On the vertex-distinguishing proper edge-colorings of graphs

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Abstract

We prove the conjecture of Burris and Schelp: a coloring of the edges of a graph of order n such that a vertex is not incident with two edges of the same color and any two vertices are incident with different sets of colors is possible using at most $n+1$ colors.

1 Introduction

In this paper we consider only undirected and simple graphs and we use the standard notation of graph theory (see [3]). Let $G = (V, E)$ be a graph with n vertices with the set of vertices V and the edge set E. We denote by $V_d(G)$ the set of vertices of degree d in G and $n_d(G) = |V_d(G)|$.

The problem in which we are interested in this paper is a particular case of the great variety of different ways of labeling a graph. The original motivation of studying this problem came from irregular networks. The idea was to weight the edges by positive integers such that the sum of the weights of edges incident to each vertex formed a set of distinct numbers. Consider a function $f : E \to \{1, \ldots, m\}$. Let $f(e)$ be the number associated to the edge e. Denote by $F(v) = \{f(e) \mid e = uv \in E\}$ the multi-set of numbers assigned to the set of edges incident to v and by $f(v) = \sum_{e \in F(v)} f(e)$. We call a function f *admissible* if the function f gives distinct values to the vertices of G. The minimum number m such that an admissible function exists for a graph G (introduced in [7]) is denoted $s(G)$ and is called the *irregularity strength* of G. In [7], an upper bound and a lower bound $s(G)$ are given and a lower bound for the irregularity strength of trees is found. They also computed the irregularity strength for paths and cycles and for others special graphs (see [11] for a survey concerning this number).

The problem that we study in this paper is a refinement of the coloring problem where the numbers associated to the edges in the above function are replaced by colors. An *edge-coloring* f of a graph G is an assignment of colors to the edges of G. A coloring f is called *vertex-distinguishing* if $F(u) \neq F(v)$ for any two vertices $u \neq v$. The minimum number of colors necessary for a vertex-distinguishing edge-coloring of a graph G (introduced in [1]) is denoted $c(G)$. In [1] and [2] the authors computed this number for some special graphs and respectively investigated the asymptotic growth of this number for k-regular graphs.

The coloring f is *proper* if no two adjacent edges have the same color. In the view of coloring, any useful constraint on a proper coloring is interesting to study. The coloring f is *vertex-distinguishing proper edge-coloring* (abbreviated VDP coloring) if it is proper and vertex-distinguishing.

The *vertex-distinguishing proper edge-coloring number* $\tilde{\chi}'(G)$ of a graph G without isolated edges and with at most one isolated vertex is the minimum number of colors required to find a VDP coloring of G. The VDP coloring number was introduced and studied by Burris and Schelp in [4] and [5] and, independently, as "observability" of a graph, by Černý, Horňák and Soták in [6]. In [4], [6], [10] and [9] the VDP coloring number is also computed for some families of graphs, such as paths P_n, cycles C_n, bipartite complete graphs $K_{m,n}$, complete graphs K_n:

$$\tilde{\chi}'(P_n) = \min\{2\lceil \tfrac{\sqrt{8n-7}-1}{4} \rceil + 1 \ , \ 2\lceil \tfrac{\sqrt{2n-5}+1}{2} \rceil\}, \quad \text{for } n \geq 3 \ ,$$

$$\tilde{\chi}'(K_{m,n}) = \begin{cases} n+1 & \text{if } n > m \geq 2 \\ n+2 & \text{for } m = n \geq 2 \end{cases} ,$$

$$\tilde{\chi}'(K_n) = \begin{cases} n & \text{if } n \text{ is odd} \\ n+1 & \text{if } n \text{ is even} \end{cases} .$$

For $k, n \geq 3$, $\tilde{\chi}'(C_n) = k$ if and only if either

1. k is odd and $n \in \left[\frac{k^2-4k+5}{2} , \frac{k^2-k-6}{2} \right] \bigcup \left\{ \frac{k^2-k}{2} \right\}$ or
2. k is even and $n \in \left[\frac{k^2-3k-2}{2} , \frac{k^2-3k}{2} \right] \bigcup \left[\frac{k^2-3k+4}{2} , \frac{k^2-2k}{2} \right]$.

Among the graphs G for which we know the value $\tilde{\chi}'(G)$, the largest value $\tilde{\chi}'(G)$ is realized when $G = K_n$ with n even.

Burris and Schelp conjectured in [4] and [5] that a graph G of order n, without isolated edges and with at most one isolated vertex has $\tilde{\chi}'(G) \leq n+1$. It is easy to see ([8]) that a graph G with n vertices without isolated edges and with at most one isolated vertex, satisfies $\tilde{\chi}'(G) \leq n + \Delta(G) - 1$. In [8] it is proved that a graph with n vertices and minimum degree $\delta \geq 5$ and maximum degree $\Delta < \frac{(2c-1)n-4}{3}$, where c is a constant with $\frac{1}{2} < c \leq 1$ has $\tilde{\chi}'(G) \leq \lceil cn \rceil$. The main result of this paper is the proof of the above conjecture.

Theorem: *A graph G with n vertices, without isolated edges and with at most one isolated vertex has $\tilde{\chi}'(G) \leq n+1$.*

In the following we shall use some additional notation. Given a proper coloring f, we denote by $B_f(v) = \{u \in V(G) - \{v\}, F(u) = F(v)\} \cup \{v\}$. A vertex v is called *good* if $B_f(v) = \{v\}$ and *bad* otherwise. A *semi-VDP coloring* is a proper coloring with $|B_f(v)| \leq 2$ for any vertex v of G. Given a proper coloring f that contains the colors α and β an (α, β)-Kempe path is a maximal path formed by the edges colored with α and β.

For a given path P denote by $\overrightarrow{P}$ one of its orientations. Then the opposite orientation is denoted by $\overleftarrow{P}$. For $v, w \in V(P)$ such that v precedes w (with respect to the fixed orientation), we denote by $v\overrightarrow{P}w$ the path starting in v and ending in w which contains all vertices of P between v and w following the orientation $\overrightarrow{P}$. Similarly, for $v, w \in V(P)$ such that w precedes v (with respect to the orientation), we denote by $v\overleftarrow{P}w$ the path which contains all vertices of P between v and w following the opposite orientation. If P is a path with a given orientation and v a vertex of P we denote by v^+ and v^- the successor and the predecessor, respectively, of the vertex v on the path P with respect to this orientation.

We will use Vizing's theorem: *Any graph G has a proper coloring with $\Delta(G)$ or $\Delta(G)+1$ colors* and also König's theorem: *Any bipartite graph G has a proper coloring with $\Delta(G)$ colors.* In the next section we shall prove some lemmas used in the proof of the main result.

2 Lemmas

Lemma 1 *If G satisfies the property $d(k-d) \geq n_d(G) - 2$ for any d, $\delta(G) \leq d \leq \Delta(G)$ where $k \geq \Delta(G)+1$, then there is a semi-VDP coloring of G with k colors.*

Proof: Since $k \geq \Delta(G)+1$ there is a proper coloring of G with k colors by Vizing's theorem. Let f be a proper coloring of G with k colors and with a minimum number of bad vertices. Suppose that f is not a semi-VDP coloring of G. Thus there exists a vertex $u \in V_d(G)$ with $|B_f(u)| \geq 3$. We give in the following a procedure to transform f to a proper coloring f' where $|B_{f'}(u)| = 2$.

There exist $k - d$ colors different from the color of an edge incident with u. So, there are $d(k - d)$ possibilities to change the color of an edge incident with u with another one such that u is not incident to two edges with the same color. Since there are at least another two vertices that are incident with the same set of colors as u, the inequality $d(k - d) \geq n_d - 2$ implies that we can choose two colors $\alpha \in F(u)$ and $\beta \notin F(u)$ such that there is no vertex v in G with $F(v) = F(u) - \{\alpha\} \cup \{\beta\}$.

Let $P_1 = u_1 \ldots v_1$ be an (α, β)-Kempe path with $u = u_1$. We transform the coloring f to another coloring f_1 by exchanging the colors α and β on the path P_1. The vertex v_1 is a bad vertex in f_1 and not in f since otherwise the coloring f_1 would be a proper coloring of G with less bad vertices than f. If $F_1(v_1) = F_1(u)$, then we take $f' = f_1$ since $|B_{f_1}(u)| = 2$. Otherwise there is another (α, β)-Kempe path $P_2 = u_2 \ldots v_2$ with $F_1(u_2) = F_1(v_1)$. We exchange the colors α and β on P_2 and we denote by f_2 this new coloring. We continue the procedure until we find an (α, β)-Kempe path $P_t = u_t \ldots v_t$ with the property that by exchanging α and β on the path P_t we obtain a coloring f_t and $F_t(v_t) = F_1(u)$.

We will prove in the following that since f is a proper coloring with a minimum number of bad vertices we can always find such a coloring. Let $\mathcal{P} = \{P_1, \ldots, P_t\}$. We observe that each vertex u in the interior of these paths has the same set of colors in each coloring $f_1, \ldots, f_t$. Also the vertices v_i and u_{i+1} exchange in f_{i+1} their sets of colors when compared with their color sets in f. For each i, v_i is a bad vertex in f_i since otherwise f_i is a coloring of G with less bad vertices than f.

It can happen that $F_i(v_i) = F_i(u_j)$ for some $j < i$, but this can only happen once for any set of colors $F_i(v_i)$. But when this happens v_i is a bad vertex under the coloring f_i so there exists another vertex w, not yet on any path constructed, such that $F_i(w) = F_i(v_i)$ and the constructive process continues. After a number of steps we reach a vertex v_t with $F_t(v_t) = F_1(u)$.

We repeat the procedure until we find a coloring f'', such that $|B_{f''}(v)| \leq 2$ for each $v \in V$. □

Let $P_1, \ldots, P_k$ be a set of vertex disjoint paths. The set $\mathcal{P} = \{P_1, \ldots, P_k\}$ is called *a long path system* if $|V(P_i)| \geq 3$ for any $i \in \{1, \ldots, k\}$. If the vertices of a graph G are covered by a long path system then $\mathcal{P}$ is called *a long path covering* of G.

The following technical lemma will be used to transform a semi-VDP coloring to a VDP coloring.

Lemma 2 *Let $\mathcal{P} = \{P_1, ..., P_k\}$ be a long path system and B a set of disjoint pairs of vertices of $\mathcal{P}$. There exists a coloring of the edges of $\mathcal{P}$ with three colors* *such that for each pair $\{x, x'\}$ of B the set of colors of the edges incident with x is different from the set of colors of the edges incident with x'.*

Proof: Fix an orientation of the paths of $\mathcal{P}$ and let $\overrightarrow{\mathcal{P}} = (\overrightarrow{P}_1, \ldots, \overrightarrow{P}_k)$ be a long path system with a given order on the paths. We denote a pair of B by (x, x') where x is the first vertex on $\mathcal{P}$ and x' is the second vertex (with respect to the orientation). Let $A = V(\mathcal{P}) - D$, where D is the set of vertices of B.

The vertices of A and the first vertices of each pair of B on $\mathcal{P}$ form *the first class* of vertices and the second vertices of the pairs form *the second class*.

We use an algorithm to color $\mathcal{P}$ with three colors α, β, γ in order to obtain a coloring where the vertices of the pairs of B are incident with different sets of colors. Let us denote this coloring by f. We color the edges of $\mathcal{P}$ in the order given by the orientation of $\overrightarrow{\mathcal{P}}$.

We start with one of the colors, say α, and we assign to the successive edges a color as follows. Suppose that the next edge to be colored is $e = uv$.

- If the vertex u belongs to the first class then
  - if u is the first vertex of a path we use for uv one of the three colors.
  - if u is an interior vertex we assign to uv one of two colors not used for u^-u.
- If u belongs to the second class then $u = x'$ and the edge or the edges incident with x have already been colored.
  - If x is an endvertex of a path and u is an interior vertex of a path P we color uv with one of the colors not used for u^-u.
  - If x is an interior vertex and u is an endvertex we color uv with one of the three colors.
  - If x and u are endvertices of a path then we color uv with one of the two colors not used for the edge incident with x.
  - If both x and u are interior vertices then we color uv in such a way that $\{f(x^-x), f(xx^+)\} \neq \{f(u^-u), f(uv)\}$.

It is easy to see that such a coloring is always possible except, possibly, in the situation where uv is the last edge on a path P where u and v belongs to the second class. For this case let $u = x'$, $v = y'$ and $w = u^-$.

Suppose first that y is the first vertex on P. Without loss of generality we can suppose that $f(yy^+) = \gamma$. Since we cannot color $x'y'$ this implies that $F(x) = \{\alpha, \beta\}$ and $f(wx') \in \{\alpha, \beta\}$ (Figure 1).

Suppose first that x lies on another path and let f' be a new coloring that is the same as f on $\mathcal{P} - \{P\}$. To color P we start by coloring yy^+ with α. There

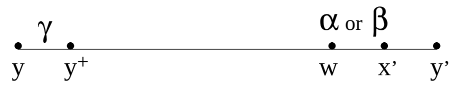

Figure 1: the path P

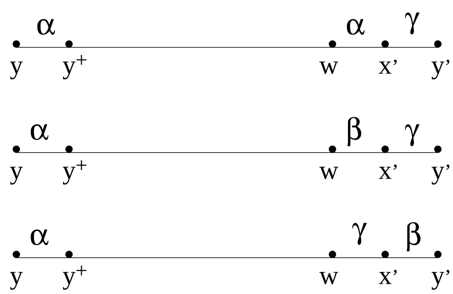

Figure 2: the path P'

are three possibilities to color P up to wx' that are illustrated in Figure 2. It is easy to see that we end up coloring $x'y'$ with a color different from α, the color of the edge incident with y.

If x belongs to the path P then we begin to modify the coloring f with the edge xx^+. We replace the color of xx^+ with γ and thus we have three cases as above. In each of these we can color $x'y'$ with a different color from the color of the edge yy^+.

Finally, if y is an endvertex of an another path P', by adding the edge yy' to the long path system we get another long path system $\mathcal{P}'$. Observe that a coloring of $\mathcal{P}'$ with three colors where the vertices of a pair of B are incident with different sets of colors induce a coloring of $\mathcal{P}$ with the same property. Recursively, we apply the quasi-algorithm to $\mathcal{P}'$ beginning with the path that contains y and y' and preserving the colors of the paths that are before P' in $\mathcal{P}$.

□

Lemma 3 *Let $G = (X, Y)$ be a bipartite graph with $|X| > |Y|$. Then there exists a proper coloring f of G with $|X|$ colors that is vertex-distinguishing on X, i.e. $F(u) \neq F(v)$ for any $u, v \in X$.*

Proof: Since G is bipartite, by König's theorem there is a proper coloring of G with $|X|$ colors. Let f be such a coloring with the minimum number of vertices in X having the same set of incident colors. Observe that $d(|X| - d) \geq |X| - 1$ for $1 \leq d \leq |Y|$. Thus, for each vertex $u \in X$ with $|B_f(u) \cap X| \geq 2$ we can choose two colors $\alpha \in F(u)$ and $\beta \notin F(u)$ such that there is no vertex v in X with $F(v) = F(u) - \{\alpha\} \cup \{\beta\}$. Let $P_1 = u_1 \ldots v_1$ be an (α, β)-Kempe path with

$u = u_1$. We transform the coloring f to another coloring f_1 by exchanging the colors α and β on the path P_1. The vertex v_1 cannot be in Y since otherwise the new coloring, f_1, would be a proper coloring of G with less vertices of X having the same set of incident colors than the coloring f, a contradiction.

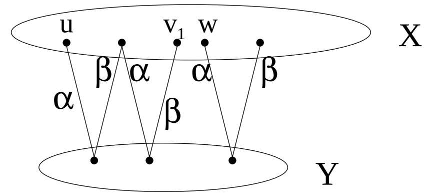

Figure 3

Thus $v_1 \in X$. By the same reasoning, this vertex cannot be the unique vertex of X incident with the set of colors $F_1(v_1)$. Thus there is a vertex $w \in X$, $w \notin V(P_1)$ with $F_1(w) = F_1(v_1)$ and another (α, β)-Kempe path $P_2 = u_2 \dots v_2$ with $w = u_2$. Now, we exchange the colors α and β on P_2 and we denote by f_2 this new coloring. The vertices v_1 and u_2 exchange in f_2 their sets of colors when compared with their color sets in f.

We continue as above to construct (α, β)-Kempe paths and to exchange their colors α and β. It is easy to see that these (α, β)-Kempe paths are vertex disjoint. Since X is a finite set the procedure must finish at a vertex $z \in X$. There exists at least another vertex $z' \in X$ incident with the same set of colors as z in this last coloring. So, z' belongs to one of the (α, β)-Kempe paths constructed before. It is easy to see that z' cannot be an interior vertex of such a path and neither an initial extremity. So, z' is a final extremity and then z was (in f) a vertex incident with the same set of colors as another vertex. In this last new coloring there is no vertex incident with the same set of colors as u. Thus this proper coloring has less vertices in X having the same sets of incident colors, a contradiction with the choice of f.

□

3 Proof of Theorem

We shall prove the theorem by induction. It is easy to see that the theorem holds for $n \leq 5$. Let G be a graph with n vertices, without isolated edges and with at most one isolated vertex. Assume that every graph H of order n', with $n' < n$, without isolated edges and with at most one isolated vertex, satisfies $\tilde{\chi}'(H) \leq n' + 1$.

Claim 1: G is a connected graph.

Proof: Suppose that G is formed by two subgraphs G_1 and G_2 with the property that there is no edge between a vertex of G_1 and a vertex of G_2. Let $n_1 = |V(G_1)|$ and $n_2 = |V(G_2)|$. By the induction hypothesis there exists a VDP coloring of G_1 and of G_2 with n_1+1 and n_2+1 colors, respectively. There is at least a color used for G_1 that is not the color of an edge with an endvertex of degree one in G_1. This color could be used in G_2 instead of another color. Thus we obtain a VDP coloring of G with $n_1 + n_2 + 1 = n + 1$ colors. □

Claim 2: G has no vertex of degree one.

Proof: If G would have such a vertex u and if the graph $G - \{u\}$ has no isolated edge, by the induction hypothesis we have a VDP coloring of $G - \{u\}$ with n colors. We color the edge incident to u with a new color and thus we obtain a VDP coloring of G with $n + 1$ colors. If $G - \{u\}$ has an isolated edge vw with $v \in N_G(u)$ then by the induction hypothesis we have a VDP coloring of $G - \{u, v, w\}$ with $n - 3 + 1$ colors. We color uv and vw with two new colors and thus we obtain a VDP coloring of G with n colors. □

Claim 3: G has no two adjacent vertices of degree two.

Proof: If G has two such vertices u and v, denote by $G' = G - \{u, v\}$. G' cannot have an isolated vertex since G is connected and G has no vertices of degree one. Also, G' cannot contain an isolated edge since G is connected and it has at least six vertices. So, G' satisfies the hypothesis of the theorem.

We apply the induction hypothesis to the graph G' and we shall use two new colors to obtain a VDP coloring of G with $n + 1$ colors as below. We distinguish two cases:

– If there is a vertex w such that $uw, vw \in E$ then we color uw and vw with two new colors and uv with a color used in G'.

– Otherwise let $x \in N_G(u)$ and $y \in N_G(v)$. We color ux and vy with two new colors. If at least one of x or y has degree two in G then $d_{G'}(x) + d_{G'}(y) \leq n - 3 + 1 = n - 2$ and thus there is a color used in G' that we can use to color uv in order to have a VDP coloring of G. If x and y have degree at least two in G' we can use any color used in G' to color uv. □

Claim 4: G has at most two vertices of degree two that are not adjacent.

Proof: If G has at least three such vertices u, v, w, denote their neighbors in G by $u_1, u_2, v_1, v_2, w_1, w_2$ (not necessary different). Using the previous claims we can suppose that $u_1, u_2, v_1, v_2, w_1, w_2$ have degree at least three in G. Let $G' = G - \{u, v, w\}$. Since G is connected, G' has at most one isolated vertex that is adjacent with u, v and w. The graph G' has at most one isolated edge that has the two endvertices among the neighbors of u, v, w since G has no isolated edge and G has no vertex of degree one. We apply the induction hypothesis to the graph obtained from G' by removing the isolated edge if such an edge exists and otherwise we apply the induction hypothesis to the graph G'. We color the isolated edge using a new color and we color the edges $uu_1, uu_2, vv_1, vv_2, ww_1, ww_2$ with another three new colors to obtain a VDP coloring of G. □

We distinguish two cases.

Case 1 G has a long path covering.

Let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be a long path covering of G with a minimum number of paths. Let $G' = G - E(\mathcal{P})$.

Since G has at most two vertices of degree two, G' has at most two isolated vertices. We show that G' has a semi-VDP coloring.

It is easy to see that $\Delta(G') \leq n-3$. A vertex that is in the interior of a path of $\mathcal{P}$ has in G' the degree at most $n-3$. If an endvertex of a path P_i has degree $n-2$ in G' then it is joined with all vertices in G. Thus $\mathcal{P}$ contains only a path since otherwise there is another long path covering of G with less paths than $\mathcal{P}$. If G is not a complete graph then we can change $\mathcal{P}$ such that vertices of degree less than $n-1$ in G become the endvertices of $\mathcal{P}$ as follows. Let one denote by $\overrightarrow{P} = u_1 \ldots u_n$ an orientation of the path of $\mathcal{P}$ and suppose that at least one of u_1 and u_n has the degree $n-1$ in G. Since G is not complete there are two vertices u_i and u_j, $i < j$ such that $u_iu_j \notin E(G)$. If u_{i+1} has the degree $n-1$ in G then we replace P by $u_i \overleftarrow{P} u_1 u_n \overleftarrow{P} u_{j+1} u_{i+1} \overrightarrow{P} u_j$ and if $d_G(u_{i+1}) < n-1$ then the path $u_i \overleftarrow{P} u_1 u_n \overleftarrow{P} u_{i+1}$ covers the vertices of G and has the endvertices of degree less than $n-1$.

We shall use Lemma 1 to show that there is a semi-VDP coloring of G' with $n-2$ colors. Using the theorem of Vizing we color the graph G' with $n-2$ colors. The restriction $d(n-2-d) \geq n-2$ is satisfied for $1 < d < n-3$ and $n \geq 6$. Also $n_1 \leq n-1$ and $n_{n-3} \leq n-1$ since otherwise if all the vertices of G' have degree $n-3$ then the interior vertices of $\mathcal{P}$ have degree $n-1$ in G and the endvertices of $\mathcal{P}$ have degree $n-2$ in G. Thus $\mathcal{P}$ has only one path since otherwise it contradicts the choice of $\mathcal{P}$ as being a long path covering of G with the minimum number of paths and the graph G is a complete graph minus one edge. In this case a VDP coloring of the complete graph is a VDP coloring of G. Thus the hypothesis of Lemma 1 are satisfied for $k = n-2$ and then G' has a semi-VDP coloring with $n-2$ colors.

Using now Lemma 2 we obtain a VDP coloring of G with $n+1$ colors.

Case 2 G has no long path covering.

Let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be a long path system that covers a maximum number of vertices of G and let denote by X_0 the set of vertices of G which do not belong to $\mathcal{P}$.

Claim 5: A vertex $v \in X_0$ is not joined with an endvertex of a path of $\mathcal{P}$ and it cannot have two neighbors on the same path. A path of $\mathcal{P}$ that contains a neighbor of X_0 is of length at most four.

Proof: Let $P = v_0 \dots v_t$, $t \geq 2$ a path of $\mathcal{P}$ and $v_j \in N_G(v), 0 \leq j \leq t$. It is easy to remark that $j \neq 0$ and $j \neq t$ since otherwise if $vv_0 \in E(G)$ then the set obtained by replacing P by $vv_0\overrightarrow{P}v_t$ covers more vertices than $\mathcal{P}$, a contradiction with the choice of $\mathcal{P}$. It is clear that the path $v_0\overrightarrow{P}v_j$ and $v_j\overrightarrow{P}v_t$ have length at most two, since otherwise we can find another set of paths that contradicts the choice of $\mathcal{P}$. Also $vv_{j+1} \notin E(G)$ and $vv_{j-1} \notin E(G)$ since if $vv_{j+1} \in E(G)$ by replacing P by $v_0\overrightarrow{P}v_jvv_{j+1}\overrightarrow{P}v_t$ the new long path system covers more vertices of G than $\mathcal{P}$. We remark that v is not joined with v_{j+2}, otherwise $j = 1$ and $t = 4$ and thus replacing P by the paths v_0v_1v and $v_2v_3v_4$ we obtain a long path system that contains more vertices than $\mathcal{P}$. □

We partition the vertices of some paths of $\mathcal{P}$ into two sets A_1 and X_1 in the following way. Let a be a neighbor of a vertex of X_0 on a path $P \in \mathcal{P}$. If P is of length two then we put a in A_1 and the endvertices of P in X_1. If P is of length three then we put the interior vertices of P in A_1 and the endvertices of P in X_1. Finally, if P is of length four then we place a and one of the vertices a^- or a^+ in A_1 and the other vertices of P in X_1. One observes that in the graph induced by X_1 there is no edge other than the edges between consecutive vertices on the same path of $\mathcal{P}$, since otherwise if there is an edge between two vertices in X_1 that are on two different paths then there is a long path system of G that covers more vertices than $\mathcal{P}$. In other words X_1 is a set of isolated vertices and edges. The same is true for $X_0 \cup X_1$. Also $|X_1| \geq |A_1|$.

Let $\mathcal{P}_0 = \mathcal{P}$ and I_1 be the set of indices of the paths of $\mathcal{P}_0$ that contain a vertex of A_1 and $\mathcal{P}_1 = \mathcal{P}_0 - \cup_{i \in I_1} P_i$. Let I_2 be the set of indices of the paths of $\mathcal{P}_1$ that contain at least a neighbor of a vertex of X_1.

Claim 6: A path with index from the set I_2 has length at most four and a vertex of X_1 is not joined with an endvertex of such a path.

Proof: Let P_1 be a path with index in I_1 and let P_2 be a path with index in I_2 that contains a neighbor v_j of a vertex u_i in $V(P_1) \cap X_1$. Denote by $\overrightarrow{P}_1 = u_0u_1 \dots u_s$ and $\overrightarrow{P}_2 = v_0v_1 \dots v_t$ two orientations of P_1 and P_2. We showed in Claim 5 that $s \leq 4$. Let u_ℓ be the vertex of P_1 adjacent with a vertex $x \in X_0$ and suppose that u_i is u_{s-1} or u_s. It is easy to see that v_j cannot be an endvertex of P_2 since otherwise if $v_j = v_0$ then the long path system obtained from $\mathcal{P}$ by replacing P_1 and P_2 by $u_0\overrightarrow{P}_1u_\ell x$ and $u_{\ell+1}\overrightarrow{P}_1u_sv_0\overrightarrow{P}_2v_t$ (if $u_i = u_s$) or $u_s\overleftarrow{P}_1u_{\ell+1}v_0\overrightarrow{P}_2v_t$ (if $u_i = u_{s-1}$) covers more vertices than $\mathcal{P}$. Also, remark that $j \leq 2$, since otherwise if $j \geq 3$ then when $u_i = u_s$, the set obtained from $\mathcal{P}$ by replacing P_1 and P_2 by the paths $u_0\overrightarrow{P}_1u_\ell x, u_{\ell+1}\overrightarrow{P}_1u_sv_j\overrightarrow{P}_2v_t$ and $v_0\overrightarrow{P}_2v_{j-1}$ forms a long path system that covers more vertices than $\mathcal{P}$. And when $u_i = u_{s-1}$, by replacing P_1 and P_2 in $\mathcal{P}$ by $u_0\overrightarrow{P}_1u_\ell x, u_su_{s-1}v_j\overrightarrow{P}_2v_t$ and $v_0\overrightarrow{P}_2v_{j-1}$ we obtain a long path system that contradicts the choice of $\mathcal{P}$. Thus $j \leq 2$ and $t \leq 4$. $\square$

Now, let us define X_2 and A_2. We consider each path P with the index in I_2. If P is of length two then we put the interior vertex in A_2 and the other two vertices in X_2. If P is of length three then we add the interior vertices in A_2 and the endvertices in X_2. Finally, if P has length four then we add two interior consecutive vertices in A_2 and the other vertices in X_2.

It is easy to see that $|X_2| \geq |A_2|$ and the graph induced by X_2 contains only isolated vertices and isolated edges. The isolated edges are only between consecutive vertices on the same path with index in I_2 since if there is an edge between two vertices in X_2 that are on two different paths then there is a long path system of G that covers more vertices than $\mathcal{P}$. Also the graph induced by $X_0 \cup X_1 \cup X_2$ contains no path of length greater than one.

Let one suppose that I_k is and $\mathcal{P}_k = \mathcal{P}_{k-1} - \cup_{i \in I_k} P_i$ and let I_{k+1} be the set of indices of the paths of $\mathcal{P}_k$ that contain at least a neighbor of a vertex of X_k.

Claim 7: A path with index in I_{k+1} has length at most four and a vertex of X_k is not joined with an endvertex of such a path.

Proof: Let $Q_1, \ldots, Q_{k+1}$ be a set of paths of $\mathcal{P}$ where Q_i is a path of $\mathcal{P}$ with index in I_{k+2-i} and Q_i contains a neighbor of a vertex of $V(Q_{i+1}) \cap X_{k+1-i}$. Thus Q_{k+1} contains a neighbor of a vertex v of X_0. Let $\overrightarrow{Q}_1 = u_0 \ldots u_s$ be an orientation of Q_1 and suppose that u_ℓ is the vertex of Q_1 with the greatest index that is joined with a path with the index in I_k and this path is Q_2. Denote by $\mathcal{Q}(u_\ell) = \{Q_1, \ldots, Q_{k+1}\}$. We prove that there is a long path system that we denote by $\mathcal{P}(u_\ell)$ that contains the vertex v of X_0 and all the vertices of the paths Q_i, $2 \leq i \leq k+1$ and the vertices u_i, $0 \leq i \leq \ell$. The proof is by induction on k. The proof of Claim 6 justifies the assertion for $k = 1$. Let $\overrightarrow{Q}_2 = v_0 \ldots v_t$ be an orientation of Q_2 and let suppose that v_j is the vertex of Q_2 with the greatest index that is joined with a path of index in I_{k-1} and this path is Q_3. Let us suppose that $u_\ell v_t \in E(G)$. The justification is similar if $u_\ell v_{t-1} \in E(G)$. Suppose the assertion is true for k. Then we add to the long path system that contains v all the vertices of the paths Q_i, $3 \leq i \leq k+1$ and the vertices v_i, for $0 \leq i \leq j$, the path $u_0 \overrightarrow{Q}_1 u_\ell v_t \overleftarrow{Q}_2 v_{j+1}$. Thus we obtain a long path system that proves the assertion for $k+1$.

Now the proof of the Claim 7 is by induction on k.

Let P be a path with index in I_k and let P' be a path with index in I_{k+1} that contains a neighbor v_j of a vertex u_i of $V(P) \cap X_k$. Let $\overrightarrow{P} = u_0 u_1 \ldots u_s$ and $\overrightarrow{P}' = v_0 v_1 \ldots v_t$ two orientations of P and P'.

Claim 6 proves Claim 7 for $k = 1$. Suppose that $s \leq 4$. Let u_ℓ be the vertex of P with the greatest index that is incident with a vertex $x \in X_{k-1}$ and suppose that u_i is u_{s-1} or u_s. It is easy to see that v_j cannot be an endvertex of P' since otherwise if $v_j = v_0$ then the long path system obtained from $\mathcal{P}$ by replacing $\mathcal{Q}(u_\ell)$ and P' by $\mathcal{P}(u_\ell)$ and $u_{\ell+1}\overrightarrow{P}u_sv_0\overrightarrow{P}'v_t$ (if $u_i = u_s$) or $u_s\overleftarrow{P}u_{\ell+1}v_0\overrightarrow{P}'v_t$ (if $u_i = u_{s-1}$) covers more vertices than $\mathcal{P}$. Also, let observe that $j \leq 2$. If $j \geq 3$ then if $u_i = u_s$, the set obtained from $\mathcal{P}$ by replacing $\mathcal{Q}(u_\ell)$ and P' by $\mathcal{P}(u_\ell)$ and $\{u_{\ell+1}\overrightarrow{P}u_sv_j\overrightarrow{P}'v_t, v_0\overrightarrow{P}'v_{j-1}\}$ forms a long path system that covers more vertices than $\mathcal{P}$ and if $u_i = u_{s-1}$ then the set obtained from $\mathcal{P}$ by replacing $\mathcal{Q}(u_\ell)$ and P' by $\mathcal{P}(u_\ell)$ and $\{u_su_{s-1}v_j\overrightarrow{P}'v_t, v_0\overrightarrow{P}'v_{j-1}\}$ is a long path system that contradicts the choice of $\mathcal{P}$. Thus $j \leq 2$ and $t \leq 4$. □

The sets X_{k+1} and A_{k+1} are defined as X_2 and A_2. By definition, $|X_{k+1}| \geq |A_{k+1}|$.

We repeat the construction until a step t when $N_{\mathcal{P}_t}(X_t) = \emptyset$. Let $X = \cup_{i=0}^{i=t} X_i$ and $A = \cup_{i=1}^{i=t} A_i$. It is easy to see that $|X| > |A|$.

Claim 8: For any k, the graph induced by X_k has no edges others than the edges between two consecutive vertices on the same path of $\mathcal{P}$. So, the graph induced by X has only isolated vertices and edges, these edges are only edges between two consecutive vertices on the same path of $\mathcal{P}$.

Proof: Suppose that there exists a path $P = u_0 \dots u_s$ with the index in I_k and two adjacent vertices from $V(P) \cap X_k$ non consecutive on P. Let u_ℓ be the vertex of P with the greatest index that is incident with a vertex in X_{k-1}. We proved in Claim 7 that there is a long path system $\mathcal{R}$ that contains a vertex of X_0 and the paths of $\mathcal{P} - P$ and the vertices $u_0, \dots, u_\ell$. By adding to $\mathcal{R}$ the path $u_0u_s\overleftarrow{P}u_{\ell+1}$ (if $u_0u_s \in E(G)$) or the path $u_0u_{s-1}u_s$ (if $u_0u_{s-1} \in E(G)$) we obtain a long path system that covers more vertices than $\mathcal{P}$, a contradiction.

Suppose now that there exist two paths $P = u_0 \dots u_s$ and $P' = v_0 \dots v_t$ with the indices in I_k with the property that a vertex $u_i \in V(P) \cap X_k$ is adjacent with a vertex $v_j \in V(P') \cap X_k$. Let u_ℓ be the vertex of P with the greatest index that is incident with a vertex in X_{k-1}. Also, as we proved in Claim 7, there is a long path system $\mathcal{R}$ that contains a vertex of X_0 and the paths of $\mathcal{P} - \{P, P'\}$ and the vertices $u_0, \dots, u_\ell$. Suppose that $u_i = u_s$. If $v_j = v_t$ then by adding to $\mathcal{R}$ the path $u_{\ell+1}\overrightarrow{P}u_sv_t\overleftarrow{P}'v_0$ we obtain a long path system that covers more vertices than $\mathcal{P}$, a contradiction. If $v_j = v_{t-1}$ then by adding to $\mathcal{R}$ the paths $u_{\ell+1}\overrightarrow{P}u_sv_{t-1}v_t$ and $v_0\overrightarrow{P}'v_{t-2}$ we obtain a long path system that contradicts the choice of $\mathcal{P}$. □

Using Lemma 3 we color the bipartite graph (X, A) with $|X|$ colors in such a way that this coloring is vertex-distinguishing on X.

Let $G' = G - X$.

Denote by $T = \{u_1v_1, \dots, u_tv_t\}$ and $S = \{w_1, \dots, w_s\}$ the sets of isolated edges and vertices of G' (it is possible that these sets are empty). The graph $G'' = G' - (S \cup T)$ has no isolated vertices and edges.

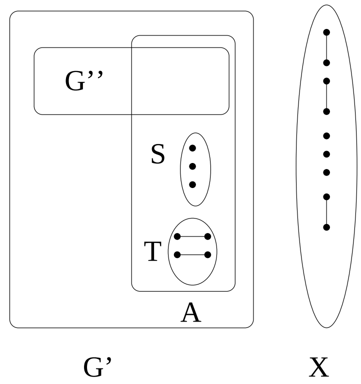

Figure 4

By the hypothesis, G has no vertices of degree one. So, for any $i \in \{1, \ldots, t\}$, u_i and v_i have at least a neighbor in X. We color each edge of T with a new color and also we change the color of one of the edges incident with u_i or v_i with another new one. Since G has no vertices of degree one, w_i has at least two neighbors x and y in X. Since the coloring is vertex-distinguishing on X, we can change the color of one of the edge $w_i x$ or $w_i y$ with a new color such that there are no two vertices incident with the same set of colors. Thus we used at most $s + 2t$ colors.

If the graph $G'' = G' - (S \cup T)$ has at least a vertex then we apply the induction hypothesis to the graph G'', thus finding a VDP coloring with $n - |X| - 2t - s + 1$ colors. Since we color G'' with $n - |X| - 2t - s + 1$ colors and G'' has $n - |X| - 2t - s$ vertices there is a color θ that is not the color of an edge incident with a vertex of degree one in G''. Finally, we choose such a color θ to color the edges in the graph induced by X.

If G'' has no vertex then since we use only $n = s + 2t + |X|$ colors we can use another color to color the edges in the graph induced by X. $\square$

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