITION, G', by adding $k-2$ isolated vertices to G. It is easy to see that G is partitionable if and only if in G' there is a partition into k nonempty parts such that each vertex has the majority of neighbors in its own part.

We reduce BALANCED SATISFACTORY PARTITION to BALANCED SUM SATISFACTORY k-PARTITION as follows. Given a graph G of order n, instance of BALANCED SATISFACTORY PARTITION, we construct an instance of BALANCED SUM SATISFACTORY k-PARTITION, G', by adding $k-2$ cliques of size $\frac{n}{2}$ to G. If G has a balanced satisfactory partition, it can be extended in G' to a k-partition where the $k-2$ remaining sets are the cliques. If G' has a balanced partition into k parts such that each vertex is satisfied, then since no clique can be cut, $k-2$ classes of this partition are the cliques and the two others induce a balanced partition in G. $\square$

Proposition 14 AVERAGE SATISFACTORY k-PARTITION *and* BALANCED AVERAGE SATISFACTORY k-PARTITION *are NP-complete for every* $k \geq 3$.

Proof: We construct a reduction from the EDGE k-COLORING problem of a k-regular graph to AVERAGE SATISFACTORY k-PARTITION. The first problem was proved to be *NP*-hard for $k = 3$ by Holyer [Hol81] and for $k \geq 3$ by Leven and Galil [LG83]. In order to illustrate our reduction, we consider $k = 3$, but the proof for general k is similar. Given a 3-regular graph $G = (V, E)$ with n vertices and $m = 3n/2$ edges, we consider as instance for AVERAGE SATISFACTORY 3-PARTITION the *complement of the line graph of* G, the graph $G' = \overline{L(G)}$. Graph G' has m vertices, and is $(m-5)$-regular. If G is edge-3-colorable, denote by E_i the set of edges colored i, for $i = 1, 2, 3$. Each set E_i has $\frac{m}{3}$ edges. Let V_i be the set of vertices of G' corresponding to the edges of E_i, V_i is a clique, thus (V_1, V_2, V_3) is a partition of G' that satisfies the property that $d_{V_i}(v) = m/3 - 1 = \lceil \frac{m-5}{3} \rceil$ for all $v \in V_i$. Conversely, given a partition (V_1, V_2, V_3) of G' with $d_{V_i}(v) \geq m/3 - 1$ for all $v \in V_i$ $(i = 1, 2, 3)$, V_i has exactly $m/3$ vertices and so all the V_i are independent sets in $L(G)$. This gives a 3-coloration of the edges of G.

This reduction is also valid for the balanced case. $\square$

Proposition 15 MAX SATISFACTORY k-PARTITION *and* BALANCED MAX SATISFACTORY k-PARTITION *are NP-complete for every* $k \geq 3$.

Proof: We use the proof of Proposition 14. If G is edge-3-colorable then, in $L(G)$, each vertex $v \in V_i$ is adjacent with exactly two vertices in each of the other two sets, and thus in G' it has more neighbors in V_i than in each of the other sets. Conversely, if (V_1, V_2, V_3)

is a partition of G' with $d_{V_i}(v) = \max_{j \in \{1,2,3\}} d_{V_j}(v)$ then $d_{V_i}(v) \geq m/3 - 1$ for all $v \in V_i$ $(i = 1, 2, 3)$. Therefore V_i has exactly $m/3$ vertices and so all the V_i are cliques in G'. This gives a 3-coloration of the edges of G as before.

This reduction is also valid for the balanced case. □

6 Concluding remarks

Our arguments proving *NP*-completeness apply reductions from the maximum clique and the edge coloring problems, both yielding vertices of high degree. On the other hand, we have seen that if the maximum degree is very small (at most 4), then both the decision and search problems for a satisfactory partition are polynomial-time solvable. It remains an open problem whether there exists a finite bound D such that Satisfactory Partition is *NP*-complete for graphs of maximum degree D.

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