Approximation of satisfactory bisection problems ⋆

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Abstract

The Satisfactory Bisection problem means to decide whether a given graph has a partition of its vertex set into two parts of the same cardinality such that each vertex has at least as many neighbors in its part as in the other part. A related variant of this problem, called Co-Satisfactory Bisection, requires that each vertex has *at most* as many neighbors in its part as in the other part. A vertex satisfying the degree constraint above in a partition is called 'satisfied' or 'co-satisfied', respectively. After stating the *NP*-completeness of both problems, we study approximation results in two directions. We prove that maximizing the number of (co-)satisfied vertices in a bisection has no polynomial-time approximation scheme (unless $P = NP$), whereas constant approximation algorithms can be obtained in polynomial time. Moreover, minimizing the difference of the cardinalities of vertex classes in a bipartition that (co-)satisfies all vertices has no polynomial-time approximation scheme either.

Key words: graph, vertex partition, degree constraints, complexity, *NP*-complete, approximation algorithm.
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1 Introduction

Gerber and Kobler introduced in [6,7] the problem of deciding if a given graph has a vertex partition into two nonempty parts such that each vertex has at least as many neighbors in its part as in the other part. This problem, called Satisfactory Partition, is proved *NP*-complete in [2]. In a non-algorithmic context and under a weaker condition, the roots of the problem go back to the paper of Thomassen [12] where, in a graph of minimum degree $12k$, each vertex is required to have at least k neighbors in its partition class.

Satisfactory Bisection is the variant of Satisfactory Partition where the parts are required to have the same cardinality. A feasible solution of this problem is called a *satisfactory bisection*. Graphs like cycles of even length and complete bipartite graphs with both vertex classes of even size trivially admit a satisfactory bisection. A graph of even order formed by two connected components of unequal size, each of which has no satisfactory partition, is an example of a graph which admits a satisfactory partition but no satisfactory bisection. Satisfactory Bisection is proved *NP*-complete in [2].

We consider also the opposite problem of deciding if a given graph has a vertex partition — that we call 'co-satisfactory' — into two parts such that each vertex has at least as many neighbors in the other part as in its own part. This problem corresponds to finding in the graph a cut which is maximal with respect to moving a vertex from its part to the other. Therefore, a graph always admits such a partition that can be found in polynomial time. However, the balanced version of this problem, called Co-Satisfactory Bisection, does not always admit a solution, e.g. for stars of even order. We prove in this paper that Co-Satisfactory Bisection is *NP*-complete.

We also study the problem of minimizing the unbalance of a satisfactory and co-satisfactory partition, corresponding to the difference between the cardinalities of the two vertex classes in such a partition. We prove for both problems that there exists no polynomial-time approximation scheme unless $P = NP$. These results are motivated by the paper [11] of Sheehan who proved upper bounds on the minimum difference in k-regular graphs on n vertices, for $k \leq 8$. Let us note at this point that if one requires that the sum of the minimum degrees inside each part should be at least as large as the minimum degree of the graph, then the situation changes substantially: minimum degree plus one (or sometimes minimum degree) is a general upper bound on the minimum unbalance in such partitions [9,10], and such a partition can be found in polynomial time [1].

When a graph has no (co-)satisfactory bisection, it is natural to ask for a bisection maximizing the number of (co-)satisfied vertices. The corresponding optimization problems are Max Satisfying Bisection and Max Co-Satisfying Bisection. We prove in this paper that Max Satisfying Bisection is 1/3-approximable, Max Co-Satisfying Bisection is 1/2-approximable, and that these two problems have no polynomial-time approximation scheme unless *P=NP*.

Let us note that the maximization version of Satisfactory Partition is uninteresting, since every connected graph on $n \geq 3$ vertices admits a partition with one vertex (e.g., of minimum degree) in one part and $n-1$ in the other part, where all vertices in the part of size $n-1$ are satisfied. This yields a trivial linear-time approximation scheme.

The paper is structured as follows. Section 2 contains notation and definitions of problems. In Section 3 we show the *NP*-completeness of Co-Satisfactory Bisection. In Section 4 we prove that Max (Co-)Satisfying Bisection has no approximation scheme, unless *P=NP*, and in Section 6 we give constant approximation algorithms for these problems. In Section 5 the non-approximability of minimum unbalance is proved, for both satisfactory and co-satisfactory partitions.

2 Preliminaries

We begin with some basic definitions concerning approximation, and then we define the problems considered.

Approximability. Given an instance x of an optimization problem A and a feasible solution y of x, we denote by $\mathrm{val}(x, y)$ the value of solution y, and by $\mathrm{opt}_A(x)$ the value of an optimum solution of x. For a function $\rho < 1$, an algorithm is a *ρ-approximation* for a maximization problem A if for any instance x of the problem it returns a solution y such that $\mathrm{val}(x, y) \geq \rho(|x|) \cdot \mathrm{opt}_A(x)$. We say that a maximization problem is *constant approximable* if, for some constant $\rho < 1$, there exists a polynomial-time ρ-approximation for it. A maximization problem has a *polynomial-time approximation scheme* (a PTAS, for short) if, for every constant $\varepsilon > 0$, there exists a polynomial-time $(1-\varepsilon)$-approximation for it.

Reductions. ([8]) Let A and A' be two maximization problems. Then A is said to be *gap-preserving reducible* to A' with parameters $(c, \rho), (c', \rho')$ (where $\rho, \rho' \leq 1$), if there is a polynomial-time algorithm that transforms any instance x of A to an instance x' of A' such that the following properties hold:

(1) $\mathrm{opt}_A(x) \geq c \Rightarrow \mathrm{opt}_B(x') \geq c'$
(2) $\mathrm{opt}_A(x) < \rho \cdot c \Rightarrow \mathrm{opt}_B(x') < \rho' \cdot c'$

Gap-preserving reductions have the following property. If it is NP-hard to decide if the optimum of an instance of A is at least c or less than $\rho \cdot c$, then it is NP-hard to decide if the optimum of an instance of A' is at least c' or less than $\rho' \cdot c'$. This NP-hardness implies that A' is hard to ρ'-approximate.

Graphs. We consider finite, undirected graphs without loops and multiple edges. For a graph $G = (V, E)$, a vertex $v \in V$, and a subset $Y \subseteq V$ we denote by $d_Y(v)$ the number of vertices in Y that are adjacent to v; and, as usual, we write $d(v)$ for the degree $d_V(v)$ of v in V. A partition (V_1, V_2) of V where $|V_1| = |V_2|$ is called a *bisection*.

The problems we are interested in are defined as follows.

SATISFACTORY BISECTION
Input: A graph $G = (V, E)$ on an even number of vertices.
Question: Is there a bisection (V_1, V_2) of V such that for every $v \in V$, if $v \in V_i$ then $d_{V_i}(v) \geq \lceil \frac{d(v)}{2} \rceil, i = 1, 2$?

Given a partition (V_1, V_2) of V, we say that a vertex $v \in V_i$ is *satisfied* if $d_{V_i}(v) \geq \lceil \frac{d(v)}{2} \rceil$. A bisection where all vertices are satisfied is called a *satisfactory bisection*.

CO-SATISFACTORY BISECTION
Input: A graph $G = (V, E)$ on an even number of vertices.
Question: Is there a bisection (V_1, V_2) of V such that for every $v \in V$, if $v \in V_i$ then $d_{V_i}(v) \leq \lceil \frac{d(v)}{2} \rceil, i = 1, 2$?

Given a partition (V_1, V_2), a vertex $v \in V_i$ is *co-satisfied* if $d_{V_i}(v) \leq \lceil \frac{d(v)}{2} \rceil$. A bisection where all vertices are co-satisfied is called a *co-satisfactory bisection*.

When a graph does not admit a (co-)satisfactory bisection, it is natural to ask for a bisection that maximizes the number of vertices that are (co-)satisfied. Therefore, we consider the following problems.

MAX SATISFYING BISECTION
Input: A graph $G = (V, E)$ on an even number of vertices.
Output: A bisection (V_1, V_2) of V that maximizes the number of satisfied vertices.

MAX CO-SATISFYING BISECTION
Input: A graph $G = (V, E)$ on an even number of vertices.
Output: A bisection(V_1, V_2) of V that maximizes the number of co-satisfied vertices.

It is also natural to ask for a (co-)satisfactory partition that minimizes the difference between the cardinalities of the two parts of the partition. Therefore, we consider the following problems.

MIN UNBALANCE (CO-)SATISFYING PARTITION
Input: A graph $G = (V, E)$ on n vertices.
Output: A (co-)satisfactory partition (V_1, V_2) of V that minimizes

$$||V_1| - |V_2|| + 1 - (n \bmod 2)$$

The previous function to be minimized was chosen such that it has value 1 for a (co-)satisfactory bisection or a quasi-bisection in a graph of even or odd degree, respectively.

3 Complexity of (Co-)Satisfactory Bisection

In this section we first establish the *NP*-completeness of CO-SATISFACTORY BISECTION. Second, we present a short polynomial reduction from CO-SATISFACTORY BISECTION to SATISFACTORY BISECTION, showing the *NP*-completeness of the latter problem. This result was established already in [2] with a different proof, but the reduction given here is based on a construction that will be used in the next section to state a non-approximability result.

Theorem 1 CO-SATISFACTORY BISECTION *is NP-complete.*

Proof: Clearly, this problem is in *NP*. We construct a polynomial reduction from a variant of INDEPENDENT SET, the problem of deciding if a graph with n vertices contains an independent set of size at least $\frac{n}{2}$, a problem stated to be *NP*-hard in [5]. Let $G = (V, E)$ be a graph with n vertices $v_1, \ldots, v_n$ and m edges, an input of this variant of INDEPENDENT SET problem. We assume that n is even, since otherwise we can add a vertex that we link with all the vertices of the graph without changing the problem. We construct a graph $G' = (V', E')$, instance of CO-SATISFACTORY BISECTION as follows (see Figure 1). The vertex set V' consists of three sets: V, the vertex set of G, $F = \{f_1, \ldots, f_{2m+1}\}$ and $T = \{t_1, \ldots, t_{2m+1}\}$. The subgraph of G' induced on V is a copy of G. The subgraph of G' induced on $F \cup T$ is a complete bipartite graph. Finally, we link each vertex $v \in V$ to $d(v)$ distinct vertices from $F \setminus \{f_1\}$ (so that each vertex from $F \setminus \{f_1\}$ is adjacent to exactly one vertex in V).

This construction is accomplished in polynomial time. All that remains to show is that G has an independent set of size at least $\frac{n}{2}$ if and only if G' admits a co-satisfactory bisection.

Suppose first that G has an independent set S of size $\frac{n}{2}$. Let $V'_1 = F \cup S$ and $V'_2 = T \cup \bar{S}$, where $\bar{S} = V \setminus S$. Let us check in the following that (V'_1, V'_2) is

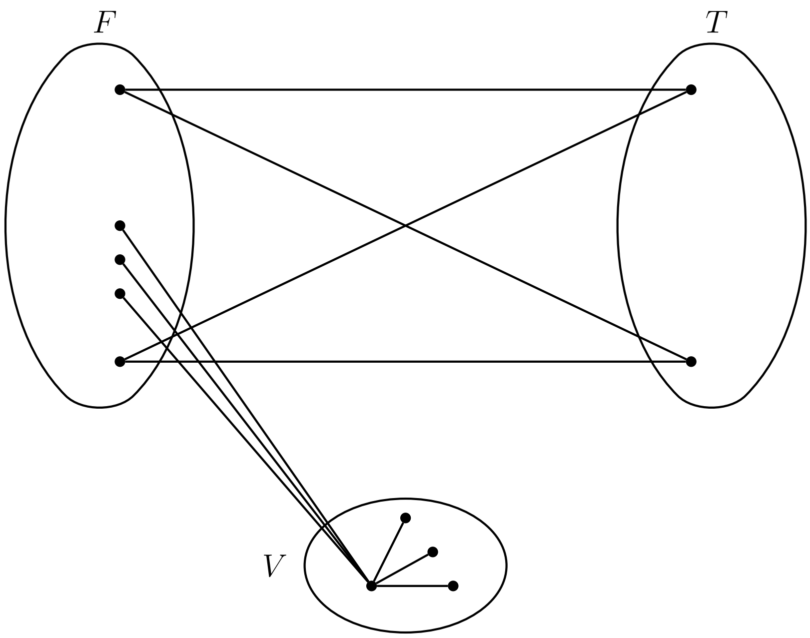

Fig. 1. Graph G' obtained from G

a co-satisfactory bisection. It is easy to see that all vertices of F and T are co-satisfied. Let $v \in S$. Since S is an independent set, v is not linked to any vertex in S. Thus, $d_{V_1'}(v) = d_{\bar{S}}(v) = d_{V_2'}(v)$ and so the vertices of S are co-satisfied. Given a vertex $v \in \bar{S}$, $d_{V_1'}(v) = 2d_S(v) + d_{\bar{S}}(v)$ while $d_{V_2'}(v) = d_{\bar{S}}(v)$, thus also the vertices of $\bar{S}$ are co-satisfied in G'.

Suppose now that G' admits a co-satisfactory bisection and let (V_1', V_2') be a co-satisfactory bisection. Observe that F and T cannot be both included in the same part of the partition since otherwise the vertices of F and T are not co-satisfied. If the partition cuts only one of the two sets F or T, suppose for example that F is cut, then the vertices of F that are in the same part of the partition as T are not co-satisfied. If the partition cuts both F and T, denote by F_1, T_1 and F_2, T_2 the sets of vertices of F and T that are included in V_1' and V_2' respectively. For vertices of T_1 to be co-satisfied, we first have $|F_1| \leq |F_2|$ whereas for vertices of T_2 to be co-satisfied, we must have $|F_2| \leq |F_1|$, that is $|F_1| = |F_2|$, which is impossible since $|F|$ is odd. Therefore, F and T are included in different parts of the partition and thus (V_1', V_2') cuts the set V into two balanced sets V_1, V_2, where $V_1' = F \cup V_1$ and $V_2' = T \cup V_2$. We show that V_1 is an independent set. A vertex $v \in V_1$ has $d_{V_1'}(v) = 2d_{V_1}(v) + d_{V_2}(v)$ and $d_{V_2'}(v) = d_{V_2}(v)$. Since v is co-satisfied in G' we have $d_{V_1'}(v) \leq d_{V_2'}(v)$ and we obtain that $d_{V_1}(v) = 0$. Thus V_1 is an independent set of size $\frac{n}{2}$. □

Theorem 2 SATISFACTORY BISECTION *is NP-complete.*

Proof: Clearly, this problem is in *NP*. We reduce CO-SATISFACTORY BISECTION to SATISFACTORY BISECTION which shows the *NP*-completeness of the latter problem by Theorem 1.

The reduction is as follows. Let G be a graph, instance of CO-SATISFACTORY BISECTION on n vertices $v_1, \dots, v_n$. The graph G', instance of SATISFACTORY BISECTION, has $2n$ vertices $v_1, \dots, v_n$, and $u_1, \dots, u_n$. G' is the complement of graph G on vertices $v_1, \dots, v_n$, and we add pendant edges (u_i, v_i), $i = 1, \dots, n$. If G admits a co-satisfactory bisection and (V_1, V_2) is such a partition, then $V_i' = V_i \cup \{u_j : v_j \in V_i\}$ is a satisfactory bisection for G'. Indeed, if $v_i \in V_1$ then $d_{V_1}(v_i) \leq d_{V_2}(v_i)$ in G. Thus, in G' we have $d_{V_1'}(v_i) = \frac{n}{2} - 1 - d_{V_1}(v_i) + 1 \geq \frac{n}{2} - d_{V_2}(v_i) = d_{V_2'}(v_i)$ and $d_{V_1'}(u_i) = 1 > d_{V_2'}(u_i) = 0$. Conversely, since in each satisfactory bisection of G', u_i is in the same set as v_i, such a partition of G' gives a co-satisfactory bisection in G. □

4 No PTAS for Max (Co-)Satisfying Bisection

In this section we prove that MAX CO-SATISFYING BISECTION and MAX SATISFYING BISECTION have no polynomial-time approximation scheme unless *P=NP*.

We introduce first a problem that we will use in the reductions.

MAX k-VERTEX COVER-B
Input: A graph $G = (V, E)$ with $|V| \geq k$ and maximum degree B.
Output: The maximum number of edges in G that can be covered by a subset $V' \subseteq V$ of cardinality k.

Theorem 3 (Petrank [8]) *There exists a constant α, $0 < \alpha < 1$ with the following property: given a graph G with n vertices and m edges, instance of* MAX k-VERTEX COVER-B *for some $k = \Theta(n)$, it is NP-hard to distinguish, whether it has opt$(G) = m$ or opt$(G) < (1 - \alpha)m$.*

Though it is not explicitly mentioned in [8], the proof of Theorem 3 yields the same conclusion for the restricted class of graphs with $m \geq \frac{n}{2}$. We prove next that Theorem 3 holds in particular for $k = \frac{n}{2}$.

Theorem 4 *There exists a constant β, $0 < \beta < 1$ with the following property: given a graph G with N vertices and M edges, instance of* MAX $\frac{N}{2}$-VERTEX COVER-B$'$, *it is NP-hard to distinguish whether it has opt$(G) = M$ or opt$(G) < (1 - \beta)M$.*

Proof: We construct a gap-preserving reduction from MAX k-VERTEX COVER-B with $k = cn$, for some constant $c < 1$, to MAX $\frac{N}{2}$-VERTEX COVER-

(2B+2). Let $G = (V, E)$ be a graph on n vertices and $m \geq \frac{n}{2}$ edges, instance of MAX k-VERTEX COVER-B. We will construct a graph G'' with N vertices and M edges such that if $\mathrm{opt}(G) = m$ then $\mathrm{opt}(G'') = M$ and if $\mathrm{opt}(G) \leq (1-\alpha)m$, for some $\alpha > 0$, then $\mathrm{opt}(G'') \leq (1-\beta)M$, for some $\beta > 0$.

First assume that $c > 1/2$. Let G'' be the graph obtained from G by inserting $2k - n$ isolated vertices. In this case, the properties of the gap-preserving reduction hold with $\beta = \alpha$.

Consider now the case $c < 1/2$. Suppose first that $n - 2k$ is a multiple of $B+1$. Let G'' be the graph that consists of a copy of G and $\frac{n-2k}{B+1}$ copies of the graph T_{B+1} which is the complete tripartite graph whose vertex classes have cardinality $B+1$ each. Observe that T_{B+1} needs $2B+2$ vertices in covering its edges (the complement of a vertex class), and if just $2B+2-t$ vertices are taken, then at least $t(B+1)$ edges remain uncovered. Thus, since G has maximum degree at most B, each subset of $\frac{N}{2}$ vertices not covering all copies of T_{B+1} is trivially improvable. Suppose first that $\mathrm{opt}(G) = m$ and let V' be a vertex cover of size k in G. Then the set V' and the vertices of two among the 3 independent sets of each of the $\frac{n-2k}{B+1}$ copies of T_{B+1} form a vertex cover of G'' of size $\frac{N}{2}$, and thus $\mathrm{opt}(G'') = M$. On the other hand, suppose $\mathrm{opt}(G) < (1-\alpha)m$. Then since $M = m + 3(B+1)(n-2k)$ and $m \geq \frac{n}{2}$, the number of edges not covered in G'' is at least $\alpha m \geq \frac{\alpha M}{1+6(B+1)(1-2c)}$ that can be viewed as βM.

Finally, if $c < 1/2$ and if $n - 2k = \ell \; mod(B+1)$, $0 < \ell \leq B$, then let G' be the graph G together with further $B+1-\ell$ isolated vertices. Now, we can transform G' to G'' as before by inserting $\frac{n-2k-\ell}{B+1} + 1$ copies of T_{B+1}. In this case we get a slightly different value for β, as the number m of edges is now compared with the modified number $n + B + 1 - \ell$ of vertices. Nevertheless, $\beta > 0$ is obtained. □

From this theorem, the following two non-approximability results can be deduced.

Theorem 5 MAX CO-SATISFYING BISECTION *has no polynomial-time approximation scheme unless P=NP.*

Proof: We construct a gap-preserving reduction between MAX $\frac{n}{2}$-VERTEX COVER-B and MAX CO-SATISFYING BISECTION. Let G be a graph instance of MAX $\frac{n}{2}$-VERTEX COVER-B on n vertices and m edges. We construct the graph G' as in the proof of Theorem 1. Denote by N the number of vertices of G'. We have $N = \Theta(m)$ and $m = \Theta(n)$, by the bounded-degree condition on G and because we may assume for MAX $\frac{n}{2}$-VERTEX COVER-B that G has fewer than $\frac{n}{2}$ isolated vertices (otherwise a vertex cover of cardinality $\frac{n}{2}$ can be found in linear time).

Suppose first that $\text{opt}(G) = m$, and let V' be a vertex cover of size $\frac{n}{2}$ of G. Then in the partition $(F \cup (V \setminus V'), T \cup V')$ all vertices are co-satisfied and thus $\text{opt}(G') = N$.

Suppose next that $\text{opt}(G) < (1 - \beta)m$. It means that, for any set V' of $\frac{n}{2}$ vertices, at least βm edges of G remain uncovered. The number of vertices incident to at least one non-covered edge is at least $\frac{2\beta m}{B}$. Hence, for some constant $c > 0$, in any bipartition of V, either the partition classes differ by at least cm in size, or each class contains at least cm vertices having at least one neighbor in the same class. Let now $(F_1 \cup V_1 \cup T_1, F_2 \cup V_2 \cup T_2)$ be any bisection. Suppose for a contradiction that as many as $N - o(m)$ vertices (that equivalently means $N - o(N)$) are co-satisfied. We may assume $|F_1| > |F_2|$ without loss of generality. Then $|T_1| = o(m)$ must hold, because no vertex of T_1 is co-satisfied. Since every $f \in F$ has the majority of its neighbors in T_2, we obtain $|F_2| = o(m)$ in a similar way. Consequently, $||V_1| - |V_2|| = o(m)$ is valid, as we started with a bisection. Thus, for some constant $c > 0$, there exist at least cm vertices in V_1 with some neighbor in the same class, V_1. To co-satisfy all but $o(m)$ of those vertices, we would need that all but $o(m)$ of them have neighbors in F_2. This would require $|F_2| \geq cm$ because each $f \in F$ has just one neighbor in V. This contradiction completes the proof. □

Theorem 6 Max Satisfying Bisection *has no polynomial-time approximation scheme unless P=NP.*

Proof: Consider the graph G' with N vertices and M edges obtained in the construction given in the proof of Theorem 5, and apply to G' the reduction given in Theorem 2. Let G'' be the graph obtained. It can be shown that if $\text{opt}(G') = N$ then $\text{opt}(G'') = 2N$ and if $\text{opt}(G') < (1 - \gamma)N$ then $\text{opt}(G'') < 2N(1 - c\gamma)$ for some constant c. □

5 No PTAS for Min Unbalance (Co-)Satisfying Partition

In this section we prove that Min Unbalance Co-Satisfying Partition and Min Unbalance Satisfying Partition have no polynomial-time approximation scheme unless *P=NP*.

Theorem 7 Min Unbalance Co-Satisfying Partition *has no polynomial-time approximation scheme unless P=NP.*

Proof: We construct a gap-preserving reduction between Max $\frac{n}{2}$-Vertex Cover-B and Min Unbalance Co-Satisfying Partition. Let G be a graph instance of Max $\frac{n}{2}$-Vertex Cover-B on n vertices and m edges. We construct the graph G' as in the proof of Theorem 1. Denote by N the number of vertices of G'. We have $N = \Theta(m)$ and $m = \Theta(n)$, by the bounded-degree condition on G and because we may assume for Max $\frac{n}{2}$-Vertex Cover-B that G has fewer than $\frac{n}{2}$ isolated vertices (otherwise a vertex cover of cardinality $\frac{n}{2}$ can be found in linear time).

Suppose first that $\text{opt}(G) = m$, and let V' be a vertex cover of size $\frac{n}{2}$ of G. Then in the bisection $(F \cup (V \setminus V'), T \cup V')$ all vertices are co-satisfied and thus $\text{opt}(G') = 1$.

Suppose next that $\text{opt}(G) < (1-\beta)m$. Assume that (V_1, V_2) satisfies all vertices. Since F (the neighbors of the vertices of G) has odd cardinality, more than $|F|/2$ vertices belong to the same class; we assume $|V_1| > |F|/2$ without loss of generality. Since T is completely joined with F, the only way to satisfy its vertices is to have $T \subseteq V_2$. But then, since each vertex of F has just one (or zero, for the specified vertex) neighbor in G, the F-vertices are satisfied only if $F \subseteq V_1$. Assume now that it is possible to generate a satisfactory partition by an efficient algorithm, such that the unbalance is as small as $o(n)$. Since $|F|$ and $|T|$ are as large as cn for some constant $c > 0$, moving as few as $o(n)$ vertices from one partition class to the other keeps the satisfied status of the F, T-vertices unchanged. Moreover, since each vertex of the large graph has only a bounded number of neighbors in G (T-vertices have none, F-vertices have at most 1, G-vertices have a bounded number by assumption), moving $o(n)$ vertices we can make only $o(n)$ vertices unsatisfied. Thus, from a $o(n)$-unbalanced partition we would derive a bisection with just $o(n)$ non-satisfied vertices in polynomial time, a contradiction. □

Theorem 8 Min Unbalance Satisfying Partition *has no polynomial-time approximation scheme unless P=NP.*

Proof: We construct a gap-preserving reduction between Min Unbalance Co-Satisfying Partition and Min Unbalance Satisfying Partition. Consider the graph G' with N vertices and M edges obtained in the construction given in the proof of Theorem 5, instance of Min Unbalance Co-Satisfying Partition, and apply to G' the reduction given in Theorem 2. Let G'' be the graph obtained. Since the two endpoints u_i, v_i of each pendant edge belong to the same vertex class in any satisfying partition of G'', we obtain that if $\text{opt}(G') = 1$ then $\text{opt}(G'') = 1$ and if $\text{opt}(G') > c$ then $\text{opt}(G'') > 2c$. □

6 Constant approximations for Max (Co-)Satisfying Bisection

We concentrate mostly on the approximation of Max Satisfying Bisection. The co-satisfying version turns out to be simpler, and will be considered at the end of the section.

We first consider graphs $G = (V, E)$ with an odd number of vertices. For such graphs, a partition (V_1, V_2) of V where $|V_1|$ and $|V_2|$ differ just by 1 is called a *quasi-bisection.*

Proposition 9 (*i*) *Any graph G with an odd number n of vertices has a quasi-bisection such that each vertex in the part of size $\frac{n+1}{2}$ is satisfied.*

(*ii*) *Any graph G with an even number n of vertices has either a bisection with all vertices satisfied, or a vertex bipartition with size distribution $(\frac{n}{2}+1, \frac{n}{2}-1)$ such that each vertex in the part of size $\frac{n}{2}+1$ is satisfied.*

(*iii*) *Such a partition can be found in polynomial time for both the odd and even case.*

Proof: For n odd, let (V_1, V_2) be a quasi-bisection of G with $|V_1| > |V_2|$. If V_1 contains a vertex v that is not satisfied, then $d_{V_1}(v) < d_{V_2}(v)$ and thus by moving v from V_1 to V_2 we obtain a quasi-bisection with a smaller value of the cut induced by (V_1, V_2). Re-labeling V_1 and V_2 to keep the inequality $|V_1| > |V_2|$ valid, the algorithm repeats this step while the largest set contains a non-satisfied vertex. Similarly, for n even, we can maintain a vertex partition (V_1, V_2) such that $\frac{n}{2} - 1 \leq |V_2| \leq |V_1| \leq \frac{n}{2} + 1$; in a bisection each part can be viewed as V_1. If some $v \in V_1$ is not satisfied, then moving v into the other class decreases the value of the cut and keeps $|V_1|, |V_2|$ in the given range. Hence, after at most $|E|$ steps we obtain a vertex partition that satisfies the requirements. □

Remark: For some graphs, such as K_n with n odd, it may happen that only the vertices in the larger part are satisfied.

We consider now graphs of even order. Given a graph on an even number n of vertices, a vertex of degree $n - 1$ is never satisfied in a bisection since it has only $\frac{n}{2} - 1$ neighbors in its own part and $\frac{n}{2}$ neighbors in the other part.

Theorem 10 *Any graph G with an even number n of vertices has a bisection satisfying at least $\lceil \frac{n-t}{3} \rceil$ vertices, where t is the number of vertices of degree $n - 1$ in G. Such a partition can be found in polynomial time.*

Proof: If G is disconnected, let us denote by $n_1, \ldots, n_k$ the numbers of vertices in its connected components $G_1, \ldots, G_k$ $(k \geq 2)$. In each G_i we find a vertex bipartition on applying Proposition 9. If there are at least two odd components, then the partitions of the components are easily combined to a bisection of G that satisfies more than $\frac{n}{2}$ vertices. For example, if n_1, n_2 are odd, n_3 is even, and G_3 is split unequally, then the equation $\frac{n_1+1}{2} + \frac{n_2+1}{2} + \frac{n_3}{2} - 1 = \frac{n_1-1}{2} + \frac{n_2-1}{2} + \frac{n_3}{2} + 1$ corresponds to such a bisection of $G_1 \cup G_2 \cup G_3$. Since pairs of unequally split components of the same parity can always be combined to a bisection of their union, the only case to be considered is when all components are even and the number of components G_i partitioned into unequal sizes $(\frac{n_i}{2}+1, \frac{n_i}{2}-1)$ is odd.

Assume that all components are even and the first three of them are split unequally. Say, $n_1 \geq n_2 \geq n_3$. Then we can combine the partitions of G_1 and G_2 to form a bisection of $G_1 \cup G_2$, and take an arbitrary bisection of G_3. This satisfies at least $\frac{n_1}{2}+\frac{n_2}{2}+2 > \frac{n_1+n_2+n_3}{3}$ vertices in $G_1 \cup G_2 \cup G_3$. Then the partitions of the remaining classes can be combined to a bisection of their union that satisfies more than half of their vertices. Altogether, more than $\frac{n}{3}$ vertices of G get satisfied. Finally, assume that G_1 is split unequally and all the other components are bisected. Then an unequal split of G_2 is obtained by moving any one vertex to the other class of its bisection. This partition of G_2 still satisfies at least $\frac{n_2}{2}$ vertices. This yields a bisection on $G_1 \cup G_2$ with more than $\frac{n_1+n_2}{2}$ satisfied vertices.

Suppose from now on that G is connected, and let H be the complement of G. We denote by $H_1, \ldots, H_q$ the connected components of H, and by n_i the number of vertices in H_i for $i = 1, \ldots, q$. Observe that if a vertex is of degree $n-1$ in G then it forms alone a connected component in H. Moreover, any two connected components of H are completely joined in G, therefore if $H = H' \cup H''$ and no connected component H_i meets both H' and H'', then in the union of bisections of H' and H'' the number of satisfied vertices in G is the sum of that in the induced subgraphs $G[V(H')]$ and $G[V(H'')]$.

Consider now a nontrivial connected component H_i, with $n_i > 1$. We will show that a (quasi-)bisection (V_1', V_2') of $V(H_i)$ can be constructed in which at least $\lceil \frac{n_i}{3} \rceil$ vertices get satisfied in any bisection of G that extends (V_1', V_2'). (In some cases, the quasi-bisection found in Proposition 9 does not work for the present purpose.) Since $\sum_{n_i>1} \lceil \frac{n_i}{3} \rceil \geq \lceil \frac{n}{3} \rceil$, the theorem will follow.

Let $M = \{(a_1, b_1), \ldots, (a_p, b_p)\}$ be a maximum matching in H_i. It can be found efficiently, using e.g. Edmonds' algorithm [3]. We distinguish two cases.

If $|M| \geq \lceil \frac{n_i}{3} \rceil$ then consider an arbitrary (quasi-)bisection (V_1'', V_2'') of $V(H_i) \setminus V(M)$. Let (V_1', V_2') be the partition of $V(H_i)$ obtained from this one by adding vertices a_j to V_1'' and vertices b_j to V_2''. While there exists a pair (a_j, b_j) where both vertices are non-satisfied in G, we exchange these two vertices. Since a_j and b_j are not linked in G, this exchange makes both a_j and b_j satisfied and decreases the value of the cut in G by at least 2. Therefore, after at most $\frac{|E|}{2}$ exchanges, we obtain a (quasi-)bisection with at least $\lceil \frac{n_i}{3} \rceil$ vertices satisfied (at least one vertex in each pair (a_j, b_j)).

If $|M| < \lceil \frac{n_i}{3} \rceil$ then using Gallai's decomposition theorem [4] we can obtain in polynomial time a vertex set S such that $2|M| = n_i - \ell + |S|$, where ℓ is the number of odd connected components of $H_i - S$. In fact, a vertex is in S if it is contained in every maximum matching of H_i and has at least one neighbor x such that some maximum matching avoids x. Algorithmically, first the set of those x can be identified and then their neighbors are found efficiently.

Let $O_1, \ldots, O_\ell$ be the odd connected components of $H_i - S$. From the conditions $\frac{n_i - \ell + |S|}{2} = |M| \leq \lceil \frac{n_i}{3} \rceil - 1$ and $\ell + |S| \leq n_i$ we see that $\ell - |S| \geq \lceil \frac{n_i}{3} \rceil$, $\ell \geq \lceil \frac{n_i}{3} \rceil$, and $|S| \leq \lceil \frac{n_i}{3} \rceil$. Let us select a vertex $v_j \in O_j$ adjacent to at least one vertex of S, for each $j = 1, \ldots, \ell$. Those v_j are mutually adjacent in G.

If $\ell \geq \lceil \frac{n_i}{2} \rceil$ then we consider the following (quasi-)bisection (V_1', V_2'): V_1' contains $\lceil \frac{n_i}{2} \rceil$ vertices from $v_1, \ldots, v_\ell$ and V_2' contains the other vertices. Observe that at least $\lceil \frac{n_i}{2} \rceil$ vertices are satisfied in G, since for $v_j \in V_1'$ we have $d_{V_1'}(v_j) = \lceil \frac{n_i}{2} \rceil - 1$ and $d_{V_2'}(v_j) \leq \lfloor \frac{n_i}{2} \rfloor - 1$.

Suppose next that $\lceil \frac{n_i}{3} \rceil \leq \ell \leq \lceil \frac{n_i}{2} \rceil$. In this case we shall find a (quasi-)bisection (V_1', V_2') of H_i such that all of $v_1, \ldots, v_\ell$ are satisfied in every bisection (V_1, V_2) of G that extends (V_1', V_2').

We begin with putting into V_1' the vertices $v_1, \ldots, v_\ell$, and into V_2' the entire S together with $\ell - |S|$ or $\ell - |S| - 1$ further vertices of H_i, depending on whether n_i is even or odd. The number of remaining vertices of H_i is even in either case. So far, $v_1, \ldots, v_\ell$ are satisfied inside the partial partition of G because they are mutually adjacent in G and each of them has at least one non-neighbor inside S in G.

While selecting the $\ell - |S|$ or $\ell - |S| - 1$ additional vertices for S, we require that from each odd component O_j an even number of vertices be taken. Hence, the remaining part of each O_j will admit a bisection, what we shall use in extending (V_1', V_2') to a bisection (V_1, V_2) of the entire G. For this, we consider $V(O_j) \setminus (\{v_j\} \cup V_2')$ for each odd connected component O_j and put half of these vertices into V_1' and half into V_2' in such a way that v_j remains satisfied. The partition in O_j does not influence the satisfied status of v_s for any $s \neq j$, therefore it can be done independently in all O_j. We complete this partition by putting half of the remaining vertices (of the even components) into V_1 and half into V_2. □

From Theorem 10 we easily obtain constant-rate approximability as follows.

Theorem 11 MAX SATISFYING BISECTION *is 1/3-approximable.*

Proof: Given a graph on n vertices, the maximum number of vertices that are satisfied in a bisection is $\mathrm{opt}(G) \leq n - t$, where t is the number of vertices of degree $n-1$. Using Theorem 10 we can obtain in polynomial time a bisection where the number of satisfied vertices is $\mathrm{val} \geq \lceil \frac{n-t}{3} \rceil \geq \frac{\mathrm{opt}(G)}{3}$. □

Remark: There is an infinite sequence of graphs for which exactly $\frac{n}{3}$ vertices can be satisfied in a bisection. Such graphs contain $6n$ vertices $V = \{u_i, v_i, w_i : i = 1, \ldots, 2n\}$. Each vertex u_i, v_i, w_i is completely linked with $V \setminus \{u_i, v_i, w_i\}$. These graphs are $(6n-3)$-regular and thus a vertex in order to be satisfied must be in the same part with at least $3n-1$ neighbors. Thus, from each triple u_i, v_i, w_i at most one could be satisfied in a bisection. It is easy to verify that in the partition with $V_1 = \{u_i : i = 1, \ldots, n\} \cup \{v_i, w_i : i = n+1, \ldots, 2n\}$ and $V_2 = V \setminus V_1$, just the vertices u_i, $i = 1, \ldots, 2n$ are satisfied.

As shown by the last theorem, co-satisfying bisections are much easier to approximate.

Theorem 12 MAX CO-SATISFYING BISECTION *is 1/2-approximable.*

Proof: Let (V_1, V_2) be a bisection of G. While there exist $v_1 \in V_1$ and $v_2 \in V_2$ that are not co-satisfied, we exchange v_1 and v_2. After this exchange the value of the cut increases by at least 2. Thus, after $\frac{|E|}{2}$ steps we obtain a bisection where at least one of the two parts contains co-satisfied vertices only. □

Remark: The lower bound $n/2$ given in the proof of Theorem 12 is nearly tight, since the star of order $n = 2t$ has just $t+1$ co-satisfied vertices in every bisection.

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References

[1] E. M. Arkin and R. Hassin, *Graph partitions with minimum degree constraints*, Discrete Mathematics 190 (1998), 55–65.

[2] C. Bazgan, Zs. Tuza and D. Vanderpooten, *The satisfactory partition problem*, Discrete Applied Mathematics 154 (2006), 1236–1245.

[3] J. Edmonds, *Paths, trees, and flowers*, Canadian Journal of Mathematics 17 (1965), 449–467.

[4] T. Gallai, *Maximale Systeme unabhängiger Kanten*, Magyar Tud. Akad. Mat. Kut. Int. Közl. 9 (1964), 401–413.

[5] M. R. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W. H. Freeman and Co., San Francisco, 1979.

[6] M. Gerber and D. Kobler, *Partitioning a graph to satisfy all vertices*, Technical report, Swiss Federal Institute of Technology, Lausanne, 1998.

[7] M. Gerber and D. Kobler, *Algorithmic approach to the satisfactory graph partitioning problem*, European Journal of Operational Research 125 (2000), 283–291.

[8] E. Petrank, *The hardness of approximation: gap location*, Computational Complexity 4 (1994), 133–157.

[9] J. Sheehan, *Graph decompositions with constraints on the minimum degree*, Discrete Mathematics 68 (1988), 299–307.

[10] J. Sheehan, *Balanced graphs with minimum degree constraints*, Discrete Mathematics 102 (1992), 307–314.

[11] J. Sheehan, *Graphical decompositions*, Discrete Mathematics 125 (1994), 347–355.

[12] C. Thomassen, *Graph decomposition with constraints on the connectivity and minimum degree*, Journal of Graph Theory 7 (1983), 165–167.