# A note on the vertex-distinguishing proper coloring of graphs with large minimum degree


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**Abstract**

We prove that the number of colors required to color properly the edges of a graph of order $n$ and $\delta(G) > n/3$ in such a way that any two vertices are incident with different sets of colors is at most $\Delta(G) + 5$.


## 1 Introduction

In this paper we consider only simple graphs and we use the standard notation of graph theory. Definitions not given here may be found in [2]. Let $G = (V, E)$ be a graph of order $n$ with the set of vertices $V$ and the edge set $E$. We denote by $V_d(G)$ the set of vertices of degree $d$ in $G$ and $n_d(G) = |V_d(G)|$. A *k-edge-coloring* $f$ of a graph $G$ is an assignment of $k$ colors to the edges of $G$. Let $f(e)$ be the color of the edge $e$. Denote by $F(v) = \{f(e) : e = uv \in E\}$ the multiset of colors assigned to the set of edges incident to $v$. The coloring

$f$ is *proper* if no two adjacent edges are assigned the same color and *vertex-distinguishing proper (VDP for short) coloring* if it is proper and $F(u) \neq F(v)$ for any two distinct vertices $u, v$.

Observe that if $G$ contains more than one isolated vertex or any isolated edges, then no coloring of $G$ is VDP. The minimum number of colors required to find a VDP coloring of a graph $G$ without isolated edges and with at most one isolated vertex is called the *vertex-distinguishing proper edge-coloring number* and denoted by $\tilde{\chi}'(G)$.

The VDP coloring number was introduced and studied by Burris and Schelp in [3] and [4] and, independently, as "observability" of a graph, by Černý, Horňák and Soták in [5], [7] and [8].

Among the graphs $G$ for which we know the value $\tilde{\chi}'(G)$, the largest value $\tilde{\chi}'(G)$ is realized when $G = K_n$ with $n$ even and equals $n + 1$.

The following result has been conjectured by Burris and Schelp in [3] and [4], and proved recently by the authors of this note in [1].

**Theorem 1** *A graph $G$ on $n$ vertices, without isolated edges and with at most one isolated vertex has $\tilde{\chi}'(G) \leq n + 1$.*

Of course, this last estimation of $\tilde{\chi}'(G)$ cannot be improved in general as it shows the example of complete graphs. However, for some families of graphs the VDP coloring number is rather closer to the maximum degree than to the order of the graph. The aim of this note is to give an example of such a situation. Using similar methods as in the proof of Theorem 1 we can prove the following theorem which is the main result of this paper.

**Theorem 2** *Let $G$ be a graph of order $n \geq 3$ without isolated edges and with at most one isolated vertex. If $\delta(G) > n/3$, then*

$$\tilde{\chi}'(G) \leq \Delta(G) + 5.$$

Recall that by Vizing's theorem, for any graph $G$, we need $\Delta(G)$ or $\Delta(G) + 1$ colors in order to color it properly.

Mention by the way that in [6] it is proved that a graph with $n$ vertices and minimum degree $\delta \geq 5$ and maximum degree $\Delta < \frac{(2c-1)n-4}{3}$, where $c$ is a constant with $\frac{1}{2} < c \leq 1$ has $\tilde{\chi}'(G) \leq \lceil cn \rceil$.

In the following we shall use some additional notation. Given a proper coloring $f$, we denote by $B_f(v) = \{u \in V(G) : F(u) = F(v)\}$. Observe that

$v \in B_f(v)$. A vertex $v$ is called *good* if $B_f(v) = \{v\}$ and *bad* otherwise. The members of $B_f(v) - \{v\}$ are called *brothers* of $v$. A *semi-VDP coloring* is a proper coloring with $|B_f(v)| \leq 2$ for any vertex $v$ of $G$. Given a proper coloring $f$ that contains the colors $\alpha$ and $\beta$, an $(\alpha, \beta)$-Kempe path is a maximal path formed by the edges colored with $\alpha$ and $\beta$.

For a given path $P$ denote by $\overrightarrow{P}$ one of its orientations. For $v, w \in V(P)$ such that $v$ precedes $w$, we denote by $v\overrightarrow{P}w$ the path starting in $v$ and ending in $w$ which contains all vertices of $P$ between $v$ and $w$ following the orientation $\overrightarrow{P}$.

Let $v$ be a vertex of $P$. We denote by $v^+$ and $v^-$ the successor and the predecessor, respectively, of the vertex $v$ on the path $P$ with respect to given orientation $\overrightarrow{P}$ (if they exist).

## 2 Two lemmas

The following two lemmas have been first proved in [1]. So, we give here only the sketches of the proofs.

**Lemma 3** *Let $G$ be a graph such that the following inequality holds for any $d$, $\delta(G) \leq d \leq \Delta(G)$:*

$$d(k - d) \geq n_d(G) - 2$$

*where $k \geq \Delta(G) + 1$.*

*Then there exists a semi-VDP coloring of $G$ with $k$ colors.*

**Proof.** By Vizing's theorem, since $k \geq \Delta(G)+1$, there is a proper coloring of $G$ with $k$ colors. Let $f$ be such a proper coloring of $G$ that has the minimum number of bad vertices, and subject to this, with the maximum number of two-element bad families. Suppose that $f$ is not a semi-VDP coloring of $G$. Thus, there exists a vertex $u \in V_d(G)$ with $|B_f(u)| \geq 3$.

There are $d$ possibilities to choose a color incident to $u$ and $k - d$ possibilities to replace it by another one. So, there are $d(k - d)$ possibilities to change the color of an edge incident with $u$ with another one such that $u$ is not incident to two edges with the same color. Recall that there are at least two other vertices that are incident with the same set of colors as $u$. By the assumption, we can choose two colors $\alpha \in F(u)$ and $\beta \notin F(u)$ in such a way that the set of colors incident to $u$ becomes unique.

Let $P$ be an $(\alpha, \beta)$-Kempe path having $u$ and $v$ as end-vertices. We transform the coloring $f$ in another coloring $f_1$ by exchanging the colors $\alpha$ and $\beta$ on the path $P$.

Observe that the vertex $v$ cannot be good with respect to $f_1$ since otherwise the coloring $f_1$ would have less bad vertices than $f$. Thus, $v$ is bad in $f_1$.

If $F_1(v) = F_1(u)$, then $v$ forms, together with $u$, a new bad family of cardinality two which contradicts the maximality of two-element families.

So, $v$ is bad in $f_1$. Let $v'$ be one of his brothers. We have $F_1(u) \neq F_1(v')$. This implies that there exists another $(\alpha, \beta)$-Kempe path, starting at $v'$. Since the set of all $(\alpha, \beta)$-Kempe paths in $G$ is finite, we get a contradiction (for more details we refer the reader to [1]). ■

Let $P_1, \ldots, P_k$ be a set of vertex disjoint paths. The set $\mathcal{P} = \{P_1, \ldots, P_k\}$ is called *a long path system* if $|V(P_i)| \geq 3$ for any $i = 1, \ldots, k$. If the vertices of a graph $G$ are covered by a long path system then $\mathcal{P}$ is called *a long path covering* of $G$.

The following lemma allows us to transform a semi-VDP coloring of a subgraph of $G$ to a VDP coloring of $G$ that uses three new colors.

**Lemma 4** *Let $\mathcal{P} = \{P_1, ..., P_k\}$ be a long path covering of $G$. If there exists a semi-VDP coloring $f$ of the edges of $G' = G - E(\mathcal{P})$ with $k$ colors, then there exists a VDP coloring of the edges of $G$ with $k+3$ colors.*

**Proof.** Let us fix an orientation of the paths of $\mathcal{P}$ and let $\overrightarrow{\mathcal{P}} = (\overrightarrow{P}_1, \ldots, \overrightarrow{P}_k)$ be a long path covering with a given order on the paths. The vertices in each bad family (with respect to $f$) are denoted by $(x, x')$ where $x$ is the first and $x'$ is the second vertex on $\mathcal{P}$ with respect to the order introduced by $\overrightarrow{\mathcal{P}}$. It suffices now to color the edges of $\mathcal{P}$ with three new colors, say $\alpha$, $\beta$, $\gamma$, in a way that distinguishes bad vertices (with respect to the coloring $f$).

We describe below a quasi-algorithm to color $\mathcal{P}$. We color the edges of $\mathcal{P}$ in the order given by the orientation $\overrightarrow{\mathcal{P}}$. We start with one of the colors, say $\alpha$, and we assign to the successive edges a color as follows:

Suppose that the next edge to be colored is $e = uu^+$. We distinguish two cases.

- The vertex $u$ is not bad or $u$ is the first bad vertex. Then
  - if $u$ is the first vertex of a path we use for $uu^+$ one of three colors;

  - if $u$ is an interior vertex we assign to $uu^+$ one of two colors not used for $u^-u$.

- The vertex $u$ is the second bad vertex. That means that is $u = x'$ for some $x$ and that the edges $x^-x$ and $xx^+$ (or one of these edges in the case where $x$ is an end-vertex) have already been colored. Then,
  - if $x$ is an end-vertex of a path and $u$ is an interior vertex of a path $P$ we color $uu^+$ with one of the colors not used for $u^-u$;
  - if $x$ is an interior vertex and $u$ is an end-vertex we color $uu^+$ with one of the three colors;
  - if both $x$ and $u$ are interior vertices then we color $uu^+$ in such a way that $\{f(x^-x), f(xx^+)\} \neq \{f(u^-u), f(uu^+)\}$;
  - if $x$ and $u$ are end-vertices of a path then we color $uu^+$ with one of two colors not used for the edge incident with $x$.

It is easy to see that such a coloring is always possible except, may be, in the situation where $uu^+$ is the last edge on a path $P$ where $u$ and $u^+$ are both second bad vertices, that is $u = x'$ and $u^+ = y'$ where $x$, $y$ are the first elements of bad families. In this case we have to modify the coloring in order to avoid such a situation. We can do it in the following way. We begin to change the colors at the second of two edges, $xx^+$ or $yy^+$ and preserve the colors of the edges that are before it (see [1] for more details). ■

# 3 Proof of Theorem 2

Observe first that by Theorem 1 our result is true for $n \leq 6$ for regular graphs, and for $n \leq 9$ for non-regular graphs.

We shall show now that $G$ contains a long path covering. Suppose, conversely, that there is no such path covering of $V(G)$.

Let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be a long path system that covers a maximum number of vertices of $G$ and let $v$ be a vertex belonging to $V(G) - V(\mathcal{P})$. We can assume additionally that the covering is chosen in such a way that the number of paths is as small as possible.

Observe that the graph induced by $V(G) - V(\mathcal{P})$ contains either isolated vertices or isolated edges. This implies, in particular, that $v$ cannot have two neighbors that are outside of $\mathcal{P}$.

Denote by $P_i$ a path of $\mathcal{P}$ that contains a neighbor of $v$ and by $a_i$, $b_i$ the end-vertices of $P_i$. Let $\overrightarrow{P_i}$ be an orientation of $P_i$ from $a_i$ to $b_i$.

We shall show that

**Claim 1** $v$ has exactly one neighbor on $P_i$.

**Proof of Claim 1.** Suppose, contrary to our claim that $v$ has many neighbors on $P_i$ and denote by $x_1$, $x_2$ the two first of them (with respect to a given orientation). Neither $x_1$ nor $x_2$ is an end-vertex of the path $P_i$ for, otherwise we could replace $P_i$ by $vx_1\overrightarrow{P_i}$ or $\overrightarrow{P_i}x_2v$ and obtain a long path system covering more vertices than $\mathcal{P}$.

It is easy to see that $x_2$ cannot be the successor of $x_1$, since the path $a_1\overrightarrow{P_i}x_1vx_2\overrightarrow{P_i}b_1$ would cover one vertex more than $P_i$.

So, there is at least one vertex on $P_i$ between $x_1$ and $x_2$. But then we can replace $P_i$ by two paths, namely $a_1\overrightarrow{P_i}x_1v$ and $(x_1)^+\overrightarrow{P_i}b_1$ and get in this way a long path system larger than $\mathcal{P}$. ■

Consider first the case where there is only one vertex outside of $\mathcal{P}$, say $v$. Since each path $P \in \mathcal{P}$ has at least three vertices, there are at most $\frac{n-1}{3}$ paths in $\mathcal{P}$. This implies, by Claim 1 that the vertex $v \notin V(\mathcal{P})$ has the degree at most $\frac{n-1}{3}$. This contradicts the fact that $\delta > \frac{n}{3}$. Thus there are at least two vertices outside of $\mathcal{P}$. If two paths of $\mathcal{P}$ are of length at least three or one path is of length at least four, then the number of paths is at most $\frac{n-4}{3}$. As above, this implies that the vertex $v \notin V(\mathcal{P})$ has the degree at most $\frac{n-4}{3} + 1$, a contradiction.

Consider now the case where all paths are of length two. Denote by $v$ one of vertices outside of the path system and by $a$, $b$ two end-vertices of the same path, say $P$. The vertex $a$ is neither joined to end-vertices on paths different from $P$ (because the number of paths is minimal) nor to vertices outside of $\mathcal{P}$. If there is no edge between $a$ and $b$, then the degree of $a$ does not exceed $\frac{n-2}{3}$, a contradiction. If $ab \in E(G)$, then the vertex $v$ cannot be connected by an edge with the path $P$. Thus its degree does not exceed $\frac{n-2}{3}$.

Finally, consider the case where all paths but one are of length two and one path is of length three. With the same notation as in the case above, it is easy to see that either the degree of $a$ or the degree of $v$ is not greater than $\frac{n}{3}$, a contradiction.

Finally we may conclude that $G$ contains a long path covering.

Denote it by $\mathcal{P}$ and let $G'$ be the graph $G - E(\mathcal{P})$. We shall show that $G'$ has a semi-VDP coloring that uses $k = \Delta + 2$ colors where $\Delta = \Delta(G)$.

In order to use Lemma 3, we have to verify the following condition

$$(*) \qquad d'(k - d') \geq n_{d'}(G') - 2$$

for any $d$, $\delta' \leq d' \leq \Delta'$, where $\delta' = \delta(G')$ and $\Delta' = \Delta(G')$. We shall consider some cases.

**Case 1.** $d' = \Delta'$.

Since $G'$ has been obtained from $G$ by removing the edges of some paths, we have two possibilities for $\Delta'$.

**Case 1a.** $\Delta' = \Delta - 1$.

Then $k - d' = 3$ and $(*)$ is implied by the inequality $(\Delta - 1)3 \geq n - 2$ since $\delta \geq \frac{n+1}{3}$.

**Case 1b.** $\Delta' = \Delta - 2$.

Then $k - d' = 4$. Moreover, observe that in this case $G$ cannot be regular. In consequence, $\Delta \geq \delta + 1 \geq \frac{n+4}{3}$. So, we have to verify that $(\Delta - 2)4 \geq n - 2$. This inequality is equivalent to $\Delta \geq \frac{n+6}{4}$ which holds for $n \geq 2$.

**Case 2.** $d' < \Delta'$.

This ensures, in particular, that $G'$ is not regular. Thus $n_{d'} \leq n - 1$. Moreover we have: $k - d' \geq 4$ and $d' \geq \delta - 2$. So, to get $(*)$ it suffices to verify that

$$(\delta - 2)4 \geq n - 3.$$

This inequality holds for $n \geq 9$. By the remark at the beginning of this section we are done if $G$ is not regular. But if $G$ is regular then $d' = \Delta - 2$ and $n_{d'} \leq n - 2$.

Therefore, we have to examine if $\Delta - 2 \geq \frac{n-4}{4}$. It is easy to see that this inequality holds for $n \geq 6$. This finishes the proof of the existence of a semi-VDP coloring of $G'$ with $\Delta + 2$ colors. By Lemma 4 this finishes also the proof of the theorem.

## 4 Concluding remark

Observe that the methods used in the proof enable us to formulate a more general result.

**Theorem 5** *If $G$ contains a long path covering $\mathcal{P}$ such that there exists a semi-VDP coloring of $G' = G - E(\mathcal{P})$ with $k$ colors, then there exists a VDP coloring of the edges of $G$ with $k+3$ colors.*

¿From this point of view, Theorem 2 can be considered as an example of the application of the above general theorem.

## References


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