Satisfactory graph partition, variants, and generalizations *

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Abstract

The SATISFACTORY PARTITION problem asks for deciding if a given graph has a partition of its vertex set into two nonempty parts such that each vertex has at least as many neighbors in its part as in the other part. This problem was introduced by Gerber and Kobler [*EJOR*, 125 (2000), 283–291] and studied further by other authors. In this paper we first review some applications and related problems. Then, we survey structural, complexity, and approximation results obtained for SATISFACTORY PARTITION and for some of its variants and generalizations. A list of open questions concludes this survey.

Keywords: combinatorial optimization, complexity theory, vertex partition, degree constraints, approximation algorithm.

MSC2000 Subject Classification: 05C85, 68R10, 05-02 (primary); 05C35, 68W25 (secondary)

1 Introduction

Gerber and Kobler introduced in [34, 35] the SATISFACTORY PARTITION problem of deciding if a given graph has a vertex partition into two nonempty parts such that each vertex has at least as many neighbors in its part as in the other part. Such a solution is called a *satisfactory partition* and a graph satisfying this property is called *partitionable*. A subset of vertices such that each vertex has at least as many neighbors in its part as outside is called a *satisfactory subset*. In this survey we also consider several variants of SATISFACTORY PARTITION. A first variant, referred to as CO-SATISFACTORY PARTITION, asks for deciding if a given graph has a vertex partition into two nonempty parts such that each vertex has at least as many neighbors in the other part as in its own part. Such a solution is called a *co-satisfactory partition*. A second variant, called SATISFACTORY BISECTION, imposes further that the partition be balanced. In the same way, we consider the balanced version of CO-SATISFACTORY PARTITION, called CO-SATISFACTORY BISECTION. We also consider generalizations of these problems

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to k parts, for $k \geq 3$, called *(co-)satisfactory k-partitions* or *balanced (co-)satisfactory k-partitions*. Other generalizations, where conditions for vertices to be satisfied are specific to each vertex, or part, are also reviewed. These problems are investigated from structural and algorithmic aspects.

The paper is structured as follows. In Section 2 we give some motivations for these problems. In Section 3 we introduce formally some definitions and notation. We start Section 4 with some examples of partitionable and non-partitionable graphs, we review sufficient conditions for the existence of a satisfactory partition and we finish with examples of graphs that admit or not a (co-)satisfactory bisection. In Section 5 we study the complexity and the approximation of (Co-)Satisfactory Partition and some balanced versions. In Section 6, we consider generalizations of Satisfactory Partition with other constraints and more than two parts. Particular classes of graphs that admit efficient algorithms for exact or approximate solutions are considered in Section 7. We conclude this paper by providing a list of open problems.

2 Applications and related problems

Detecting a satisfactory partition, or some of its variants, arises in a variety of applications and contexts.

Identifying *communities* within social or biological networks, or within the web graph, is a major and fashionable concern. In the web context, a community is defined by Flake et al. [31] as a set of web pages that links to more web pages in the community than to pages out of community. Interesting weighted extensions are studied in [32]. The problem of partitioning into communities, so as to identify clusters, amounts to determining a satisfactory k-partition, for a given or suitably chosen k.

Another stream of applications deals with *alliances* in graphs, originally introduced by Kristiansen et al. [46], and further studied by Shafique [61] and other authors. The purpose is to form coalitions of vertices able to defend each other (defensive alliances) or attack non-allied vertices (offensive alliances). Alliances can be formed between nations in a security context, between companies in a business context, or between people wishing to gather by affinity. Various types of alliances were formally defined, among which the concept of strong defensive alliance used in [46] strictly corresponds to a satisfactory subset. Trying to establish a partition into strong defensive alliances, i.e. a satisfactory (k-)partition, is natural when aiming at stabilizing a region by favoring alliances between neighboring countries. In the context of persons trying to gather by affinity, Gerber and Kobler [35] suggest the problem of conference organizers that propose a sightseeing tour on two boats and try to assign the participants to one of the boats so that each participant is satisfied if he knows at least as many persons on his boat as on the other. In this case, an even more appropriate assignment would correspond to a satisfactory bisection.

A co-satisfactory partition is also known in the literature as an *unfriendly partition*. This problem was studied, e.g., in [1, 11, 52]. It was readily observed that any finite graph admits an unfriendly 2-partition or co-satisfactory partition (see section 4.4). The main research emphasis was on infinite graphs. It was proved in [52] that all graphs admit an unfriendly 3-partition, but also that not all infinite graphs admit an unfriendly 2-partition. The counterexample involves uncountably many vertices, and the question for countably infinite graphs remains open. The emphasis in our paper is different, since we consider finite graphs but balanced versions of the Co-Satisfactory Partition problem, for which the existence is not guaranteed.

Satisfactory Partition and Co-Satisfactory Partition are also related to a problem in Artificial Intelligence concerning the study of a connectionist model of the human brain, known as a binary coherent system [42] or a stable configuration in neural networks in the Hopfield model [60]. The problem can be formally stated as follows. Given an edge-weighted undirected graph $G = (V, E)$ and a threshold value t_v for each vertex $v \in V$, find a vertex assignment $\sigma : V \to \{-1, 1\}$ such that for each vertex $v \in V$ the energy $E(v)$ is non-negative, where

$$E(v) = \sigma(v)t_v + \sum_{e=(u,v)\in E} \sigma(u)\sigma(v)w(e)\,.$$

The Happynet problem defined in [55] is a particular case of the previous problem where $t_v = 0$ for each vertex $v \in V$. One may observe that, when every $w(e)$ is a positive constant, the two vertex classes in a solution of Satisfactory Partition correspond to the vertices having $\sigma = 1$ and $\sigma = -1$, respectively. It may happen, however, that the maximum energy of the system is attained when all signs $\sigma(v)$ are the same, which is not accepted for Satisfactory Partition as a solution.

Thus, Satisfactory Partition (resp. Co-Satisfactory Partition) corresponds to Happynet when the weight function is a positive (resp. negative) constant, but with the additional constraint that the assignment must be nontrivial (i.e., $\sigma \neq (1, \ldots, 1)$ and $\sigma \neq (-1, \ldots, -1)$).

It is interesting to notice that Happynet always admits a solution, which entails that it cannot be *FNP*-hard unless *FP*=*FNP* [51, 55] (where *FP*, *FNP* are the search classes associated with the decision classes *P*, *NP*). On the other hand, no polynomial-time algorithm is known for Happynet so far. The *NP*-completeness of Satisfactory Partition, proved in [6], shows that the particular case of Happynet when function w is a positive constant becomes difficult as soon as a nontrivial solution is looked for. However, when function w is a negative constant and we search for a nontrivial solution, Happynet, which then corresponds to Co-Satisfactory Partition, becomes polynomial.

Another interpretation comes from the area of network reliability. If we model a communication network with a connected graph, edge sets whose removal still keeps the graph connected correspond to sets of line failures under which the entire network is still capable to transfer messages from any origin to any destination. In this way, networks *without* a satisfactory partition remain operating even if up to half of the local connections at each node fail down.

A lower level of fault tolerance, namely where the network remains connected when at most one line failure occurs per node, has been studied in detail in [30]. We shall refer to this situation as graphs without *matching cutsets*.

On the other hand, in the context of graph drawing, in [57] and [26] graphs *with* matching cutsets are considered. It is pointed out in those papers that matching cutsets (if they are already at hand) can be applied to make a three-dimensional orthogonal drawing more efficiently in running time, and also more effectively in terms of volume occupation, number of bends, and average edge length.

Actually, the first motivation for the study of graphs with/without matching cutsets appears to be some kind of optimal binary codes, as mentioned in [38].

A different kind of motivation comes from the engineering problem of structural process control. The dynamic behavior of a process system can be described with a system of differential equations

$$\frac{dy_j}{dt} = f_j(\boldsymbol{y}, \boldsymbol{x}) \quad (j = 1, \ldots, n), \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0$$

called state equation (see e.g. [44]), where $\boldsymbol{y} = \boldsymbol{y}(t) = (y_1(t), \ldots, y_n(t))$ and $\boldsymbol{x} = \boldsymbol{x}(t) = (x_1(t), \ldots, x_k(t))$ is the vector of states and of the input, respectively, at any time t. The structure of this system can be represented with the 'equation-variable graph' that is a directed graph $G = (X \cup Y, E)$ with two types of vertices x_i and y_j representing the input and state variables ($i = 1, \ldots, k$, $j = 1, \ldots, n$). There is an arc $(x_i, y_j) \in E$ or $(y_i, y_j) \in E$ if variable x_i or y_i appears in the argument of f_j with nonzero effect.

A standard way to modify the behavior of the system (e.g., to stabilize it) is to use static state feedback controllers, which amounts to associating a subset Y_i of state variables with input variable x_i, and manipulating the input x_i according to the states in Y_i at any time t. The sets Y_i are assumed to be mutually disjoint. Hence, taking k controllers in the distributed control system structure corresponds to a vertex partition into k classes. Then the edges between different classes occur as disturbances from one class to the other. The primary aim usually is to design an optimal distribution of controllers. However, the *degree of coupling*, defined as the total number of edges joining the distinct classes of the partition, may also be of importance. Therefore, a satisfactory k-partition yields a locally optimal solution for the problem of finding a stabilizing structure whose degree of coupling is kept at a low level. Complexity results on controllers with small $|Y_i|$, and also the description of a more general model where the edges are weighted, can be found e.g. in [39, 40].

3 Notation and problem definitions

The following notation will be used in the rest of the paper. For a graph $G = (V, E)$, a vertex $v \in V$, and a subset $Y \subseteq V$ we denote by $d_Y(v)$ the number of vertices in Y that are adjacent to v; and, as usual, we write $d(v)$ for the degree $d_V(v)$ of v in V. The minimum and maximum degree of G will be denoted by $\delta(G)$ and $\Delta(G)$, respectively. For any subgraph G' of G, $V(G')$ and $E(G')$ denote, respectively, the set of vertices and edges of G'. A partition (V_1, V_2) of V is said to be *nontrivial* if both V_1 and V_2 are nonempty. For basic notions on graphs not defined in this paper, we refer to the textbooks such as [15] and [27].

The main problem we are interested in is defined as follows.

Satisfactory Partition
Input: A graph $G = (V, E)$.
Question: Is there a nontrivial partition (V_1, V_2) of V such that for every $v \in V$, if $v \in V_i$ ($i = 1, 2$) then $d_{V_i}(v) \geq \lceil \frac{d(v)}{2} \rceil$?

We consider also the *balanced version* of the Satisfactory Partition problem, where feasible solutions are restricted to *bisections*, that means vertex partitions satisfying the condition $|V_1| = |V_2|$.

Satisfactory Bisection
Input: A graph $G = (V, E)$ on an even number of vertices.

Question: Is there a nontrivial partition (V_1, V_2) of V such that $|V_1| = |V_2|$ and for every $v \in V$, if $v \in V_i$ $(i = 1, 2)$ then $d_{V_i}(v) \geq \lceil \frac{d(v)}{2} \rceil$?

We also consider the following dual version of SATISFACTORY PARTITION.

CO-SATISFACTORY PARTITION
Input: A graph $G = (V, E)$.
Question: Is there a nontrivial partition (V_1, V_2) of V such that for every $v \in V$, if $v \in V_i$ $(i = 1, 2)$ then $d_{V_i}(v) \leq \lfloor \frac{d(v)}{2} \rfloor$?

The balanced version is defined formally as follows.

CO-SATISFACTORY BISECTION
Input: A graph $G = (V, E)$ on an even number of vertices.
Question: Is there a nontrivial partition (V_1, V_2) of V such that $|V_1| = |V_2|$ and for every $v \in V$, if $v \in V_i$ $(i = 1, 2)$ then $d_{V_i}(v) \leq \lfloor \frac{d(v)}{2} \rfloor$?

Given a partition (V_1, V_2) of V, a vertex $v \in V_i$ is *satisfied* if $d_{V_i}(v) \geq \lceil \frac{d(v)}{2} \rceil$ and *co-satisfied* if $d_{V_i}(v) \leq \lfloor \frac{d(v)}{2} \rfloor$. A set $A \subset V$ is a *satisfactory subset* if every vertex of A is satisfied in $(A, V \setminus A)$. A partition where every vertex is (co-)satisfied is called a *(co-)satisfactory partition*. If a (co-)satisfactory partition (A, B) has the property $|A| = |B|$ then it will be called a *(co-)satisfactory bisection*.

4 Existence conditions

Most of the known results deal with satisfactory partitions. These results are presented in the next three subsections. Other types of partitions are reviewed in the last subsection.

For short, we call a graph *partitionable* if it admits a satisfactory partition.

4.1 Necessary and sufficient conditions for general graphs

We first observe that disconnected graphs are trivially partitionable. Therefore, in the following, we only consider connected graphs.

We provide now simple observations which give necessary conditions for the existence of a satisfactory partition.

Proposition 1 *Let $G = (V, E)$ a connected graph, with $|V| = n$. If G is partitionable then each of the following conditions is satisfied:*

- (i) *One of the parts is of size at least $\lceil \frac{\Delta(G)}{2} \rceil + 1$ and the other is of size at least $\lceil \frac{\delta(G)}{2} \rceil + 1$.*
- (i') $n \geq 4$.
- (ii) *Each vertex of degree 1 must be in the same part as its neighbor.*
- (iii) *All vertices of degree $n - 1$ must belong to the same part.*
- (iii') *G contains at most $\lceil \frac{n}{2} \rceil - 2$ vertices of degree $n - 1$.*

Using this proposition, we can easily observe that the following graphs are not partitionable: stars (since (ii) implies that one part of the partition is empty) and complete graphs (since none of (i), (iii), and (iii') is satisfied).

In the following we give two characterizations of partitionable graphs, which provide additional insights on the SATISFACTORY PARTITION problem. The first one is a consequence of the more general Proposition 17 .

Proposition 2 *A connected graph is partitionable if and only if it contains two disjoint satisfactory subsets.*

Considering two disjoint satisfactory subsets A and B, one can check indeed that one obtains a satisfactory partition by adding iteratively each outside vertex either to A (or B) if it is satisfied in A (or B), and finally adding all remaining vertices either to the current subset A or the current subset B.

The second characterization highlights relations between the connectivity of a graph and the existence of a satisfactory partition.

Proposition 3 *[62] A connected graph is partitionable if and only if it has a locally minimal nontrivial edge cut.*

Here 'locally minimal' means that the size of the cut cannot be decreased by moving any single vertex to the other side of the cut. Some properties of a graph guarantee the existence of a locally minimal nontrivial edge cut. They give rise to several sufficient conditions for the existence of a satisfactory partition.

Proposition 4 *A graph with any of the following properties is partitionable:*

(i) *([34]) There is a non-pendant cut-edge,*

(ii) *([35]) There is a cut that consists of mutually disjoint edges and contains at least one non-pendant edge,*

(iii) *([34]) There is a cut-vertex not incident with pendant edges.*

Using this proposition, we can easily observe that the following graphs are partitionable : trees which are not stars (since (i) is satisfied), cycles of length at least 4 (since (ii) is satisfied).

Interesting relations with vertex and edge connectivity can also be described:

Proposition 5 *Any graph with one of the following properties is partitionable:*

(i) *([34]) the edge connectivity is smaller than the minimum degree,*

(ii) *([62]) the vertex connectivity is not larger than half of the minimum degree.*

Since vertex and edge connectivity can be determined by network flow techniques and a smallest cut can be found efficiently, under these conditions one can also find efficiently a satisfactory partition.

We consider now the class of complete bipartite graphs $K_{p,q}$. Observing that each satisfactory subset should contain at least $\lceil \frac{p}{2} \rceil + \lceil \frac{q}{2} \rceil$ vertices, it appears that if p or q is odd then these graphs are not partitionable. If both p and q are even, then taking V_1 as a subset of $\frac{p}{2}$ vertices in the first part together with a subset of $\frac{q}{2}$ vertices in the second part, we obtain a satisfactory partition $(V_1, V \setminus V_1)$. In fact, this is also a satisfactory bisection, a co-satisfactory partition, and a co-satisfactory bisection.

4.2 Sufficient conditions for graphs with small degrees only

In the context of satisfactory partitions, graphs of small maximum degree were studied independently in two papers. In [62], which appeared first, the existence of a satisfactory partition for 3- and 4-regular graphs is investigated, without addressing algorithmic issues. An efficient algorithm was designed in [5]. On the other hand, for 3-regular graphs, the existence of a satisfactory partition is equivalent to the existence of a 'matching cutset' (this is not true for graphs of maximum degree 3). Under this formulation, part (*i*) below already appeared in [54], whose proof can also be made algorithmic. The simple approach used in [5], which works both for (*i*) and (*ii*), is to start from a shortest cycle and either find a satisfactory partition within a few steps or conclude that the graph in question is one of the specified ones.

Theorem 6 ([54, 62, 5]) *The following graphs are partitionable in polynomial time:*

(*i*) *All cubic graphs except* K_4 *and* $K_{3,3}$*.*

(*ii*) *All 4-regular graphs except* K_5*.*

These results cannot be extended for regular graphs with degree greater than 4 since there are 5-regular graphs, different from K_6 and $K_{5,5}$ that are not partitionable, and there are 6-regular graphs different from K_7 that are not partitionable (see Figure 1).

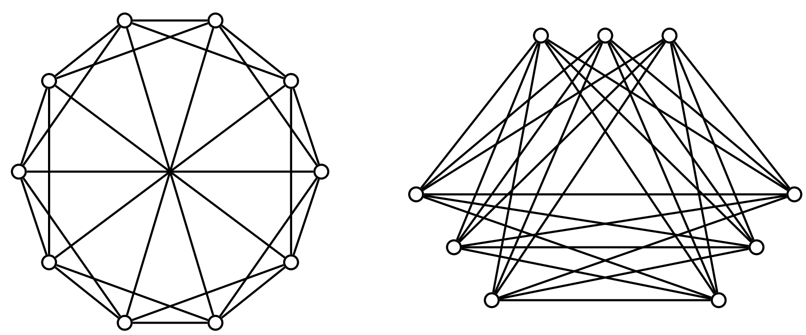

Figure 1: Non-partitionable 5-regular and 6-regular graphs

As noted by Regen [58], known results on *bisection width* (i.e., minimum cut size over all bisections) can also be applied to obtain asymptotic estimates for cubic and 4-regular graphs of large orders. Indeed, on graphs with n vertices, any bisection whose cut size is less than $n/2$ leads to a satisfactory partition because moving any non-satisfied vertex to the other part decreases the size of the cut and hence the procedure terminates before any class becomes empty. Even better, in a 4-regular graph, a non-satisfied vertex has at least three neighbors in the other part, thus moving it there decreases the cut by at least 2, and consequently any bisection with cut size smaller than n will do. The existence of such bisections were proved by Clark and Entringer [23] ($n/3 + 46$ for degree 3) and by Hromkovič and Monien [43] ($n/2 + 1$ for degree 4), respectively. This method also gives information on the unbalance (difference between the two vertex classes) in a possible satisfactory partition; namely, bisections with smaller cut size yield smaller unbalance. The currently best estimates on bisection width appear to be $n/6 + o(n)$ for cubic and $2n/5 + o(n)$ for 4-regular graphs, proved in [53]. This approach does not work for high vertex degree, however, because by the results of Bollobás

[13] the bisection width of almost all d-regular graphs of order n is at least $nd/4 - n\sqrt{d \ln 2}/2$. (The estimate $nd/4 - \Theta(n\sqrt{d})$ is also a universal upper bound, see Alon [2]).

Concerning graphs with maximum degree at most 4, the following result can be proved. In its part (ii), a *wheel* means a graph consisting of a cycle and one further vertex which is adjacent to all vertices of the cycle.

Theorem 7 *Let G be a graph with $\Delta(G) \leq 4$.*

- (i) *([5]) One can decide in polynomial time if G is (not) partitionable, and find a satisfactory partition of G if it exists.*
- (ii) *([62]) If $\delta(G) \geq 3$ holds, too, then G is partitionable if and only if $|V| \geq 6$ and G is neither a wheel nor a graph containing $|V| - 3$ independent vertices of degree 3.*

In fact, for $\Delta(G) \leq 4$, the problem is reducible to finding two vertex-disjoint cycles in a slightly extended graph [5], while under the further condition $\delta(G) \geq 3$ the two vertex-disjoint cycles should occur in the graph itself [7]. In this way, the algorithmic result of Bodlaender [12] on finding disjoint cycles can be applied to obtain a solution efficiently.

4.3 Sufficient conditions for line graphs

We recall that the *line graph* $L(G)$ of a graph G is a graph whose vertex set corresponds to the edge set of G, and whose edge set is defined by pairs of vertices whose corresponding edges are adjacent in G.

The characterization of partitionable line graphs appears to be an open problem. Nevertheless, there are several partial results presenting sufficient conditions.

Theorem 8 *The line graph $L(G)$ of a graph G is partitionable under any of the following conditions.*

- (i) *([37]) G is triangle-free, and it is neither the star nor a tree of radius 2 obtained from a star by subdividing some, but not all, edges to paths of length 2, or subdividing all edges of the star if its central vertex has odd degree.*
- (ii) *([34]) $\delta(G) \geq 2$, and G contains a vertex x of maximum degree such that any two adjacent neighbors of x have degree sum at least 6.*
- (iii) *([62]) G contains a vertex of maximum degree, which is not adjacent to*
  - (a) *another vertex of degree $\Delta(G)$, or to*
  - (b) *any vertex of degree 2, and G is not a star.*

As a matter of fact, part (i) completely characterizes the 'triangle-free' case, as the condition described there is not only sufficient but also necessary for triangle-free graphs to be partitionable. It implies, in particular, that the stars are the only triangle-free graphs without vertices of degree 2 whose line graphs are not partitionable.

4.4 Other types of partitions

In this subsection, we give examples of graphs admitting or not satisfactory bisections, co-satisfactory partitions, or co-satisfactory bisections.

4.4.1 Satisfactory bisection

One can easily see that many graphs, e.g. cycles of even length and complete bipartite graphs with both vertex classes of even size, trivially admit satisfactory bisections. On the other hand, a disconnected graph of even order formed by two non-partitionable connected components of unequal size is an example that is partitionable but not into two equal parts.

We saw that all cubic graphs except K_4 and $K_{3,3}$ are partitionable. However, there exist such graphs that have no satisfactory bisection. Such examples are cubic Hamiltonian graphs on $n = 4k+2$ vertices ($k \geq 1$), with a Hamiltonian cycle $v_1, v_2, \ldots, v_{4k+2}, v_1$ and the chords (v_i, v_{2k+1+i}), $1 \leq i \leq 2k+1$ (see Figure 2 for $k = 2$).

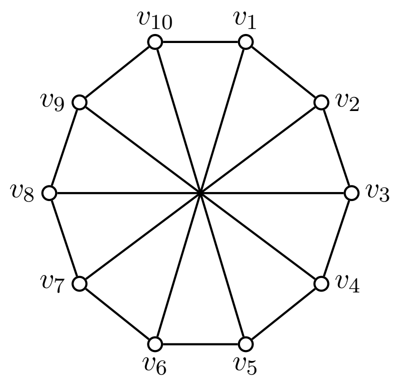

Figure 2: Cubic graph different from K_4 and $K_{3,3}$ that has no satisfactory bisection

4.4.2 Co-satisfactory partition

Co-Satisfactory Partition corresponds to finding a cut that is maximal with respect to moving a vertex from its part to the other. Therefore, a graph always admits such a partition that can be found in polynomial time. On the other hand, those locally optimal cuts can be rather far from globally optimal ones (even when local vertex switching is allowed in a much wider sense), as proved in [21].

4.4.3 Co-satisfactory bisection

The sequence of cubic graphs described in Subsection 4.4.1 has a co-satisfactory bisection except for the first member $K_{3,3}$. As we mentioned before, every graph is co-satisfactory partitionable but not every graph has a co-satisfactory bisection. Actually, some of the graphs that admit a satisfactory bisection do not admit a co-satisfactory bisection; a simple example is the disconnected graph with two components K_4 and $K_{1,3}$ of order four each. Other types of graphs admit neither a satisfactory bisection nor a co-satisfactory bisection (stars of even order, complete bipartite graphs $K_{2k+1,2\ell+1}$ with $k \neq \ell$).

5 Complexity and approximation

5.1 Complexity results

Complexity issues were already considered in the first papers [34, 35], but *NP*-completeness of Satisfactory Partition remained an open problem until the works [6, 7]. Gerber and

Kobler proved in [34] (and re-stated without proof in [35]) the strong *NP*-hardness of two 'weighted' generalizations. In the first one, the *vertices* are weighted and one asks for a nontrivial vertex partition such that, for each vertex, the sum of weights of the neighbors in the same part is at least as large as the sum of weights of the neighbors in the other part. In another generalization, the *edges* are weighted and the analogous condition is required on each vertex for the sum of weights of edges incident with it. More explicitly, the following complexity results are known.

Theorem 9 ([34]) *The following problems are NP-complete:*

(*i*) *The vertex-weighted generalization of* Satisfactory Partition *on chordal graphs.*

(*ii*) *The edge-weighted generalization of* Satisfactory Partition *on complete bipartite graphs.*

The complexity of the original (unweighted) problem was settled several years later; in some sense it is the main negative result on the subject.

Theorem 10 ([6, 7]) Satisfactory Partition *is NP-complete.*

We quote here a further result though not one on complexity, which is related to the subject along the line of hard-to-characterize classes.

Theorem 11 ([62]) *Neither the partitionable graphs, nor the non-partitionable ones, admit a characterization in terms of forbidden induced subgraphs.*

We consider next the balanced version, of the Satisfactory Partition problem. Algorithmically, both this problem and its dual version are intractable.

Theorem 12 ([6, 7]) Satisfactory Bisection *is NP-complete.*

Theorem 13 ([6, 7]) Co-Satisfactory Bisection *is NP-complete.*

5.2 Approximating the number of (co-)satisfied vertices

When a connected graph on at least three vertices has no satisfactory partition, then it admits a partition where all vertices except one are satisfied: we can put a vertex of minimum degree into the first vertex class and all the other vertices in the second class. However, such a partition is extremely unbalanced.

When a graph admits no (co-)satisfactory bisections, it is natural to ask for a bisection maximizing the number of (co-)satisfied vertices. The corresponding optimization problems are called Max (Co-)Satisfying Bisection and they are defined as follows.

Max (Co-)Satisfying Bisection
Input: A graph $G = (V, E)$ on an even number of vertices.
Output: A partition (V_1, V_2) of V, such that $|V_1| = |V_2|$, that maximizes the number of (co-)satisfied vertices.

Theorem 14 ([6]) *Unless P=NP, neither* Max Satisfying Bisection *nor* Max Co-Satisfying Bisection *admit a polynomial-time approximation scheme; but* (*i*) Max Satisfying Bisection *is 3-approximable, and*

(*ii*) Max Co-Satisfying Bisection *is 2-approximable.*

An important tool in the proof is the Gallai–Edmonds structure theorem [33, 29] concerning largest matchings, which is here applied for the *complements* of certain subgraphs of the input graph.

5.3 Approximating minimum unbalance

If the input graph is partitionable but does not admit a satisfactory bisection, it is an interesting question to estimate the minimum unbalance in a satisfactory partition. A convenient measure for a solution (V_1, V_2) is the value $||V_1| - |V_2|| + 1$, where the explanation for '+1' is that it makes comparisons possible even if the graph in question has a satisfactory bisection in which case $|V_1| - |V_2| = 0$ occurs.

The problem of minimum unbalance was first studied by Sheehan [63] under an assumption weaker than requiring all vertices to be satisfied. He proved that every graph with n vertices and minimum degree δ admits, for *some* $i \le \delta/2$, a vertex partition (V_1, V_2) such that $|V_1| = \lceil n/2 \rceil + i$, $|V_2| = \lfloor n/2 \rfloor - i$, $d_{V_1}(v) \ge \lceil \delta/2 \rceil + i$ for all $v \in V_1$, and $d_{V_2}(v) \ge \lfloor \delta/2 \rfloor - i$ for all $v \in V_2$. That is, the unbalance is guaranteed to be quite small (with the measure introduced above, it is never larger than $2\delta + 1$), but the degrees inside V_2 are allowed to be quite small, or even zero if $i = \lfloor \delta/2 \rfloor$ happens to occur. In a subsequent paper [64] this case of zero was in fact proved to be avoidable for every connected graph, and later Arkin and Hassin proved in [3] that a solution satisfying the requirements can be found in polynomial time.

The minimum unbalance of partitions of connected graphs was addressed by Sheehan [65]. Assume that G is k-regular and admits a partition (V_1, V_2) such that $d_{V_1}(v) \ge \lceil k/2 \rceil$ for all $v \in V_1$, and $d_{V_2}(v) \ge \lfloor k/2 \rfloor$ for all $v \in V_2$. If k is even, this corresponds exactly to the condition for a satisfactory partition. Upper bounds of the form $c'_k n + c''_k$ were proved on $||V_1| - |V_2||$ in such partitions for $k \le 7$, where c'_k and c''_k are absolute constants. For example, if $n \ge 24$ is even and $k = 7$, then the values $c'_7 = 17/33$ and $c''_7 = 356/33$ describe a valid upper bound. Moreover, as mentioned in Section 4.2, estimates on minimum unbalance for cubic and 4-regular graphs can also be derived from results on bisection width.

Upper bounds of this kind may be tight in some cases. However, approximating minimum unbalance for both the satisfactory and co-satisfactory partitions of the input graph turns out to be intractable. This fact is extracted in the following result.

Theorem 15 ([10]) *Unless $P = NP$, there is no polynomial-time approximation scheme for* $\min(||V_1| - |V_2|| + 1)$*, where minimum is taken over*

(*i*) *all satisfactory partitions of the input graph that is required to be partitionable, or*

(*ii*) *all co-satisfactory partitions of the input graph.*

6 Some generalizations

6.1 (a, b)-partitions

Given a graph $G = (V, E)$ and integer-valued functions a, b on V, a nontrivial vertex partition (A, B) of V with the property

$$d_A(v) \ge a(v) \quad \forall\, v \in A \qquad \text{and} \qquad d_B(v) \ge b(v) \quad \forall\, v \in B$$

is called an (a,b)*-partition.*

In order to avoid trivialities, it will be assumed throughout that the functions $a,b : V \to \mathbb{N}$ satisfy $a(v) \le d(v)$ and $b(v) \le d(v)$ for all vertices $v \in V$. Partitions (A,B) of V will be assumed nontrivial (i.e., $A \ne \emptyset$ and $B \ne \emptyset$) without explicitly mentioning this condition at each occurrence.

In this section we are interested in the following problem:

(a,b)-PARTITION
Input: A graph $G = (V,E)$, and two functions $a,b : V \to \mathbb{N}$.
Question: Does G have an (a,b)-partition ?

General sufficient conditions can be summarized as follows.

Theorem 16 *Let G be a graph, and a,b two integer-valued functions on its vertex set. Then G admits an (a,b)-partition under any of the following conditions.*

(i) *([66]) G is unrestricted, $a,b : V \to \mathbb{N}$, and $d(v) \ge a(v)+b(v)+1$ for every $v \in V$.*

(ii) *([45]) G is triangle-free, $a,b : V \to \mathbb{N} \setminus \{0\}$, and $d(v) \ge a(v)+b(v)$ for every $v \in V$.*

(iii) *([28]) G has girth at least five, $a,b : V \to \mathbb{N} \setminus \{0,1\}$, and $d(v) \ge a(v)+b(v)-1$ for every $v \in V$.*

Moreover,

(iv) *([9]) in all these cases, an (a,b)-partition of G can be found in polynomial time.*

Result (i) was motivated by a problem raised in [67], where it was proved that every graph of minimum degree $12k$ has an (a,b)-partition for $a = b = k$. Parts (ii) and (iii) were originally stated just for constants a, b instead of functions $a(v)$, $b(v)$; but in fact the proofs work for the general case without any substantial changes. The original proofs of parts (i)–(iii) contain nonconstructive steps, which were made constructive for the search version of the problem as described in part (iv).

The exclusion of 0 in part (ii) and $\{0,1\}$ in part (iii) is necessary: a cycle of any length with $a(v) = 2$ for all vertices v (and $b(v) = 0$ or $b(v) \le 1$, respectively) admits no nontrivial (a,b)-partition, because $V_1 \ne \emptyset$ would imply $V_2 = \emptyset$.

In order to establish Theorem 16, Stiebitz gives a necessary and sufficient condition for the existence of an (a,b)-partition. While originally stated with the condition $d(v) \ge a(v) + b(v)+1$, its proof still works for a slightly weaker condition. Moreover, the sufficiency part of this condition can be proved constructively using a polynomial-time algorithm, as in [9]. The assertion refers to the concept of *feasible pair*, which is a pair (A,B) of disjoint, nonempty vertex subsets $A,B \subseteq V$ such that $d_A(v) \ge a(v)$ for all $v \in A$ and $d_B(v) \ge b(v)$ for all $v \in B$.

Proposition 17 ([66, 9]) *Let $G = (V,E)$ be a graph and $a,b : V \to \mathbb{N}$ integer-valued functions such that $d(v) \ge a(v)+b(v)-1$ for every $v \in V$. G admits an (a,b)-partition if and only if it admits a feasible pair. Moreover, if a feasible pair (A,B) is given, then an (a,b)-partition can be found in polynomial time.*

Most of the algorithms for Theorem 16 are not simple. To give a flavor of these results, we present the algorithm for the triangle-free case for which a concise description can be given. Algorithm 1 (recalled from [9]) exhibits for this case a polynomial-time procedure that finds a feasible pair. Using then Proposition 17, an efficient way of finding an (a, b)-partition is obtained.

To formulate the algorithm in a concise way it is convenient to introduce some terminology.

Let G be a graph and $f : V \to \mathbb{N}$ be a function. Graph G is said to be *f-degenerate* if every nonempty $Y \subseteq V$ contains some $v \in Y$ with $d_Y(v) \leq f(v)$. Thus, if G is not $(f-1)$-degenerate, then there exists a subset A such that $d_A(v) \geq f(v)$ for each vertex $v \in A$. Such a set A will be called an *f-satisfactory subset*.

Algorithm 1 Determination of a feasible pair; triangle-free, $d(v) \geq a(v) + b(v)$

Require: a triangle-free graph G such that $d(v) \geq a(v) + b(v)$ for every $v \in V$
Ensure: a feasible pair (A, B)

```
Find A ⊆ V, a minimal a-satisfactory subset
B ← V \ A
while G[B] is (b − 1)-degenerate do
   Let x ∈ B such that d_B(x) < b(x)
   A ← A ∪ {x}; B ← B \ {x}
   while there is y ∈ A such that d_A(y) ≤ a(y) and G[A \ {y}] is not (a − 1)-degenerate
   do
      A ← A \ {y}; B ← B ∪ {y}
```

The key point in proving that the main **while** loop is executable, is that after each iteration — and also before running the loop for the first time — the partition (A, B) currently at hand satisfies the following properties:

- the set A is a-satisfactory,
- there exists a $v_1 \in A$ such that $d_A(v_1) = a(v_1)$,
- the set $A \setminus \{v_1\}$ is $(a-1)$-degenerate.

These conditions ensure the existence of a neighbor $v_2 \in A$ of v_1 with the same properties. Then, applying the assumption that G is triangle-free, one can select some $x \in B$ for which one of v_1 and v_2 is a suitable choice for y.

There is an important tool for evaluating the quality of a partition along the way towards finding a satisfactory partition. This has been used in the proofs of all parts of Theorem 16. As introduced by Stiebitz [66], a vertex partition (A, B) is associated with the weight

$$w(A, B) = |E(G[A])| + |E(G[B])| + \sum_{v \in A} b(v) + \sum_{v \in B} a(v)$$

Then, for example, the finiteness and efficiency of Algorithm 1 can be proved by showing that $w(A, B)$ increases after each iteration of the main **while** loop. This certainly ensures that the number of iterations within this loop is at most $O(|E|)$.

Consider now the particular case where $a(v) = b(v)$ for all $v \in V$. In the extreme case where $a = b = d$, d being the vertex degree function, a graph G contains an (a, b)-partition if and only if G contains at least two connected components. Thus, the (a, b)-Partition problem is polynomial-time solvable for $a = b = d$.

Motivated by certain binary codes, Graham [38] introduced the decomposition problem of 2-coloring the vertices of a graph in such a way that each vertex has at most one neighbor with a different color. Answering one of his questions, Chvátal in [22] proved the *NP*-hardness of this problem for graphs with minimum degree $\delta(G) = 3$ and maximum degree $\Delta(G) = 4$. This result implies the following assertion for our problem.

Theorem 18 ([22]) (a, b)-Partition *is NP-complete when* $a = b = d - 1$*, even for graphs with* $\delta(G) = 3$ *and* $\Delta(G) = 4$*.*

This means the *NP*-completeness of (a, b)-Partition for the entire range $\lfloor \frac{d}{2} \rfloor + 1 \leq a = b \leq d - 1$. Putting this fact together with Theorem 10 and with the positive result of Theorem 16 (*iv*), the separation between easy and hard instances of degree functions can be summarized as follows. The situation is illustrated in Figure 3.

Theorem 19 ([5]) *Consider input graphs* G *with* $\Delta(G) \geq 5$*. Then:*

(*i*) *The decision problem* (a, b)-Partition *for* $\lceil \frac{d}{2} \rceil \leq a = b \leq d - 1$ *is NP-complete.*

(*ii*) *The search version of* (a, b)-Partition *for* $a = b \leq \lceil \frac{d}{2} \rceil - 1$ *is polynomial-time solvable.*

(*iii*) *The search version of* (a, b)-Partition *for* $a = b = d$ *is linear-time solvable.*

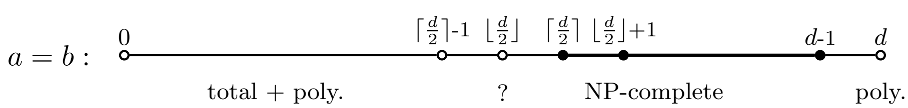

Figure 3: Complexity of (a, b)-Partition for $a = b$ vs. vertex degrees

As indicated earlier, the case of $\Delta(G) \leq 4$ is different, due to *NP*-completeness for $a = b = d - 1$ in Theorem 18 and polynomial-time solvability for $a = b = \lceil \frac{d}{2} \rceil$ in Theorem 7.

The case $a = b = d - 1$ has been studied independently under the name Matching Cutset (graphs admitting a matching cutset are sometimes called *decomposable*). Its *NP*-completeness has been re-proved in [56], and intractability remains valid on bipartite graphs where one class contains only vertices of degree 2 [54] or one vertex class is 3-regular and the other class is 4-regular [49], and also if the input graph is planar with maximum degree 4 or planar without cycles shorter than 5 [16]. On the other hand, the problem is solvable in polynomial time on graphs of maximum degree 3 [22], line graphs and graphs without induced cycles longer than 4 [54], series-parallel graphs [56], claw-free graphs and planar graphs without cycles shorter than 7 [16], graphs satisfying $d(u) + d(v) \leq 6$ for all edges (u, v) [49], graphs of diameter 2 [20] and graphs whose line graphs are planar [48].

Theorems 18 and 19 also have consequences on the complexity of determining the so-called 3-consecutive coloring number, introduced in [59] (a 3-consecutive coloring of a graph G is a vertex coloring such that in each path xyz of length two in G, at least one of x and z has the same color as the middle vertex y; the goal is to maximize the number of colors).

Weakening the inequality in the definition of co-satisfied vertex to the condition $d_{V_i}(v) \leq \lceil \frac{d(v)}{2} \rceil$, for vertices of odd degree there is more flexibility, and stronger results are valid. Recently, Borowiecki et al. [19] proved that every graph of maximum degree three admits a vertex partition (V_1, V_2) such that each V_i induces a subgraph of maximum degree two, and the edge set joining V_1 and V_2 contains no cycle; moreover, such a partition can be found in polynomial time. On the other hand, the analogous decision problem for 4-regular graphs, i.e. determining if a 4-regular graph admits a co-satisfactory partition (V_1, V_2) such that the edge set between V_1 and V_2 contains no cycle, is *NP*-complete. Hence there is substantial difference between even and odd degrees.

6.2 Partitions into more than two classes

6.2.1 (a, b)-partitions

Stiebitz has observed that Theorem 16 implies the following result by induction:

Corollary 20 ([66]) *Let G be a graph, and $f_1, \ldots, f_k : V \to \mathbb{N}$ be $k \geq 2$ functions. Assume that $d(v) \geq f_1(v) + \ldots + f_k(v) + k - 1$ for every vertex $v \in V$. Then there is a partition $(V_1, \ldots, V_k)$ of V into k nonempty subsets such that*

$$d_{V_i}(v) \geq f_i(v) \qquad \forall\ 1 \leq i \leq k, \quad \forall\ v \in V_i$$

A partition $(V_1, \ldots, V_k)$ of V into k nonempty subsets such that $d_{V_i}(v) \geq f_i(v)$ for all $1 \leq i \leq k$ and all $v \in V_i$ is called an *$(f_1, \ldots, f_k)$-partition.* Using this terminology, analogues of the results of the previous subsection can be formulated as follows:

Theorem 21 ([9]) *Consider a graph G, an integer $k = k(n) \geq 2$ (i.e., possibly depending on the number n of vertices), and k functions $f_1, \ldots, f_k : V \to \mathbb{N}$. An $(f_1, \ldots, f_k)$-partition of G exists and can be found in polynomial time under any of the following conditions:*

(i) G is unrestricted, $f_i : V \to \mathbb{N}$ $(i = 1, \ldots, k)$, and $d(v) \geq f_1(v) + \ldots + f_k(v) + k - 1$ for every $v \in V$.

(ii) G is triangle-free, $f_i : V \to \mathbb{N} \setminus \{0\}$ $(i = 1, \ldots, k)$, and $d(v) \geq f_1(v) + \ldots + f_k(v)$ for every $v \in V$.

(iii) G has girth at least five, $f_i : V \to \mathbb{N} \setminus \{0, 1\}$ $(i = 1, \ldots, k)$, and $d(v) \geq f_1(v) + \ldots + f_k(v) - k + 1$ for every $v \in V$.

6.2.2 (Co-)satisfactory partitions

The complexity of generalizations of Satisfactory Partition has also been studied, where a partition into k nonempty parts is requested, for $k \geq 3$. In this case, the condition for a vertex to be satisfied can be stated in several ways. The three following conditions were investigated, where we consider that a vertex is satisfied if :

- it has at least as many neighbors in its part as in all the other parts together,
- it has at least $1/k$ proportion of its neighbors in its own part,
- it has at least as many neighbors in its own part as in each of the other parts.

For $k = 2$ all these conditions boil down to the unique standard condition.

Formally, these three extensions can be stated as follows.

Sum Satisfactory k-Partition
Input: A graph $G = (V, E)$.
Question: Is there a partition $(V_1, \ldots, V_k)$ of V into k nonempty parts such that, for all $v \in V$, if $v \in V_i$ $(i = 1, \ldots, k)$ then $d_{V_i}(v) \geq \lceil \frac{d(v)}{2} \rceil$?

Average Satisfactory k-Partition
Input: A graph $G = (V, E)$.
Question: Is there a partition $(V_1, \ldots, V_k)$ of V into k nonempty parts such that, for all $v \in V$, if $v \in V_i$ $(i = 1, \ldots, k)$ then $d_{V_i}(v) \geq \lceil \frac{d(v)}{k} \rceil$?

Max Satisfactory k-Partition
Input: A graph $G = (V, E)$.
Question: Is there a partition $(V_1, \ldots, V_k)$ of V into k nonempty parts such that, for all $v \in V$, if $v \in V_i$ $(i = 1, \ldots, k)$ then $d_{V_i}(v) = \max_{1 \leq j \leq k} d_{V_j}(v)$?

The Sum and Max versions were introduced by Gerber and Kobler in [34]. They proved the strong *NP*-hardness of generalizations of these problems where there are weights on the vertices or edges, and they left as an open question the complexity of the unweigthed case.

The balanced versions of the three k-partition problems are obtained by imposing the further condition that all partition classes be of the same cardinality (or any two of them differ by at most 1 if $|V|$ is not a multiple of k). As it could be expected, all these problems are *NP*-complete for every value of k.

Proposition 22 ([7]) *The following problems are NP-complete for every* $k \geq 3$.

- Sum Satisfactory k-Partition
- Average Satisfactory k-Partition
- Max Satisfactory k-Partition
- Balanced Sum Satisfactory k-Partition
- Balanced Average Satisfactory k-Partition
- Balanced Max Satisfactory k-Partition

For *co-satisfactory* k-partitions, these problems are defined in a similar way, except for the third problem where one asks for a k-partition where each vertex is required to have at most as many neighbors in its own part as in each of the other parts. For the first three problems, the polynomial-time solvability is easily seen by taking any locally maximal k-cut, that is a partition into k vertex classes such that moving any single vertex to another class does not increase the number of edges that join distinct classes. More generally, the following analogue of Corollary 20 is valid.

Theorem 23 ([17, 47, 11]) *Let G be a graph, $k = k(n) \geq 2$ an integer, and $f_1, \ldots, f_k : V \to \mathbb{N}$ be k functions. If $d(v) < f_1(v) + \ldots + f_k(v)$ for every vertex $v \in V$, then there exists a vertex partition $(V_1, \ldots, V_k)$ into k nonempty parts such that*

$$d_{V_i}(v) < f_i(v) \qquad \forall\ 1 \leq i \leq k, \quad \forall\ v \in V_i$$

The version where each f_i is constant and $f_1 + \ldots + f_k > \Delta(G)$ holds, appeared first in [50]. Although it is not mentioned in the papers cited above, a k-partition satisfying the conditions of Theorem 23 can be determined in polynomial time. Moreover, it was proved in [18] that if G is a *connected* graph with $\Delta(G) = \Delta \geq 3$ and $G \neq K_{\Delta+1}$, then for any constants f_i with $f_1 + \ldots + f_k \geq \Delta$ there exists a partition $(V_1, \ldots, V_k)$ with $d_{V_i}(v) \leq f_i$ for all $1 \leq i \leq k$ and all $v \in V_i$, and each connected component of the subgraph induced by any V_i contains a vertex of degree smaller than f_i (the result for $f_1 = \ldots = f_k = \lceil \Delta(G)/k \rceil$ was proved already in [14]). Also, non-constant f_i were considered in [18].

Probably, the balanced versions of most of the above problems are algorithmically harder. But this is not always the case; e.g., every 3-regular graph admits a balanced average co-satisfactory 3-partition, which can be determined efficiently by switching iteratively vertices from different parts that are not co-satisfied.

7 Exact or approximate solutions on particular classes of graphs

In this section we first consider two classes of graphs that behave nicely concerning many problems that are *NP*-hard in general. Then, an application to planar graphs is presented. The published results deal with satisfactory (a, b)-partition (bisection) but the methods can be adjusted for the co-satisfactory case too.

7.1 Bounded tree-width or clique-width

Graphs of bounded tree-width or bounded clique-width are of great interest because they admit efficient solutions for fairly large problem classes.

The simplest definition for graphs of tree-width less than k is that they are subgraphs of chordal graphs with *clique number* at most k. For algorithmic purposes, however, an equivalent but more technical definition using the concept of 'tree decomposition' is usually more convenient. A *tree decomposition* of a graph $G = (V, E)$ is a tree $T = (X, F)$ with vertex set X and edge set F, together with subsets $H_x \subseteq V$ indexed with the vertices $x \in X$ of T. It is required that every edge $(u, v) \in E$ be contained in some H_x, and that for every $v \in V$, the vertices x with $v \in H_x$ induce a connected subgraph (i.e., subtree) in T. The host tree T is then a powerful tool to organize computation for various problems concerning G. The *width* of a tree decomposition is the largest $|H_x|$ minus 1, and G has tree-width at most k if and only if it admits a tree decomposition of width at most k.

On the other hand, graphs of clique-width at most k have a less transparent structure. These graphs are generated recursively in terms of three operations. For this purpose, vertex labels $\{1, \ldots, k\}$ are used. The one-vertex graph with any label between 1 and k has clique-width at most k. Then, by definition, a graph has clique-width at most k if it can be constructed from these singletons by vertex-disjoint union, the complete join of all vertices with label i to all vertices with label j (for an arbitrarily chosen pair of labels $1 \leq i \neq j \leq k$), and re-labeling all vertices of label i to label j.

The following important facts were proved by Courcelle *et al.* in [24] and [25], respectively:

- Every problem expressible in monadic second-order logic with quantifiers on vertex and edge subsets is solvable in linear time on graphs of bounded tree-width.
- Every problem expressible in monadic second-order logic with quantifiers on vertex subsets is solvable in linear time on graphs of bounded clique-width.

These results do not immediately apply to vertex partitions with general degree constraints, because an unrestricted type of functions a, b on V may need a monadic expression whose length increases with the order of the input graph. Nevertheless, the following results can be proved by applying methods different from those in monadic logic.

Theorem 24 ([8]) (a, b)-Partition *and* (a, b)-Bisection *are solvable in polynomial time on graphs G of bounded tree-width, for any integer-valued functions $a, b : V \to \mathbb{N}$.*

Theorem 25 ([36]) Satisfactory Partition *is solvable in polynomial time on graphs of bounded clique-width.*

In fact, the result of [36] is much more general than cited above. Although it cannot solve (a, b)-Partition for arbitrary a and b, it is applicable to a number of other important problems, including Independent Set, Graph k-Coloring for k fixed, Independent Dominating Set, and many more. Also, Theorem 24 is a particular case of a more general result of [8] that involves a condition on the cardinalities of partition classes.

7.2 Planar graphs

The polynomial-time solvability of (a, b)-Bisection on graphs of bounded tree-width can be applied to designing an approximation scheme on planar graphs for maximizing the number of satisfied vertices in bisections [8]. The idea is to decompose, for an increasing value of k, any planar input graph into subgraphs of tree-width less than k in several different ways, in order to maximize the number of satisfied vertices in bisections of each decomposition, and to show that the largest among these solutions is not very far from the optimum on the entire input graph.

This method dates back to the paper of Baker [4] and was further applied by many authors. A more general setting — applicable not only for PTAS but also for approximations of guaranteed performance ratio — is presented in [8].

8 Open problems

In this concluding section we list some of the problems that remain open.

(1) Characterize interesting subclasses of partitionable graphs.

(2) Suppose that a graph (or a class of graphs) F is given as 'forbidden' subgraph(s). How does the complexity of Satisfactory Partition and the other problems introduced above depend on F when the input is restricted to graphs not containing (any member of) F as a subgraph, or as an induced subgraph?

(3) Prove or disprove: For every k there is an $n = n(k)$ such that every graph with at least n vertices and maximum degree at most k is partitionable.

(It is known that $n(2) = 4$, $n(3) \leq 9$, and $n(4) \leq 13$.)

(4) If $n = n(k)$ in the previous question does not exist for k large, then determine the complexity of Satisfactory Partition on graphs of bounded degree.

(5) Can an (a, b)-partition be found in *linear* time if:

- $d(v) = a(v) + b(v) + 1$ for all $v \in V$, or
- $d(v) = a(v) + b(v)$ for all $v \in V$ and G is triangle-free, or
- $d(v) = a(v) + b(v) - 1$ for all $v \in V$ and G has girth at least 5 ?

(6) Are there some stronger versions of the existence theorems of Stiebitz, Kaneko, or Diwan valid if $a(v)$ is much smaller than $b(v)$ for every vertex v ?

(7) Determine the complexity of the (a, b)-Partition problem for

(i) $(a, b) = (\lfloor \frac{d}{2} \rfloor, \lfloor \frac{d}{2} \rfloor)$

(ii) $(a, b) = (\lfloor \frac{d}{2} \rfloor, \lceil \frac{d}{2} \rceil)$

(iii) $(a, b) = (\lceil \frac{d}{2} \rceil - 1, \lceil \frac{d}{2} \rceil)$

(From the results presented above, we know that the problem for $(a, b) = (\lceil \frac{d}{2} \rceil - 1, \lfloor \frac{d}{2} \rfloor)$ is solvable in polynomial time, while for $(a, b) = (\lceil \frac{d}{2} \rceil, \lceil \frac{d}{2} \rceil)$ it is *NP*-complete.)

(8) Determine the complexity of Co-Satisfactory (a, b)-Partition for $1 \leq a = b \leq \lfloor \frac{d}{2} \rfloor - 1$.

(9) For $d \geq 2$, determine the largest integer $k = k(d)$ such that every graph with minimum degree at least d and vertex connectivity at most k is partitionable.

(10) Design a polynomial-time $(3 - c)$-approximation for the maximum number of satisfied vertices in a bisection (for as large $c > 0$ as possible).

(11) Design a polynomial-time $(2 - c)$-approximation for the maximum number of co-satisfied vertices in a bisection (for as large $c > 0$ as possible).

(12) For every natural number k, determine the smallest (or infimum) constant c'_k for which there exists a constant c''_k such that every k-regular partitionable connected graph admits a satisfactory partition (V_1, V_2) with $|\,|V_1| - |V_2|\,| \leq c'_k n + c''_k$.

(13) Determine stronger (in)approximability results for the problems on minimum unbalance.

(14) Determine the threshold probability $p(n)$ such that the random graph $G_{n,p}$ of order n and edge probability p is almost surely partitionable if p is under $p(n)$, and is almost surely non-partitionable if p is above $p(n)$; and estimate the concentration of the probability of being (non-)partitionable around $p(n)$.

(It was claimed without proof in [35], just based on simulation results, that graphs of small resp. large edge density are partitionable resp. non-partitionable with high probability.)

(15) Determine the complexity of Balanced Sum / Average / Min Co-Satisfactory k-Partition for $k \geq 3$.

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