Differential approximation for optimal satisfiability and related problems

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Abstract

We study the differential approximability of several optimization satisfiability problems. We prove that, unless $\mathbf{co}-\mathbf{RP}=\mathbf{NP}$, Min Sat is not differential $1/m^{1-\varepsilon}$-approximable for any $\varepsilon > 0$, where m is the number of clauses. We also prove that any differential approximation algorithm for Max Minimal Vertex Cover can be transformed into a differential approximation algorithm for Min kSat achieving the same differential performance ratio. This leads us to study the differential approximability of Max Minimal Vertex Cover and Min Independent Dominating Set. Both of them are equivalent for the differential approximation. For these problems we prove a strong inapproximability result, informally, unless $\mathbf{P} = \mathbf{NP}$, any approximation algorithm has worst-case approximation ratio equal to 0.

Keywords: Combinatorial optimization, Complexity theory, Heuristics.

1 Introduction

In this paper we deal with the approximation of classical optimization satisfiability problems as Max and Min Sat, Max and Min DNF, as well as of restrictive versions of these problems as the ones where the size of any clause is bounded, or/and the number of the occurrences of any literal is bounded. We also deal with some related graph-problems as Max and Min Independent Dominating Set and Max and Min Minimal Vertex Cover. We study the approximability of all these problems (formally defined in section 2) using the so-called *differential approximation ratio* which, informally, for an instance I *measures the relative position of the value of an approximated solution in the interval [worst-value feasible solution of* I*, optimal-value solution of* I*]*.

Optimization satisfiability problems are of interest from both theoretical and practical points of view. On the one hand, the satisfiability problem (Sat) is the first complete problem for **NP**. On the other hand, many problems in mathematical logic and in artificial intelligence can be expressed in terms of versions of Sat; constraints satisfaction is one such version. Also problems in database integrity constraints or in knowledge bases can be seen as optimization satisfiability problems. Finally, some approaches to inductive inference can be modeled as Max Sat problems ([9, 10]).

All the problems dealt in this paper have no polynomial time approximation schemata for the standard approximation (where one measures the ratio between the value of the approximate solution of an instance and the value of an optimal one). The Sat problems admit algorithms achieving constant standard approximation ratio, while algorithms for the DNF ones do not guarantee such ratios (more details about the standard approximability of all these problems can be found in [4, 5]). The Min Vertex Cover (called Min Minimal Vertex Cover

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in this paper) is standard 2-approximable, while the MAX INDEPENDENT SET (called MAX INDEPENDENT DOMINATING SET in the paper) cannot be approximated within $n^{1-\varepsilon}$, for any $\varepsilon > 0$, unless $\mathbf{co} - \mathbf{RP} = \mathbf{NP}$ ([8]). On the other hand, MIN INDEPENDENT DOMINATING SET is standard approximable within B where, B is the maximum graph-degree ([11]), while the MAX MINIMAL VERTEX COVER has, to our knowledge, not been studied yet in the standard approximation.

The initial objective of the paper was to study the differential approximability of some optimization satisfiability problems defined in section 2. This study has exhibited an interesting relationship between MIN kSAT and MIN MINIMAL VERTEX COVER which can be informally described as follows: *any differential approximation algorithm for* MIN MINIMAL VERTEX COVER *can be transformed into a differential approximation algorithm for* MIN kSAT *achieving the same differential performance ratio*. On the other hand, as we will see just below, MAX MINIMAL VERTEX COVER is equivalent, for the differential approximation, to the well-known MIN INDEPENDENT DOMINATING SET. We are so led to study differential approximation results for MAX MINIMAL VERTEX COVER and MIN INDEPENDENT DOMINATING SET.

All the problems we deal with in this paper have the characteristic that computation of both their optimal and worst solutions is **NP**-hard (for example, considering an instance φ of MAX kSAT, its worst solution is an assignment satisfying the minimum number of the clauses of φ, i.e., an optimal solution for MIN kSAT on φ). Remark also that, given a graph $G = (V, E)$, the complement, with respect to V of a vertex cover (resp., independent set) is an independent set (resp., vertex cover) of G. In other words, the objective values of MIN (MAX) MINIMAL VERTEX COVER and of MIN (MAX) INDEPENDENT DOMINATING SET are linked by affine transformations. On the other hand, the differential approximation ratio is stable for the affine transformation, in the sense that pairs of problems, the objective values of which are linked by affine transformations, are differential equivalent. Hence the following fact holds: MIN (MAX) MINIMAL VERTEX COVER *and* MAX (MIN) INDEPENDENT DOMINATING SET *are differential equivalent.*

In what follows, we first study differential approximation preserving reductions for several optimization satisfiability problems. Combining them with a general result linking approximability of maximization problems in differential and standard approximations, we obtain interesting differential inapproximability results for optimal satisfiability. We also prove that MIN kSAT$(B, \bar{B})$ and MAX kSAT$(B, \bar{B})$ reduce to MIN MINIMAL VERTEX COVER-B' and MIN INDEPENDENT DOMINATING SET-B', respectively, where $B' = kB$. These reductions lead us to study the differential approximation of MIN INDEPENDENT DOMINATING SET. For this problem we prove a strong inapproximability result, informally, *unless* $\mathbf{P} = \mathbf{NP}$*, any approximation algorithm has worst-case approximation ratio equal to 0.* To our knowledge, no such result was previously known for the differential approximation.

2 Preliminaries

We first recall a few definitions about differential and standard approximabilities. Given an instance I of an optimization problem and a feasible solution S of I, we denote by $m(I, S)$ the value of the solution S, by $\text{opt}(I)$ the value of an optimal solution of I, and by $\omega(I)$ the value of a worst solution of I. The *standard performance, or approximation, ratio* of S is defined as

$$r(I, S) = \max\left\{\frac{m(I, S)}{\text{opt}(I)}, \frac{\text{opt}(I)}{m(I, S)}\right\}$$

while the *differential performance, or approximation, ratio* of S is defined as

$$\rho(I, S) = \frac{|m(I, S) - \omega(I)|}{|\text{opt}(I) - \omega(I)|}.$$

It is easy to see that the differential approximation ratio (originally defined in [3] and further depthened in [7]) is stable for the affine transformation of the objective function of a problem, while this does not hold for the standard approximation ratio.

For a function f, $f(n) > 1$, an algorithm is a *standard $f(n)$-approximation algorithm* for a problem Π if, for any instance I of Π, it returns a solution S such that $r(I,S) \leqslant f(|I|)$, where $|I|$ is the size of I. We say that an optimization problem is *standard constantly approximable* if, for some constant $c > 1$, there exists a polynomial time standard c-approximation algorithm for it. An optimization problem has a *standard polynomial time approximation schema* if it has a polynomial time standard $(1+\varepsilon)$-approximation, for every constant $\varepsilon > 0$. Similarly, for a function f, $f(n) < 1$, an algorithm is a *differential $f(n)$-approximation algorithm* for a problem Π if, for any instance I of Π, it returns a solution S such that $\rho(I,S) \geqslant f(|I|)$. We say that an optimization problem is *differential constantly approximable* if, for some constant $\delta < 1$, there exists a polynomial time differential δ-approximation algorithm for it. An optimization problem has a *differential polynomial time approximation scheme* if it has a polynomial time differential $(1+\varepsilon)$-approximation, for every constant $\varepsilon > 0$. We say that two optimization problems are *differential equivalent* if a differential δ-approximation algorithm for one of them implies a differential δ-approximation algorithm for the other one.

Given two problems Π and Π', we say that Π' *is differential reducible to* Π, iff any differential δ-approximation algorithm for Π can be used to approximately solve Π' within differential ratio δ.

In this paper, we study the differential approximability of the following **NP**-hard optimal satisfiability problems.

MAX (MIN) SAT
Input: a set of clauses (i.e., disjunctions) $C_1, \ldots, C_m$ on n variables $x_1, \ldots, x_n$.
Output: a truth assignment to the variables that maximizes (minimizes) the number of clauses satisfied.

MAX (MIN) DNF
Input: a set of conjunctions $C_1, \ldots, C_m$ on n variables $x_1, \ldots, x_n$.
Output: a truth assignment to the variables that maximizes (minimizes) the number of conjunctions satisfied.

For a constant $k \geqslant 2$, MAX kSAT, MAX kDNF, MIN kSAT, MIN kDNF are the versions of MAX SAT, MAX DNF, MIN SAT, MIN DNF where each clause or conjunction has size at most k. For two constants $B, B' \geqslant 1$, MAX kSAT$(B, \bar{B}')$, MAX kDNF$(B, \bar{B}')$, MIN kSAT$(B, \bar{B}')$, MIN kDNF$(B, \bar{B}')$, MAX SAT$(B, \bar{B}')$, MAX DNF$(B, \bar{B}')$, MIN SAT$(B, \bar{B}')$, MIN DNF$(B, \bar{B}')$ are the versions of these problems where any positive literal appears at most B times and any negative one appears at most B' times.

MAX NAE 3SAT
Input: a set of conjunctions $C_1, \ldots, C_m$ of three literals on n variables $x_1, \ldots, x_n$.
Output: a truth assignment to the variables that maximizes the number of conjunctions satisfied in such a way that any one of them has at least one true literal and at least one false literal.

MIN (MAX) MINIMAL VERTEX COVER
Input: a graph $G = (V, E)$.
Output: a minimal vertex cover (a set $S \subseteq V$ such that, $\forall (u,v) \in E$, $u \in S$ or $v \in S$) of minimum (maximum) size.

MIN (MAX) INDEPENDENT DOMINATING SET
Input: a graph $G = (V, E)$.
Output: a maximal independent set (a set $S \subseteq V$ such that, $\forall u, v \in S$, $(u,v) \notin E$ and $\forall u \notin S, \exists v \in S$, $(u,v) \in E$) of minimum (maximum) size.

In what follows, we denote by Min (Max) Independent Dominating Set-B and Min (Max) Minimal Vertex Cover-B the versions of the above problems on graphs with maximum degree bounded by B.

Finally, note that the satisfiability problems dealt here have, under the differential approximation ratio, a natural interpretation. For instance, Max Sat can be seen as the problem of determining a truth assignment that *minimizes the number of falsified clauses.*

3 Satisfiability problems

3.1 Approximation preserving reductions for optimization satisfiability

We first prove the differential equivalence for Max Sat and Min DNF and for Min Sat and Max DNF.

Theorem 1. Max Sat *and* Min DNF*, as well as* Min Sat *and* Max DNF *are differential equivalent.*

Proof. We construct a reduction from Max Sat to Min DNF that preserves the differential approximation ratio. Let I be an instance of Max Sat on n variables and m clauses. The instance I' of Min DNF contains m clauses and the same set of n variables. With each clause $\ell_1 \vee \ldots \vee \ell_t$ of I we associate in I' the conjunction $\bar{\ell}_1 \wedge \ldots \wedge \bar{\ell}_t$, where $\bar{\ell}_i = \bar{x}_j$ if $\ell_i = x_j$ and $\bar{\ell}_i = x_j$ if $\ell_i = \bar{x}_j$. It is easy to see that $\mathrm{opt}(I') = m - \mathrm{opt}(I)$ and $\omega(I') = m - \omega(I)$. Also, if $m(I', y)$ is the value of the solution y in I', then the same solution y has in I the value $m(I, y) = m - m(I', y)$. Thus, $\rho(I, y) = \rho(I', y)$. The reduction from Min DNF to Max Sat is the same.

By an exactly similar reduction, one can prove that Min Sat and Max DNF are also approximate equivalent. ∎

By the proof of theorem 1 one easily can deduce that for each constant $k \geqslant 2$, Max kSat and Min kDNF as well as Min kSat and Max kDNF are differential equivalent.

Consider an instance I of a maximization problem Π, an approximation algorithm A for Π and denote by S a feasible solution of Π computed by A in I. Then,

$$\frac{m_A(I,S) - \omega(I)}{\mathrm{opt}(I) - \omega(I)} \geqslant \delta \Longrightarrow \frac{m_A(I,S)}{\mathrm{opt}(I)} \geqslant \delta + (1-\delta)\frac{\omega(I)}{\mathrm{opt}(I)} \overset{\omega(I),\mathrm{opt}(I) \geqslant 0}{\Longrightarrow} \frac{m_A(I,S)}{\mathrm{opt}(I)} \geqslant \delta$$

and the following proposition immediately holds.

Proposition 1. *Approximation of a maximization problem* Π *within differential approximation ratio* δ*, implies approximation of* Π *within standard approximation ratio* $1/\delta$*.*

Combining the results of theorem 1 and proposition 1 with the fact that *for* $k \geqslant 2$*,* $B \geqslant 2$ *and* $B' \geqslant 1$*,* Max kSat$(B, \bar{B}')$ *and* Max kDNF$(B, \bar{B}')$ *have no standard polynomial time approximation schemata* ([2]), one deduces the following.

Corollary 1. *For* $k \geqslant 2$*, and* $(B, B') \geqslant (2, 1)$*, or* $(B, B') \geqslant (1, 2)$*,* Max kSat$(B, \bar{B}')$, Max kDNF$(B, \bar{B}')$, Min kSat$(B, \bar{B}')$ *and* Min kDNF$(B, \bar{B}')$ *have no differential polynomial time approximation schemata, unless* $\mathbf{P} = \mathbf{NP}$*.*

3.2 Min Sat and Min Vertex Cover

Min Vertex Cover is as the Min Minimal Vertex Cover defined in section 2 modulo the fact that the feasible solutions for the former are not mandatorily minimal. In what follows, by reduction from Min Vertex Cover, we establish an inapproximability result for Min Sat.

Theorem 2. *Unless* $\mathbf{co}-\mathbf{RP}=\mathbf{NP}$*,* Min Sat *is not differential* $1/m^{1-\varepsilon}$*-approximable for any* $\varepsilon>0$*, where* m *is the number of clauses of the instance.*

Proof. Let $G=(V,E)$ be a graph on n vertices and denote by $V=\{1,\ldots,n\}$ its vertex set. In order to construct an instance I of Min Sat, at each edge $(i,j)\in E, i<j$ we associate a variable x_{ij}. For each vertex i we define a clause C_i, where

$$C_i=\bigvee_{j:(i,j)\in E\wedge i<j} x_{ij}\vee\bigvee_{j:(i,j)\in E\wedge i>j}\bar{x}_{ji}.$$

From a vertex cover C of G we define an assignment as follows. For each $i\notin C$ and each $(i,j)\in E$, $x_{ji}=1$ if $i>j$ and $x_{ij}=0$ if $i<j$. Since C is a vertex cover, this definition is not contradictory. If $i\notin C$, then C_i is not satisfied and so $\text{opt}(I)\leqslant\text{opt}(G)$.

Given an assignment v of I, let $C=\{i:C_i \text{ is satisfied}\}$. Note that set C is a vertex cover since for $(i,j)\in E$, at least one of C_i and C_j is satisfied and so at least one of the vertices i,j appears in C. So, at each assignment v of I, we associate in G a vertex cover C with $m(G,C)=m(I,v)$. This also proves that $\text{opt}(I)=\text{opt}(G)$.

Finally, using $\omega(I)\leqslant\omega(G)$, it is easy to show that $\rho(G)\geqslant\rho(I)$.

We have seen that Min Vertex Cover is differential equivalent to Max Independent Set (which is as Max Independent Dominating Set modulo the fact that the independent set to compute has not to be minimal). On the other hand since the worst solution for Max Independent Set is the empty set (in other words, $\omega(I)=0$, $\forall I$), standard and differential approximation ratios coincide. Furthermore, Max Independent Set is not differential $1/n^{1-\varepsilon}$-approximable for any $\varepsilon>0$, unless $\mathbf{co}-\mathbf{RP}=\mathbf{NP}$ ([8]). Consequently, Min Vertex Cover is not differential $1/n^{1-\varepsilon}$-approximable for any $\varepsilon>0$, unless $\mathbf{co}-\mathbf{RP}=\mathbf{NP}$ and the result claimed follows. ∎

From the above proof the following corollary is also deduced. In what follows, we denote by Min or Max Sat$(B,\bar{B})$ the versions of Min or Max Sat$(B,\bar{B}')$ with $B=B'$.

Corollary 2. Min Sat$(B,\bar{B})$ *for* $B\geqslant 1$ *is not differential* $1/m^{1-\varepsilon}$*-approximable for any* $\varepsilon>0$*, unless* $\mathbf{co}-\mathbf{RP}=\mathbf{NP}$.

3.3 A positive differential approximation result for Max NAE 3Sat

We show in this section that a restrictive version of Max NAE 3Sat, the one on satisfiable instances is differential constantly approximable by the standard 1.096-approximation algorithm of [13].

Theorem 3. Max NAE 3Sat *on satisfiable instances is differential 0.649-approximable.*

Proof. Consider a satisfiable instance φ of Max NAE 3Sat defined on m clauses; obviously, $\text{opt}(\varphi)=m$. Run the standard 1.096-approximation algorithm of [13] on φ to obtain a solution C satisfying $m(\varphi,C)\geqslant m/1.096$. On the other hand any random assignment by values in $\{0,1\}$ of the variables of φ, where any of the two values is assigned with probability $1/2$, will feasibly satisfy $3m/4$ clauses (in other words, the assignments $(1,1,1)$ and $(0,0,0)$ are to be excluded from the eight possible assignments for each 3-clause); consequently, $\omega(\varphi)\leqslant 3m/4$.

Using the values for $\text{opt}(\varphi)$, $m(\varphi,C)$ and $\omega(\varphi)$, and the fact that the differential approximation ratio decreases with ω when dealing with maximization problems, we finally get

$$\frac{m(\varphi,C)-\omega(\varphi)}{\text{opt}(\varphi)-\omega(\varphi)}\geqslant\frac{\frac{m}{1.096}-\frac{3}{4}m}{m-\frac{3}{4}m}=\frac{0.712}{1.096}\geqslant 0.649. \quad ∎$$

On the other hand, using proposition 1 and the result of [13] that Max NAE 3Sat is not standard approximable within 1.090, unless $\mathbf{P}$=$\mathbf{NP}$, the following is deduced.

Proposition 2. MAX NAE 3SAT *is not differential 0.917-approximable.*

4 Optimal satisfiability and MIN INDEPENDENT DOMINATING SET

We now show that MIN kSAT$(B, \bar{B})$ is differential reducible to MIN MINIMAL VERTEX COVER-B'. Note that an analogous result, dealing with standard approximation, is presented in [6] between MIN SAT and MIN VERTEX COVER. But this result does not work for the differential approximation.

Theorem 4. MIN kSAT$(B, \bar{B})$ *is differential reducible to* MIN MINIMAL VERTEX COVER-B', *where* $B' = kB$.

Proof. Let I be an instance of MIN kSAT$(B, \bar{B})$ with n variables and m clauses. In the instance G of MIN MINIMAL VERTEX COVER, with each clause C_i of I we associate a vertex i. We draw an edge between i and j if there is a variable x such that C_i contains x and C_j contains $\bar{x}$. The vertex-degrees of the so constructed graph are bounded above by $B' = kB$.

From an assignment v of I we define a vertex cover C as the set of vertices that correspond to clauses satisfied by v. So, $\text{opt}(G) \leq \text{opt}(I)$.

From a vertex cover C of G we define a partial assignment v as follows: if $i \notin C$ and $x_j \in C_i$ then $x_j = 0$, and if $i \notin C$ and $\bar{x}_j \in C_i$ then $x_j = 1$. Hence, if $i \notin C$ then C_i is not satisfied by v. By the way v has been defined, the number of the non satisfied clauses in I is greater than, or equal to, the number of vertices that are not in C, i.e., $m(I, v) \leq m(G, C)$. This, together with $\text{opt}(G) \leq \text{opt}(I)$ proved just above, implies $\text{opt}(G) = \text{opt}(I)$.

If C is a minimal vertex cover (for each $i \in C$ there exists $j \notin C$ such that $(i, j) \in E$), then $m(I, v) = m(G, C)$ since the clause C_i is satisfied by v when $i \in C$. Consequently, in particular, $\omega(I) = \omega(G)$ and this concludes the proof of the theorem. ∎

By a proof similar to the one of theorem 4, one can show that MAX kSAT$(B, \bar{B})$ reduces to MAX MINIMAL VERTEX COVER-B'. Since the latter is differential equivalent to MIN INDEPENDENT DOMINATING SET-B' the following theorem concludes the section.

Theorem 5. MAX kSAT$(B, \bar{B})$ *is differential reducible to* MIN INDEPENDENT DOMINATING SET-B', *where* $B' = kB$.

5 MIN INDEPENDENT DOMINATING SET

The results of section 4 naturally bring us to study the differential approximation of MIN INDEPENDENT DOMINATING SET. In the following theorem we establish a strongly negative differential approximation result showing that *any polynomial approximation algorithm for* MIN INDEPENDENT DOMINATING SET *has (worst-case) differential approximation ratio equal to 0.*

Theorem 6. *If* **P** $\neq$ **NP**, *then, for any decreasing* $\delta : \mathbb{N} \rightarrow (0, 1)$, MIN INDEPENDENT DOMINATING SET *on graphs of order* n *is not differential* $\delta(n)$*-approximable.*

Proof. We show that, for any $\delta(n) \in (0, 1)$, a polynomial time differential $\delta(n)$-approximation algorithm A for MIN INDEPENDENT DOMINATING SET, could distinguish in polynomial time if an instance of SAT on n variables is satisfiable or not.

Given an instance φ of SAT with n variables $x_1, \ldots, x_n$ and m clauses $C_1, \ldots, C_m$ we construct a graph G, instance of MIN INDEPENDENT DOMINATING SET as follows. With each positive literal x_i we associate a vertex u_i and for each negative literal $\bar{x}_i$ we associate a vertex v_i. For $i = 1, \ldots, n$ we draw edges (u_i, v_i). For any clause C_j we add in G a vertex w_j and an edge between w_j and each vertex corresponding to a literal contained in C_j. Finally, we add edges in G in order to obtain a complete graph on $w_1, \ldots, w_m$.

Remark that an independent set of G contains at most $n + 1$ vertices since it contains at most one vertex among $w_1, \ldots, w_m$ and at most one vertex among u_i and v_i for $i = 1, \ldots, n$. An independent dominating set containing the vertices corresponding to true literals of a non satisfiable assignment and one vertex corresponding to a clause not satisfied by this assignment is a worst solution of G of size $n + 1$.

If φ is satisfiable then $\mathrm{opt}(G) = n$ since the set of vertices corresponding to the true literals of an assignment satisfying φ is an independent dominating set (each vertex w_j is dominated by a vertex corresponding to a true literal of C_j) of minimum size. On the other hand, if φ is not satisfiable then $\mathrm{opt}(G) = n + 1$.

In fact any independent dominating set of G has cardinality either n, or $n + 1$. Hence, if A computes a solution of value n then φ is satisfiable, otherwise φ is not satisfiable. ∎

As we have already mentioned, an interesting consequence of theorem 6 above is that *unless* $\mathbf{P} = \mathbf{NP}$*, any polynomial time approximation algorithm for* MIN INDEPENDENT DOMINATING SET *has worst-case differential approximation ratio equal to 0.* This makes MIN INDEPENDENT DOMINATING SET one of the hardest problems for the differential approximation. Let us note that, to our knowledge, no problem verifying a statement as the one of theorem 6 were known until now for the differential approximation. Moreover, theorem 6 has also the following interesting corollary.

Corollary 3. *Any approximation algorithm for* MIN INDEPENDENT DOMINATING SET-B *that achieves approximation ratio $\delta(B)$, for any decreasing function $\delta : \mathbb{N} \to (0, 1)$, has time-complexity exponential in B.*

Consider the refinement, due to Arora et al.([1]), of Cook's theorem on the **NP**-hardness of 3SAT.

Theorem 7. *([1]) Let $\mathcal{L}$ be a language in* **NP**. *There exists a polynomial-time algorithm and a constant $0 < \varepsilon < 1$ such that, given any input x, the algorithm constructs an instance φ_x of* 3SAT *which satisfies the following properties:*

1. *if $x \in \mathcal{L}$, then φ_x is satisfiable;*
2. *if $x \notin \mathcal{L}$, then no assignment satisfies more than a fraction $(1 - \varepsilon)$ of the clauses.*

Using now the L-reduction of [12] from MAX 3SAT to MAX 3SAT$(4, \bar{4})$, and observing that satisfiable instances are mapped into satisfiable instances, theorem 7 holds also if we replace 3SAT with 3SAT$(4, \bar{4})$ and ε with some constant ε'. This allows us to provide an upper bound for the differential approximation ratio of any algorithm polynomially solving MIN INDEPENDENT DOMINATING SET-B.

Theorem 8. MIN INDEPENDENT DOMINATING SET-B *is not differential $f(B)$-approximable, for $f(B) = 1 - (2\varepsilon'(B - 5)/(2B - 5))$, unless* $\mathbf{P} = \mathbf{NP}$.

Proof. We show that if MIN INDEPENDENT DOMINATING SET-B was differential $f(B)$-approximable, then we could distinguish in polynomial time if an instance of MAX 3SAT$(4, \bar{4})$ is satisfiable or at most a fraction $(1 - \varepsilon')$ of the clauses are satisfied.

Given an instance φ of 3SAT(4,$\bar{4}$) with n variables $x_1, \ldots, x_n$ and m clauses $C_1, \ldots, C_m$, we construct a graph G, instance of MIN INDEPENDENT DOMINATING SET-B, as follows. With each positive literal x_i we associate a vertex u_i, and with each negative literal $\bar{x}_i$ we associate a vertex v_i. For $i = 1, \ldots, n$ we draw in G the edges $u_i v_i$. Also with each clause C_j we associate $c = \lfloor (B - 1)/4 \rfloor$ vertices $w_{j1}, \ldots, w_{jc}$. For each clause C_j we add in G an edge between each w_{jk}, $k = 1, \ldots, c$ and any vertex corresponding to a literal contained in C_j.

Suppose that each literal appears at least once. Remark that an independent set of G contains at most $m \cdot c$ vertices. An independent dominating set containing the vertices corresponding to the m clauses of φ is a worst solution of size $m \cdot c$.

If φ is satisfiable then $\text{opt}(G) = n$ since the set of vertices corresponding to the true literals of an assignment satisfying φ is an independent dominating set (each vertex w_{jk} is dominated by a vertex corresponding to a true literal of C_j) of minimum size. On the other hand, if the optimal value of φ is $m' \leqslant (1-\varepsilon')m$ then $\text{opt}(G) = n + (m - m') \cdot c \geqslant n + \varepsilon' \cdot m \cdot c$.

We show that a differential $f(B)$-approximation algorithm A for Min Independent Dominating Set-B with $f(B) = 1 - (2\varepsilon'(B-5)/(2B-5))$ gives in the case where φ is satisfiable a solution of value less that the value of the optimum solution in the case where φ is not satisfiable.

Denote by val the value of the solution computed by A. Then, $(m \cdot c - \text{val})/(m \cdot c - n) \geqslant f(B)$. Since $c \leqslant (B-1)/4$ and $m \leqslant 8n/3$, $\text{val} \leqslant n + (m \cdot \varepsilon'(B-5)/4) < n + m \cdot \varepsilon' \cdot c$, q.e.d. ∎

6 Discussion

We have given in this paper differential inapproximability results for optimal satisfiability problems, as well as for Min Independent Dominating Set. For this problem we have shown that any polynomial time approximation algorithm has worst-case differential approximation ratio 0. This result brings Min Independent Dominating Set to the status of one of the hardest problems for the differential approximation.

Differential approximation for optimal satisfiability misses until now in positive results. Despite our efforts, the only one we have been able to produce is the one of section 3.3 on a class of instances of Max NAE 3Sat, the satisfiable ones. It would be interesting to produce non-trivial such results and this is a major open problem. However, it seems to us that, in the opposite of the standard approximation, obtaining constant differential approximation ratios for optimal satisfiability is a rather hard task.

As we have already mentioned, results as the one of theorem 6 have not been produced until now. However such strongly negative results are very interesting since they draw the hardest of the NP-hard problems classes in the differential approximability hierarchy. Establishing such results for other problems is an equally interesting open problem.

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