Approximation with a fixed number of solutions of some multiobjective maximization problems[*]

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Abstract

We investigate the problem of approximating the Pareto set of some multiobjective optimization problems with a given number of solutions. Our purpose is to exploit general properties that many well studied problems satisfy. We derive existence and constructive approximation results for the biobjective versions of MAX SUBMODULAR SYMMETRIC FUNCTION (and special cases), MAX BISECTION, and MAX MATCHING and also for the k-objective versions of MAX COVERAGE, HEAVIEST SUBGRAPH, MAX COLORING of interval graphs.

1 Introduction

In multiobjective combinatorial optimization a solution is evaluated considering several objective functions and a major challenge in this context is to generate the set of efficient solutions or the Pareto set (see [12] about multiobjective combinatorial optimization). However, it is usually difficult to identify the efficient set mainly due to the fact that the number of efficient solutions can be exponential in the size of the input and moreover the associated decision problem is NP-complete even if the underlying single-objective problem can be solved in polynomial time (e.g. shortest path [12]). To handle these two difficulties, researchers have been interested in developing approximation algorithms with an a priori provable guarantee such as polynomial time constant approximation algorithms. Considering that all objectives have to be maximized, and for a positive $\rho \leq 1$, a ρ-approximation of Pareto set is a set of solutions that includes, for each efficient solution, a solution that is at least at a factor ρ on all objective values. Intuitively, the larger the size of the approximation set, the more accurate it can be.

It has been pointed out by Papadimitriou and Yannakakis [28] that, under certain general assumptions, there always exists a $(1 - \varepsilon)$-approximation, with any given accuracy $\varepsilon > 0$, whose size is polynomial both in the size of the instance and in $1/\varepsilon$ but exponential in the number of criteria. In this result, the accuracy $\varepsilon > 0$ is given explicitly and a general upper

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bound on the size of the approximation set is given. When the number of solutions in the approximation set is limited, not every level of accuracy is possible. So, once the number of solutions is fixed in the approximation set of a multiobjective problem, the following questions are raised: What is the accuracy for which an approximation is guaranteed to exist? Which accuracy can be obtained in polynomial time?

In this paper we are interested in establishing for some multiobjective maximization problems the best approximation ratio of the set of efficient solutions when the size of the approximation set is given explicitly. We give three approaches that deal with biobjective and k-objective problems that allow us to obtain approximations of the set of efficient solutions with one or several solutions. More precisely, in a first approach, we consider a general maximization problem and establish a sufficient condition that guarantees the construction of a constant approximation of the Pareto set with an explicitly given number of solutions. As a corollary, we can construct a $(1 - \varepsilon)$-approximation of the Pareto set with $O(\frac{1}{\varepsilon})$ solutions. In a second approach, we establish a necessary and sufficient condition for the construction of a constant approximation of the Pareto set with one solution. In a third approach, we establish a sufficient condition for the construction of one solution with approximation guarantee for k-objective selection problems. In these three approaches, if the corresponding solutions can be found in polynomial time then the biobjective or k-objective selection problems admits polynomial time approximation with one solution.

Properties defined in these three approaches apply to several problems previously studied in single-objective approximation. Thus we derive polynomial time constant approximations with one solution for Biobjective MAX BISECTION, Biobjective MAX PARTITION, Biobjective MAX CUT, Biobjective MAX SET SPLITTING, Biobjective MAX MATCHING and k-objective HEAVIEST SUBGRAPH, k-objective MAX q COLORABLE SUBGRAPH and k-objective MAX COVERAGE, but also for k-objective versions of some particular cases of MAX COVERAGE: MAX q VERTEX COVER and MAX q SELECTION. Some instances show that the given biobjective approximation ratios are the best we can expect. In addition Biobjective MAX PARTITION, Biobjective MAX CUT, Biobjective MAX SET SPLITTING admit a $(1 - \varepsilon)$-approximation of the Pareto set with $O(\frac{1}{\varepsilon})$ solutions.

Several results exist in the literature on the approximation of multiobjective combinatorial optimization problems. One can mention the existence of fully polynomial time approximation schemes for biobjective shortest path [19, 34, 33], knapsack [13, 8], minimum spanning tree [28], scheduling problems [6], randomized fully polynomial time approximation scheme for matching [28], and polynomial time constant approximation for max cut [4], a biobjective scheduling problem [32] and the traveling salesman problem [5, 26]. Note that [4], [26] and [32] are approximations with a single solution.

This article is organized as follows. In Section 2, we introduce basic concepts about multiobjective optimization and approximation. Section 3 is devoted to an approach for approximating some biobjective problems with one or several solutions. Section 4 presents a necessary and sufficient condition for approximating some biobjective problems with one solution within a constant factor. In Section 5 we establish an approach for approximating some multiobjective selection problems. Conclusions are provided in a final section.

2 Preliminaries on multi-objective optimization and approximation

Consider an instance of a multi-objective optimization problem with k criteria or objectives where X denotes the finite set of feasible solutions. Each solution $x \in X$ is represented in the objective space by its corresponding objective vector $w(x) = (w_1(x), \ldots, w_k(x))$. We assume that each objective has to be maximized.

From these k objectives, the dominance relation defined on X is defined as follows: a feasible solution x dominates a feasible solution x' if and only if $w_i(x) \geq w_i(x')$ for $i = 1, \ldots, k$ with at least one strict inequality. A solution x is *efficient* if and only if there is no other feasible solution $x' \in X$ such that x' dominates x, and its corresponding objective vector is said to be *non-dominated*. Usually, we are interested in finding a solution corresponding to each non-dominated objective vector. The set of all such solutions is called Pareto set.

For any $0 < \rho \leq 1$, a solution x is called a ρ-approximation of a solution x' if $w_i(x) \geq \rho \cdot w_i(x')$ for $i = 1, \ldots, k$. A set of feasible solutions X' is called a ρ-approximation of the set of all efficient solutions if, for every feasible solution $x \in X$, X' contains a feasible solution x' that is a ρ-approximation of x. If such a set exists, we say that the multi-objective problem admits a ρ-approximate Pareto set with $|X'|$ solutions.

An algorithm that outputs a ρ-approximation of a set of efficient solutions in polynomial time in the size of the input is called a ρ-approximation algorithm. In this case we say that the multi-objective problem admits a polynomial time ρ-approximate Pareto set.

Consider in the following a single-objective maximization problem P defined on a ground set $\mathcal{U}$. Every element $e \in \mathcal{U}$ has a non negative weight $w(e)$. The goal is to find a feasible solution (subset of $\mathcal{U}$) with maximum weight. The weight of a solution S must satisfy the following scaling hypothesis: if $opt(I)$ denotes the optimum value of I, then $opt(I') = t \cdot opt(I)$, where I' is the same instance as I except that $w'(e) = t \cdot w(e)$. For example, the hypothesis holds when the weight of S is defined as the sum of its elements' weights, or $\min_{e \in S} w(e)$, etc.

In the k-objective version, called k-objective P, $k \geq 2$, every element $e \in \mathcal{U}$ has k non negative weights $w_1(e), w_2(e), \ldots, w_k(e)$ and the goal is to find a Pareto set within the set of feasible solutions. Given an instance I of k-objective P, we denote by $opt_i(I)$ (or simply opt_i) the optimum value of I restricted to objective i, $i = 1, \ldots, k$. Here, the objective function on objective 1 is not necessarily of the same kind as on objective 2, but both satisfy the scaling hypothesis. For example, one objective can be additive (sum of element's weight) and the other can be bottleneck (min or max of element's weights).

3 Approximation with a given number of solutions for some biobjective problems

Papadimitriou and Yannakakis [28] proved the existence of a $(1-\varepsilon)$-approximation of size polynomial in the size of the instance and $\frac{1}{\varepsilon}$. In this general result, the accuracy $\varepsilon > 0$ is given explicitly but the size of the approximation set is roughly bounded. In this section we consider a general maximization problem Π and establish a sufficient condition that guarantees the construction of a constant approximation of the Pareto set with an explicitly given number of solutions for Π. This result allows to construct a $(1-\varepsilon)$-approximation of the Pareto set with $O(\frac{1}{\varepsilon})$ solutions but not necessarily in polynomial time. Moreover, if the single objective problem is polynomial time constant approximable and the above construction is done in poly-

nomial time then the biobjective version is also polynomial time constant approximable with one solution. Thus we obtain constant approximations and polynomial time constant approximations with one solution for Biobjective SUBMODULAR SYMMETRIC FUNCTION and also for Biobjective MAX PARTITION, Biobjective MAX CUT, Biobjective MAX SET SPLITTING but also for Biobjective MAX MATCHING.

In the following, we are interested in biobjective maximization problems, Biobjective Π, which satisfy the following property.

Property 1 *Given any two feasible solutions S_1 and S_2, and any real α satisfying $0 < \alpha \leq 1$, if $w_2(S_1) < \alpha w_2(S_2)$ and $w_1(S_2) < \alpha w_1(S_1)$ then there exists a feasible solution S_3 which satisfies $w_1(S_3) > (1-\alpha)w_1(S_1)$ and $w_2(S_3) > (1-\alpha)w_2(S_2)$.*

We say that *Biobjective Π satisfies Property 1 polynomially* if S_3 can be constructed in polynomial time.

Property 1 means that if S_1 is not an α-approximation of S_2 and S_2 is not an α-approximation of S_1 for both objective functions w_1 and w_2, then there exists a feasible solution S_3 which simultaneously approximates S_1 and S_2 with performance guarantee $1 - \alpha$.

Given a positive integer ℓ, consider the equations $x^{2\ell} = 1 - x^\ell$ and $x^{2\ell-1} = 1 - x^\ell$. Denote by α_ℓ and β_ℓ their respective solutions in the interval $[0, 1)$. Note that $\alpha_\ell = \left(\frac{\sqrt{5}-1}{2}\right)^{1/\ell}$. Moreover $\alpha_\ell < \beta_{\ell+1} < \alpha_{\ell+1}, \ell \geq 1$. Indeed, since $\beta_\ell \in (0, 1)$, we have $1 - \beta_{\ell+1}^\ell < 1 - \beta_{\ell+1}^{\ell+1} = \beta_{\ell+1}^{2\ell+1} < \beta_{\ell+1}^{2\ell}$. Since the function $f_\ell(x) = x^{2\ell} + x^\ell - 1$ is strictly increasing when $x \in (0, 1)$, for any $\ell \geq 1$ and $f_\ell(\beta_{\ell+1}) > 0 = f_\ell(\alpha_\ell)$, we have $\beta_{\ell+1} > \alpha_\ell$.

Theorem 1 *If Biobjective Π satisfies Property 1, then it admits a β_ℓ-approximate Pareto set (resp. an α_ℓ-approximate Pareto set) containing at most p solutions, where p is a positive odd integer such that $p = 2\ell - 1$ (resp. a positive even integer such that $p = 2\ell$).*

Proof: Let S_1 (resp. S_2) be a solution optimal for the first objective (resp. second one). In the following, *opt* denotes the optimal value on the first objective and also on the second objective. This can be assumed without loss of generality because a simple rescaling can make the optimal values coincide (e.g. we can always assume that $opt_2 \neq 0$, thus by multiplying each weight $w_2(e)$ by $\frac{opt_1}{opt_2}$ we are done; since the result is only existential, the time complexity for the determination of opt_1 and opt_2 is not taken into account). Then $w_1(S_1) = w_2(S_2) = opt$. Consider first the case where p is odd. Let $\rho = \beta_\ell$ with $p = 2\ell - 1$. Subdivide the bidimensionnal value space with coordinates $\{0\} \cup \{\rho^i opt : 0 \leq i \leq p\}$. See Figure 1 for an illustration.

Given $i, 1 \leq i \leq p$, the *strip* $s(i, \cdot)$ is the part of the space containing all couples (w_1, w_2) satisfying $\rho^i opt < w_1 \leq \rho^{i-1} opt$ and $0 \leq w_2 \leq opt$. The strip $s(p+1, \cdot)$ is the part of the space containing all couples (w_1, w_2) satisfying $0 \leq w_1 \leq \rho^p opt$ and $0 \leq w_2 \leq opt$. Given $j, 1 \leq j \leq p$, the *strip* $s(\cdot, j)$ is the part of the space containing all couples (w_1, w_2) satisfying $\rho^j opt < w_2 \leq \rho^{j-1} opt$ and $0 \leq w_1 \leq opt$. The strip $s(\cdot, p+1)$ is the part of the space containing all couples (w_1, w_2) satisfying $0 \leq w_2 \leq \rho^p opt$ and $0 \leq w_1 \leq opt$.

Suppose that $w_2(S_1) < \rho^p opt$ and $w_1(S_2) < \rho^p opt$. In other words $S_1 \in s(1, \cdot) \cap s(\cdot, p+1)$ and $S_2 \in s(\cdot, 1) \cap s(p+1, \cdot)$. Using Property 1 there exists a solution S_3 satisfying $w_1(S_3) > (1 - \rho^p) opt$ and $w_2(S_3) > (1 - \rho^p) opt$. Moreover, $1 - \rho^p = 1 - \beta_\ell^{2\ell-1} = \beta_\ell^\ell = \rho^\ell$. Then S_3 is a ρ-approximation of any solution S satisfying $\max\{w_1(S), w_2(S)\} \leq \rho^{\ell-1} opt$.
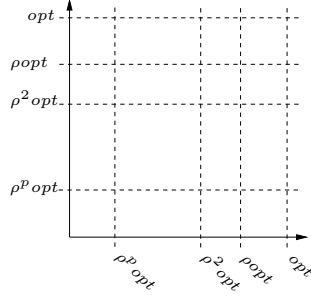
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Figure 1: Illustration of Theorem 1

One can construct a ρ-approximate Pareto set P as follows: $P = \{S_3\}$ at the beginning and for $j = \ell - 1$ down to 1, pick a feasible solution S with maximum weight w_1 in $s(\cdot, j)$ (if $s(\cdot, j)$ contains at least one value of a feasible solution) and set $P = P \cup \{S\}$. Afterwards, for $i = \ell - 1$ down to 1, pick a feasible solution S with maximum weight w_2 in $s(i, \cdot)$ (if $s(i, \cdot)$ contains at least one value of a feasible solution) and set $P = P \cup \{S\}$. For every strip the algorithm selects a solution which ρ-approximates (on both objective functions) any other solution in the strip. Since the solutions of P approximate the whole bidimensionnal space, P is a ρ-approximate Pareto set containing at most $p = 2\ell - 1$ solutions.

Now suppose that $w_2(S_1) \geq \rho^p opt$ (the case $w_1(S_2) \geq \rho^p opt$ is treated similarly). Solution S_1 must be in $s(\cdot, j^*)$ for $1 \leq j^* \leq p$. Since $w_1(S_1) = opt$, S_1 is a ρ-approximation of any solution S in $s(\cdot, p) \cup s(\cdot, p + 1)$. One can build an ρ-approximate Pareto set P as follows: $P = \{S_1\}$ at the beginning and for $j = j^* - 1$ down to 1, pick a feasible solution S with maximum weight w_1 in $s(\cdot, j)$ (if $s(\cdot, j)$ contains at least one value of a feasible solution) and set $P = P \cup \{S\}$. Since the strips form a partition of the space, the algorithm returns an ρ-approximate Pareto set containing at most p solutions.

The proof is similar in the second case where p is even by considering $\rho = \alpha_\ell$ with $p = 2\ell$.
$\square$

Corollary 1 *If Biobjective Π satisfies Property 1, then it admits a $(1-\varepsilon)$-approximate Pareto set containing $O(\frac{1}{\varepsilon})$ solutions.*

Proof: According to Theorem 1, we need at most 2ℓ solutions where $(\frac{\sqrt{5}-1}{2})^{1/\ell} \geq 1 - \varepsilon$ in order to obtain a $(1 - \varepsilon)$-approximate Pareto set. Thus $\ell = O(\frac{1}{\varepsilon})$. $\square$

Property 1 can be relaxed in the following way:

Property 2 *Given any two feasible solutions S_1 and S_2, and a real α satisfying $0 < \alpha \leq 1$. If $w_2(S_1) < \alpha w_2(S_2)$ and $w_1(S_2) < \alpha w_1(S_1)$ then there exists a feasible solution S_3 which satisfies $w_1(S_3) > (c - \alpha)w_1(S_1)$ and $w_2(S_3) > (c - \alpha)w_2(S_2)$, where $0 < c \leq 1$ is a constant.*

We say that *Biobjective Π satisfies Property 2 polynomially* if S_3 can be constructed in polynomial time.

Given a positive integer ℓ, consider the equations $x^{2\ell} = c - x^\ell$ and $x^{2\ell-1} = c - x^\ell$. Denote by γ_ℓ and δ_ℓ their respective solutions in the interval $[0, 1)$. Note that $\gamma_\ell = (\frac{\sqrt{1+4c}-1}{2})^{1/\ell}$ and $\gamma_\ell < \delta_\ell < \gamma_{\ell+1}, \ell \geq 1$.

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Theorem 2 *If Biobjective Π satisfies Property 2, then it admits a δ_ℓ-approximate Pareto set (resp. a γ_ℓ-approximate Pareto set) containing at most p solutions, where p is a positive odd integer such that $p = 2\ell - 1$ (resp. a positive even integer such that $p = 2\ell$).*

Proof: The proof is similar to the proof of Theorem 1. Suppose that $w_2(S_1) < \rho^p opt$ and $w_1(S_2) < \rho^p opt$. Using Property 2 there exists a solution S_3 satisfying $w_1(S_3) > (c-\rho^p)opt$ and $w_2(S_3) > (c-\rho^p)opt$. For the case $\rho = \delta_\ell$ and $p = 2\ell-1$, we get that $c-\rho^p = c-\delta_\ell^{2\ell-1} = \delta_\ell^\ell = \rho^\ell$. For the case $\rho = \gamma_\ell$ and $p = 2\ell$, we get that $c - \rho^p = c - \gamma_\ell^{2\ell} = \gamma_\ell^\ell = \rho^\ell$. Then S_3 is a ρ-approximation of any solution S satisfying $\max\{w_1(S), w_2(S)\} \leq \rho^{\ell-1} opt$. $\qquad\square$

The previous results of this section consider the construction, not necessarily in polynomial time, of an approximate Pareto set with a fixed number of solutions. We give in the following some conditions on the construction in polynomial time of an approximate Pareto set with one solution.

Proposition 1 *If Π is polynomial time ρ-approximable and Biobjective Π satisfies Property 1 polynomially (resp. 2), then Biobjective Π is polynomial time $\frac{\rho}{2}$-approximable (resp. $\frac{c\rho}{2}$-approximable) with one solution.*

Proof: Consider first the case where Biobjective Π satisfies Property 2 polynomially. Let S_1 (resp. S_2) be a polynomial time ρ-approximation solution for the first objective (resp. second one). In the following, opt_1 (resp. opt_2) denotes the optimal value on the first objective (resp. second one). If $w_2(S_1) \geq \frac{cw_2(S_2)}{2}$ then $w_2(S_1) \geq \frac{c\rho}{2} opt_2$ and thus S_1 is a $\frac{c\rho}{2}$-approximate Pareto set. If $w_1(S_2) \geq \frac{cw_1(S_1)}{2}$ then $w_1(S_2) \geq \frac{c\rho}{2} opt_1$ and thus S_2 is a $\frac{c\rho}{2}$-approximate Pareto set. Otherwise, $w_2(S_1) < \frac{cw_2(S_2)}{2}$ and $w_1(S_2) < \frac{cw_1(S_1)}{2}$ and since Biobjective Π satisfies Property 2 polynomially, we can construct in polynomial time a feasible solution S_3 which satisfies $w_1(S_3) \geq \frac{cw_1(S_1)}{2}$ and $w_2(S_3) \geq \frac{cw_2(S_2)}{2}$, that is a $\frac{c\rho}{2}$-approximate Pareto set. The proposition also holds in the case where Biobjective Π satisfies Property 1 polynomially by replacing c by 1. $\qquad\square$

In the following, we say that Property 1 is *tight* if there exist S_1, S_2 and α such that $w_2(S_1) < \alpha w_2(S_2)$ and $w_1(S_2) < \alpha w_1(S_1)$ and there is no $\alpha' < \alpha$ and a feasible solution S_4 satisfying $w_1(S_4) > (1 - \alpha')w_1(S_1)$ and $w_2(S_4) > (1 - \alpha')w_2(S_2)$.

We consider in Sections 3.1 and 3.2 several examples of problems Π that satisfy the scaling hypothesis and such that Biobjective Π satisfy Property 1 or Property 2.

3.1 Max Submodular Symmetric Function

A function f defined on the power set of some ground set U is called

- *submodular* if $f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$, for all $X, Y \subseteq U$;

- *symmetric* if $f(X) = f(U \setminus X)$ for all $X \subseteq U$;

- *non negative* if $f(X) \geq 0$ for all $X \subseteq U$.

MAX SUBMODULAR SYMMETRIC FUNCTION is a combinatorial optimization problem which consists of maximizing a non negative, symmetric, and submodular function f computable in polynomial time. The problem is known to generalize many NP-hard problems

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[14, 24]. Feige, Mirrokni and Vondràk [14] give a $(\frac{1}{2} - o(1))$-approximation algorithm for MAX SUBMODULAR SYMMETRIC FUNCTION.

Lemma 1 *Biobjective* MAX SUBMODULAR SYMMETRIC FUNCTION *satisfies Property 1 polynomially.*

Proof: Let $\alpha \in (0, 1]$ and S_1, S_2 two solutions of an instance of Biobjective MAX SUBMODULAR SYMMETRIC FUNCTION satisfying the inequalities: $w_2(S_1) < \alpha w_2(S_2)$ and $w_1(S_2) < \alpha w_1(S_1)$. Consider $S_3 = S_1 \Delta S_2 = (S_1 \setminus S_2) \cup (S_2 \setminus S_1)$, where Δ is the symmetric difference. We have

$$
\begin{aligned}
w_1(S_3) + w_1(S_2) &\geq w_1(S_1 \cup S_2) + w_1(S_2 \setminus S_1) \\
&= w_1(U \setminus (S_1 \cup S_2)) + w_1(S_2 \setminus S_1) \\
&\geq w_1(U \setminus S_1) + w_1(\emptyset) \\
&\geq w_1(S_1) \quad\quad\quad\quad\quad\quad\quad\quad (1)
\end{aligned}
$$

where submodularity is used on the first and third lines, symmetry is used on the second and third lines and non negativity is used on the third line. Using $w_1(S_2) < \alpha w_1(S_1)$ into Inequality (1), we get that $w_1(S_3) \geq w_1(S_1) - w_1(S_2) \geq (1-\alpha)w_1(S_1)$. With similar arguments we get that $w_2(S_3) \geq w_2(S_2) - w_2(S_1) \geq (1 - \alpha)w_2(S_2)$. □

Corollary 2 *Biobjective* MAX SUBMODULAR SYMMETRIC FUNCTION *admits a*
    (i) β_ℓ-*approximate Pareto set (resp. an* α_ℓ-*approximate Pareto set) containing at most* p *solutions, where* $p = 2\ell - 1$ *(resp.* $p = 2\ell$).
    (ii) $(1 - \varepsilon)$-*approximate Pareto set containing* $O(\frac{1}{\varepsilon})$ *solutions.*

As indicated above, Corollary 2 deals with the possibility to reach some approximation bounds when the number of solutions in the Pareto set is fixed. We give in the following an approximation bound that we can obtain in polynomial time with one solution.

Corollary 3 *Biobjective* MAX SUBMODULAR SYMMETRIC FUNCTION *admits a polynomial time* $(\frac{1}{4} - o(1))$-*approximate Pareto set with one solution.*

Proof: The results follows from Lemma 1 and Proposition 1 with $\rho = \frac{1}{2} - o(1)$. □

Now we review some special cases of MAX SUBMODULAR SYMMETRIC FUNCTION and use Proposition 1 to derive the existence of an approximate Pareto set with one solution for the biobjective version.

Special cases of MAX SUBMODULAR SYMMETRIC FUNCTION

MAX POS NAE consists of a set of clauses $\mathcal{C}$ defined on a set U of boolean variables $x_1, \ldots, x_n$. The clauses are composed of two or more positive variables and they are endowed with a non negative weight. The MAX POS NAE problem consists of finding an assignment of the variables such that the total weight of the clauses that are satisfied is maximum, where a positive clause is satisfied by an assignment if it contains at least a true variable and at least a false variable. It is NP-hard and 0.7499-approximable [36]. MAX POS NAE is also known under the name MAX SET SPLITTING or MAX HYPERGRAPH CUT [36]. The special case in

which every clause contains exactly k variables is denoted MAX POS kNAE. MAX POS 3NAE is 0.908-approximable [37]. For $k \geq 4$, MAX POS kNAE is $(1 - 2^{1-k})$-approximable [3, 22] and this is the best possible since it is hard to approximate within a factor of $1 - 2^{1-k} + \varepsilon$, for any constant $\varepsilon > 0$ [20]. Another special case of MAX POS NAE in which every clause contains exactly 2 variables corresponds to MAX CUT (given a graph $G = (V, E)$ with non negative weights on its edges, find $V' \subset V$ such that the total weight of the edges having exactly one extremity in V' is maximum) which is 0.878-approximable [16].

The MAX PARTITION problem consists of a set J of n items $1, \ldots, n$ where each item j has a non negative weight $w(j)$. A solution S is a bipartition $J_1 \cup J_2$ of the items. The goal is to find a solution S such that $w(S) = \min\{\sum_{j \in J_1} w(j), \sum_{j \in J_2} w(j)\}$ is maximized. This NP-hard problem was also studied in the context of scheduling, where the number of partitions is not fixed, and consists of maximizing the earliest machine completion time [35]. MAX PARTITION is a special case of the MAX SUBSET SUM problem. An input of MAX SUBSET SUM is formed by a set J of n items $1, \ldots, n$, each item j has a non negative weight $w(j)$, and an integer t. The problem consists of finding a subset S of J whose sum $w(S)$ is bounded by t and maximum. MAX SUBSET SUM has an FPTAS [10]. We can obtain a FPTAS for MAX PARTITION using the previous FPTAS for $t = \sum_{i=1}^{n} w(i)/2$.

Observe that Biobjective MAX PARTITION is not $(1/2 + \varepsilon)$-approximable with one solution. In order to see this, consider 3 items of weights $w_1(1) = 2, w_2(1) = 1, w_1(2) = 1, w_2(2) = 2, w_1(1) = 1, w_2(3) = 1$. The two efficient solutions S_i, $i = 1, 2$ consists of placing i in a part and the other items in the other part and have weights $w_1(S_1) = 2, w_2(S_1) = 1, w_1(S_2) = 1, w_2(S_2) = 2$. Any other solution is either dominated by one of these two or has weights equal to 1 on both criteria.

Using Proposition 1, we obtain the following approximation ratios for the the biobjective versions with one solution of special cases of MAX SUBMODULAR SYMMETRIC FUNCTION:

Problem	approx. ratio
MAX POS NAE	0.374
MAX POS 3NAE	0.454
MAX POS kNAE, $k \geq 4$	$1/2 - 1/2^k$
MAX CUT	0.439
MAX PARTITION	$1/2 - \varepsilon$

Note that the result about MAX CUT is the same as the one given in [4]. Moreover, Biobjective MAX CUT is not $(1/2 + \varepsilon)$-approximable with one solution [4], meaning that we are close to the best possible approximation result.

3.2 Max Matching

Given a complete graph $G = (V, E)$ with non negative weights on the edges, the MAX MATCHING problem is to find a matching of the graph of total weight maximum. MAX MATCHING is solvable in polynomial time [11]. We study in this part the biobjective MAX MATCHING problem and consider instances where the graph is a collection of complete graphs inside which the weights satisfy the triangle inequality, since otherwise the biobjective MAX MATCHING problem is not at all approximable with one solution. In order to see this, consider a complete graph on 3 vertices with weights $(1, 0), (0, 1), (0, 0)$. The optimum value on each

objective is 1. Nevertheless, any solution has value 0 with respect to at least one objective. Clearly Property 1 is not satisfied in this case.

Biobjective MAX MATCHING problem is NP-hard [30]. It remains NP-hard even on instances where the graph is a collection of complete graphs inside which the weights satisfy the triangle inequality. In order to see this, we reduce in polynomial time PARTITION (proved NP-hard in [23]) to our problem. Given an instance I of PARTITION with $2n$ non negative integers $a_1, \ldots, a_{2n}$ such that $\sum_{i=1}^{n} a_i = 2B$, we construct a graph G, instance of our problem as follows: G contains $6n$ vertices, at each integer a_i we associate a triangle with vertices v_i^1, v_i^2, v_i^3 and weights $w_1(v_i^1, v_i^2) = 2a_i, w_2(v_i^1, v_i^2) = a_i, w_1(v_i^2, v_i^3) = a_i, w_2(v_i^2, v_i^3) = a_i, w_1(v_i^1, v_i^3) = a_i, w_2(v_i^1, v_i^3) = 2a_i$. For every edge between 2 different triangles we associate an weight 0 on both objectives. Clearly, there is a partition of the $2n$ integers into two sets of sum B if and only if there is a matching in G of weight at least $3B$ on each objective. It remains an open problem to decide if Biobjective MAX MATCHING remains NP-hard on complete graphs where the weights satisfy the triangle inequality.

Lemma 2 *Biobjective* MAX MATCHING *satisfies Property 2 polynomially with $c = 1/3$.*

Proof: Let $\alpha \in (0, 1]$ and S_1, S_2 two solutions of an instance of biobjective MAX MATCHING satisfying the inequalities: $w_2(S_1) < \alpha w_2(S_2)$ and $w_1(S_2) < \alpha w_1(S_1)$. The set of edges of $S_1 \cup S_2$ constitutes several connected components: cycles of size 4ℓ, cycles of size $4\ell + 2$, $\ell \geq 1$, and paths of length $\ell \geq 1$. In order to construct S_3 we proceed in two steps. Firstly, starting from $S_1 \cup S_2$, we construct in polynomial time a collection of vertex disjoint paths $\mathcal{P}$ of length at most 2 such that $w_i(S_i \cap \mathcal{P}) \geq \frac{w_i(S_i)}{3}$. Secondly, we construct S_3 from $\mathcal{P}$.

Given such a collection $\mathcal{P}$ of paths of length at most 2, we construct S_3 as follows: for each path $P_j = v_1, v_2$ of length 1 from $\mathcal{P}$, we add (v_1, v_2) to S_3; for each path $P_j = v_1, v_2, v_3$ of length 2 from $\mathcal{P}$, we add (v_1, v_3) in S_3. It is easy to see, since $\mathcal{P}$ is vertex disjoints, that S_3 is a matching. In this last case, using triangle inequality, we have

$$w_i(S_{3-i} \cap P_j) + w_i(S_3 \cap P_j) \geq w_i(S_i \cap P_j) \quad \text{for } i = 1, 2$$

and thus making the sum over all paths $P_j \in \mathcal{P}$, we obtain

$$w_i(S_{3-i} \cap \mathcal{P}) + w_i(S_3) \geq w_i(S_i \cap \mathcal{P}) \quad \text{for } i = 1, 2.$$

Moreover, $w_i(S_{3-i} \cap \mathcal{P}) < w_i(S_{3-i}) < \alpha w_i(S_i)$. Thus

$$w_i(S_3) \geq w_i(S_i \cap P_j) - \alpha w_i(S_i) \geq \frac{w_i(S_i)}{3} - \alpha w_i(S_i) = \left(\frac{1}{3} - \alpha\right) w_i(S_i) \quad \text{for } i = 1, 2.$$

We show in the following how to construct the collection $\mathcal{P}$.

Consider a cycle of size 4ℓ, $C_j = v_1, v_2, \ldots, v_{4\ell}$ from $S_1 \cup S_2$ and suppose $S_1 \cap C_j = \{(v_{2i+1}, v_{2i+2}), i = 0, \ldots, 2\ell - 1\}$. Let S_1' be the matching of maximum weight w_1 between $M = \{(v_{4i+1}, v_{4i+2}), i = 0, \ldots, \ell - 1\}$ and $(S_1 \cap C_j) \setminus M$. Thus $w_1(S_1') \geq \frac{1}{2} w_1(S_1 \cap C_j)$. Similarly we construct S_2' obtaining $w_2(S_2') \geq \frac{1}{2} w_2(S_2 \cap C_j)$. $S_1' \cup S_2'$ consists of paths of length two and we add $S_1' \cup S_2'$ to $\mathcal{P}$. Thus, $w_1(S_1' \cup S_2') \geq \frac{1}{2} w_1(S_1 \cap C_j)$ and $w_2(S_1' \cup S_2') \geq \frac{1}{2} w_2(S_2 \cap C_j)$.

On a path of length $\ell \geq 1$ from $S_1 \cup S_2$, we proceed as in the previous case, obtaining the same inequalities as before.

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Consider a cycle C_j of size $4\ell + 2$ from $S_1 \cup S_2$. We remove from C_j the edge with the smallest w_1 from $S_1 \cap C_j$ and the one with the smallest w_2 from $S_2 \cap C_j$. On the path or the two remaining paths we proceed as in the previous case for the construction of S_1' and S_2'. $S_1' \cup S_2'$ consists of paths of length at most two and we add $S_1' \cup S_2'$ to $\mathcal{P}$. In this case, $w_1(S_1') \geq \frac{1}{2}(1 - \frac{1}{2\ell+1})w_1(S_1 \cap C_j) \geq \frac{1}{3}w_1(S_1 \cap C_j)$ and $w_2(S_2') \geq \frac{1}{2}(1 - \frac{1}{2\ell+1})w_2(S_2 \cap C_j) \geq \frac{1}{3}w_2(S_2 \cap C_j)$.

Thus, summing all these inequalities we obtain $w_i(S_i \cap \mathcal{P}) \geq \frac{w_i(S_i)}{3}$ for $i = 1, 2$. □

Corollary 4 *Biobjective* MAX MATCHING *admits a δ_ℓ-approximate Pareto set (resp. an γ_ℓ-approximate Pareto set) containing at most p solutions, where $p = 2\ell - 1$ (resp. $p = 2\ell$).*

Corollary 5 *Biobjective* MAX MATCHING *admits a polynomial time $\frac{1}{6}$-approximate Pareto set with one solution.*

Proof: It follows from Lemma 2 and Proposition 1 considering $\rho = 1$. □

4 Approximation with one solution for biobjective problems

In this section, we establish a necessary and sufficient condition for constructing, not necessarily in polynomial time, a constant approximation with one solution of the Pareto set for biobjective maximization problems. Moreover, if the construction can be done in polynomial time and the single-objective problem is polynomial time constant approximable, then the biobjective version is polynomial time constant approximable with one solution. Using this condition, we establish a polynomial time 0.174-approximation with one solution for Biobjective MAX BISECTION.

In the following, we are interested in biobjective maximization problems, Biobjective Π, which satisfy the following property.

Property 3 *We can construct three solutions S_1, S_2, S_3 such that S_i is a ρ_i-approximation for problem Π on objective i, $i = 1, 2$, and S_3 is such that $w_1(S_2) + w_1(S_3) \geq \alpha \cdot w_1(S_1)$ and $w_2(S_1) + w_2(S_3) \geq \alpha \cdot w_2(S_2)$ for some fixed constant $\alpha \leq 1$.*

We say that *Biobjective Π satisfies Property 3 polynomially* if S_1, S_2, S_3 can be constructed in polynomial time.

The aim of solution S_3 in Property 3 is to compensate the potential inefficiency of S_i on criterion $3 - i$, $i = 1, 2$.

Theorem 3 *Biobjective Π is constant approximable with one solution if and only if it satisfies Property 3. Moreover, if Biobjective Π is polynomial time constant approximable with one solution then it satisfies Property 3 polynomially. More precisely, if Biobjective Π satisfies Property 3 polynomially such that S_i is a polynomial time ρ_i-approximation for problem Π on objective i, $i = 1, 2$, then Biobjective Π admits a polynomial time $\alpha \frac{\min\{\rho_1, \rho_2\}}{2}$-approximation algorithm with one solution.*

Proof: Suppose that Biobjective Π is ρ-approximable with one solution. Let S_3 be this solution and S_1 and S_2 any two solutions. Then $w_1(S_3) \geq \rho \cdot opt_1 \geq \rho \cdot w_1(S_1)$ and thus by setting $\alpha = \rho$ we have $w_1(S_2) + w_1(S_3) \geq \alpha \cdot w_1(S_1)$. The second inequality holds also.

Suppose now that Biobjective Π satisfies Property 3. Since S_i is a ρ_i-approximation for problem Π on objective i, $i = 1, 2$, we have $w_1(S_1) \geq \rho_1 \cdot opt_1$ and $w_2(S_2) \geq \rho_2 \cdot opt_2$.

Since Property 3 is satisfied, we can construct S_3 such that

$$w_1(S_2) + w_1(S_3) \geq \alpha \cdot w_1(S_1) \tag{2}$$

and

$$w_2(S_1) + w_2(S_3) \geq \alpha \cdot w_2(S_2) \tag{3}$$

Now, we study different cases:

- If $w_1(S_2) \geq \frac{\alpha}{2} w_1(S_1)$, then we deduce that S_2 is a good approximation of the Pareto set. From the hypothesis, we have $w_1(S_2) \geq \frac{\alpha}{2} w_1(S_1) \geq \alpha \cdot \frac{\min\{\rho_1, \rho_2\}}{2} opt_1$. On the other hand, we also have $w_2(S_2) \geq \rho_2 \cdot opt_2 \geq \alpha \frac{\min\{\rho_1, \rho_2\}}{2} opt_2$.

- If $w_2(S_1) \geq \frac{\alpha}{2} w_2(S_2)$, then we deduce that S_1 is a good approximation of the Pareto set. From the hypothesis, we have $w_2(S_1) \geq \frac{\alpha}{2} w_2(S_2) \geq \alpha \cdot \frac{\min\{\rho_1, \rho_2\}}{2} opt_2$. On the other hand, by the construction of S_1 we also have $w_1(S_1) \geq \rho_1 \cdot opt_1 \geq \alpha \cdot \frac{\min\{\rho_1, \rho_2\}}{2} opt_1$.

- If $w_1(S_2) \leq \frac{\alpha}{2} w_1(S_1)$ and $w_2(S_1) \leq \frac{\alpha}{2} w_2(S_2)$, then it is S_3 which is a good approximation of the Pareto set. Indeed, from inequality (2), we deduce $w_1(S_3) \geq \frac{\alpha}{2} w_1(S_1) \geq \alpha \cdot \frac{\min\{\rho_1, \rho_2\}}{2} opt_1$ and on the other hand, from inequality (3), we also get $w_2(S_3) \geq \frac{\alpha}{2} w_2(S_2) \geq \alpha \cdot \frac{\min\{\rho_1, \rho_2\}}{2} opt_2$.

In any of these three cases, we obtain a $\alpha \cdot \frac{\min\{\rho_1, \rho_2\}}{2}$-approximation with one solution.

Clearly, if S_1, S_2, S_3 are computable in polynomial time, then Biobjective Π is approximable in polynomial time. $\square$

Note that we can extend Theorem 3 to the case where ρ_i are not constant.

The interest of Property 3 is to find a simple method in order to construct a polynomial time constant approximation for Biobjective Π. This method does not allow us to obtain the best polynomial time constant approximation for Biobjective Π with one solution, but only to prove the fact that the problem is polynomial time constant approximable with one solution.

In Proposition 1 we prove that if a problem Π is (resp. polynomial time) constant approximable and if Biobjective Π satisfies (resp. polynomially) Property 1, then Biobjective Π is (resp. polynomial time) constant approximable with one solution, and thus Biobjective Π satisfies (resp. polynomially) Property 3 by Theorem 3. Thus all problems studied in Section 3 satisfy Property 3.

There exist problems which are polynomial time constant approximable and thus satisfy Property 3 and do not satisfy Property 1. One example is Biobjective Max TSP, which is polynomial time $\frac{7}{27}$-approximable with one solution [26] and does not satisfy Property 1.

Proposition 2 *Biobjective Max TSP does not satisfy Property 1.*

Proof: Consider the complete graph K_5 where a fixed K_4 is decomposable into 2 edge-disjoint Hamiltonian paths P_1 and P_2. For every edge $e \in E(K_5)$, set $w_1(e) = 1$ and $w_2(e) = 0$ if

$e \in P_1$, $w_1(e) = 0$ and $w_2(e) = 1$ if $e \in P_2$ and $w_1(e) = 0$ and $w_2(e) = 0$ if $e \notin P_1 \cup P_2$. We can check that there are four non-dominated tours T_i, $i = 1, \ldots, 4$ with $w_1(T_1) = 3$, $w_2(T_1) = 0$, $w_1(T_2) = 0$, $w_2(T_2) = 3$, $w_1(T_3) = 2$, $w_2(T_3) = 1$ and $w_1(T_4) = 1$, $w_2(T_4) = 2$. Consider $S_i = T_i$, $i = 1, 2$ and $\alpha = 1/2$. Clearly $w_2(S_1) < \alpha w_2(S_2)$ and $w_1(S_2) < \alpha w_1(S_1)$. Moreover there is no solution S_3 such that $w_1(S_3) > (1 - \alpha)w_1(S_1)$ and $w_2(S_3) > (1 - \alpha)w_2(S_2)$. $\qquad \square$

We consider in the following a problem that satisfies Property 3 and for which we are not able to prove that it satisfies Property 1.

4.1 Max Bisection

Given a graph $G = (V, E)$ with non negative weights on the edges, the Max Bisection problem consists of finding a bipartition of the vertex set V into two sets of equal size such that the total weight of the cut is maximum. We establish in this part a polynomial time $\frac{\rho}{4}$-approximation algorithm for Biobjective Max Bisection where ρ is any polynomial time approximation ratio given for Max Bisection. Max Bisection is NP-hard [23] and the best approximation ratio known for Max Bisection is $\rho = 0.701$ [17].

Lemma 3 *Biobjective* Max Bisection *on graphs with $4n$ vertices satisfies Property 3 polynomially with $\alpha = 1$ and $\rho_1 = \rho$ and $\rho_2 = \frac{\rho}{2}$, where ρ is any polynomial time approximation ratio given for* Max Bisection.

Proof: Formally, let $S_1 = (V_1, V_2)$ be a bisection of $I = (G, w_1)$ given by a polynomial time ρ-approximation algorithm for Max Bisection. Let G_i be the subgraph of G induced by V_i, $i = 1, 2$ and let (A, B) (resp., (C, D)) be a bisection of $I_1 = (G_1, w_2)$ (resp., $I_2 = (G_2, w_2)$) given by a polynomial time ρ-approximation algorithm. We produce two other bisections S_2 and S_3 of G described by $(A \cup C, B \cup D)$ and $(A \cup D, B \cup C)$. W.l.o.g., assume $w_2(S_2) \geq w_2(S_3)$. We show in the following that S_1, S_2, S_3 satisfy inequalities of Property 3 with $\alpha = 1$.

Let $S^* = (V_1^*, V_2^*)$ be an optimal bisection on (G, w_2) using edge set E^*. Thus, $w_2(S^*) = w_2(E^*) = opt_2$. Let $V_1' = V_1^* \cap (A \cup B)$ (resp., $V_1'' = V_1^* \cap (C \cup D)$) and $V_2' = V_2^* \cap (A \cup B)$ (resp., $V_2'' = V_2^* \cap (C \cup D)$). Let E_1^* (resp., E_2^*) be the edge set given the cut $S_1' = (V_1', V_2')$ (resp., $S_2' = (V_1'', V_2'')$). Hence, for $i = 1, 2$, E_i^* is the restriction of S^* on $G_i = (V_i, E_i)$. Finally, let E_3^* be the remaining edges, $E_3^* = E^* \setminus (E_1^* \cup E_2^*)$. Note that E_3^* belongs to the crossing edges between G_1 and G_2, i.e., are in the cut S_1. Thus, we get:

$$opt_2 = w_2(E_1^*) + w_2(E_2^*) + w_2(E_3^*) \tag{4}$$

Since $F_1 = (A, B)$ (resp., $F_2 = (C, D)$) is the solution given by a polynomial time ρ-approximation algorithm on $I_1 = (G_1, w_2)$ (resp., $I_2 = (G_2, w_2)$), we obtain for $i = 1, 2$:

$$w_2(F_i) \geq \rho \cdot opt_2(I_i) \geq \frac{\rho}{2} w_2(E_i^*) \tag{5}$$

Actually, consider the cut $S_1' = (V_1', V_2')$ and assume $|V_1'| \geq |V_2'|$. Let $U = argmin\{w_2(U, V_2') : U \subset V_1'$ and $|U| = n - |V_2'|\}$. Since $|V_1'| + |V_2'| = 2n$, we get $2|U| \leq |V_1'|$. Thus, we deduce $2w_2(U, V_2') \leq w_2(V_1', V_2') = w_2(E_1^*)$. Now, observe that $\widetilde{S_1} = (V_1' \setminus U, V_2' \cup U)$ is a bisection of G_1. Hence $opt_2(I_1) \geq w_2(\widetilde{S_1})) \geq w_2(E_1^*) - w_2(U, V_2') \geq \frac{1}{2} w_2(E_1^*)$. Obviously, the same arguments for the cut $S_2' = (V_1'', V_2'')$ on G_2 lead to a similar conclusion.

On the other hand, since $w_2(S_2) \geq w_2(S_3)$, we have:

$$2(w_2(A, D) + w_2(B, C)) \geq w_2(S_1) \geq w_2(E_3^*) \tag{6}$$

Finally, using $w_2(S_2) = w_2(F_1) + w_2(F_2) + w_2(A, D) + w_2(B, C)$, equality (4), (5), (6) and $\rho \leq 1$, we obtain that S_2 is a $\rho/2$-approximation.

Due to the construction of S_1, S_2 and S_3, we get $w_1(S_1) + w_1(S_2) + w_1(S_3) = 2(w_1(S_1) + w_1(A, B) + w_1(C, D))$ and $w_2(S_1) + w_2(S_2) + w_2(S_3) = 2(w_2(S_2) + w_2(A, C) + w_2(B, D))$. Thus, we deduce:

$$w_1(S_2) + w_1(S_3) \geq w_1(S_1) \tag{7}$$

and

$$w_2(S_1) + w_2(S_3) \geq w_2(S_2) \tag{8}$$

□

Corollary 6 *Biobjective* MAX BISECTION *on graphs with $4n$ vertices admits a polynomial time 0.174-approximate Pareto set with one solution.*

Proof: The results follows from Theorem 3 and Lemma 3 and using the polynomial time 0.701-approximation algorithm for MAX BISECTION [17]. □

5 Approximation with one solution for some multiobjective selection problems

In this section we establish a sufficient condition for the construction, not necessarily in polynomial time, of a solution with performance guarantee for k-objective selection problems. Moreover, if all steps of the construction are feasible in polynomial time, then the k-objective version is polynomial time approximable with one solution. The general result is applied to the k-objective versions of three problems: MAX COVERAGE (and its special cases MAX q VERTEX COVER and MAX q SELECTION), HEAVIEST SUBGRAPH and MAX COLORING of interval graphs.

Consider a general single-objective maximization problem Π defined as follows: a set $\mathcal{U}$ of elements, a non negative weight $w(e)$ for every element $e \in \mathcal{U}$, a family $\mathcal{F}$ of subsets of $\mathcal{U}$ (either $\mathcal{F}$ is given explicitly or there is a function that can decide if a subset $S \in \mathcal{F}$), a positive integer q, a covering function c such that $c(S) \subseteq \mathcal{U}$ for every subset $S \subseteq \mathcal{F}$. It is assumed that $c(\mathcal{F}) = \mathcal{U}$ and $S \subseteq S' \Rightarrow c(S) \subseteq c(S')$. A feasible solution is a set S of q subsets of $\mathcal{F}$. Its weight, to be maximized, is denoted by $w(S)$ and defined as $\sum_{e \in c(S)} w(e)$.

We are interested in problems Π which satisfy the following property.

Property 4 *For any integer t, $1 \leq t \leq q$, there exists a function $\rho(t, q) \in (0, 1]$ such that for any feasible solution S, one can always select t subsets among the q subsets of S such that the weight of these t subsets is at least $\rho(t, q)w(S)$.*

We say that Π *satisfies Property 4 polynomially* if the t subsets can be found in polynomial time. In the k-objective version of Π, every element e has k non negative weights $w_1(e), \ldots, w_k(e)$.

Theorem 4 *If Π satisfies Property 4, then k-objective Π admits a $\rho(\lfloor q/k \rfloor, q)$-approximate Pareto set with one solution. Moreover, if Π satisfies Property 4 polynomially and if Π has a polynomial time r-approximation algorithm then k-objective Π has a $r\rho(\lfloor q/k \rfloor, q)$-approximate Pareto set with one solution computable in polynomial time.*

Proof: Using $S \subseteq S' \Rightarrow c(S) \subseteq c(S')$ and the fact that every element e has a non negative weight $w_i(e)$ for every i, $1 \leq i \leq k$, we deduce that $S \subseteq S' \Rightarrow w_i(S) \leq w_i(S')$. Hence when a partial solution containing less than q subsets is completed with additional subsets, its weight cannot decrease.

Let S_i (resp. $\hat{S}_i$) be an optimal (resp. r-approximate) solution with respect to w_i. Using Property 4 one can choose $\lfloor \frac{q}{k} \rfloor$ subsets in each S_i (resp. $\hat{S}_i$) to build a new solution S_0 (resp. $\hat{S}_0$) containing at most q subsets. We complete, if necessary, S_0 (resp. $\hat{S}_0$) in order to obtain a solution with exactly q subsets. Thus, $w_i(S_0) \geq \rho(\lfloor q/k \rfloor, q) w_i(S_i)$ (resp. $w_i(\hat{S}_0) \geq \rho(\lfloor q/k \rfloor, q) w_i(\hat{S}_i) \geq r\rho(\lfloor q/k \rfloor, q) w_i(S_i))$ for every i, $1 \leq i \leq k$.

Clearly, if $\hat{S}_i$ is computable in polynomial time and Π satisfies Property 4 polynomially, then $\hat{S}_0$ is a polynomial time approximation for k-objective Π. $\square$

Several examples of Π which satisfy Property 4 are given in Sections 5.1, 5.2 and 5.3.

5.1 Max Coverage and special cases

The input of MAX COVERAGE is a set $\mathcal{U}$ of elements, a non negative weight $w(e)$ for every element $e \in \mathcal{U}$, a family $\mathcal{F}$ of subsets of $\mathcal{U}$ and a positive integer q. The goal is to select q elements of $\mathcal{F}$ so that the total weight of the elements of $\mathcal{U}$ covered by the union of these q elements of $\mathcal{F}$ is maximum. MAX COVERAGE is NP-hard and $(1 - 1/e)$-approximable in polynomial time [1]. Clearly, MAX COVERAGE is a special case of Π and it is not difficult to prove that MAX COVERAGE satisfies Property 4 polynomially for $\rho(t,q) = \frac{t}{q}$.

MAX q VERTEX COVER is a special case of MAX COVERAGE. It consists of finding q vertices from an undirected and edge-weighted graph $G = (V, E)$, where $q \leq |V|$, such that the total edge weight covered by the q vertices is maximized. MAX q VERTEX COVER is NP-hard and $\frac{3}{4}$-approximable [18].

MAX q SELECTION is another special case of MAX COVERAGE. It consists of finding q items from a set of n weighted items of maximum weight. MAX q SELECTION is trivial but k-objective MAX q SELECTION is NP-hard even for $k = 2$ [7].

Using Theorem 4, we get that k-objective MAX COVERAGE, MAX q VERTEX COVER and MAX q SELECTION admit a $\frac{\lfloor q/k \rfloor}{q}$-approximate Pareto set with one solution. In addition, a $\rho \frac{\lfloor q/k \rfloor}{q}$-approximate Pareto set with one solution can be computed in polynomial time where $\rho = (1 - 1/e)$, $\rho = 3/4$ and $\rho = 1$ for MAX COVERAGE, MAX q VERTEX COVER and MAX q SELECTION respectively.

5.2 Heaviest Subgraph

The input of HEAVIEST SUBGRAPH is a complete graph $G = (V, E)$ with a non negative weight $w(e)$ for each edge $e \in E$ and a positive integer $q \leq |V|$. The goal is to select q nodes of V such that the total weight of the subgraph induced by these q nodes is maximum. The weight of a subgraph induced by $V' \subset V$ is denoted by $w(V')$ and defined as $w(V') = \sum_{\{(i,j): i \in V', \, j \in V', \, i < j\}} w((i,j))$. HEAVIEST SUBGRAPH is an NP-hard problem and it remains NP-hard even when the weights satisfy the triangle inequality [29]. It is also known under

the name max edge-weighted clique problem and densest subgraph problem (when all the edge weights are equal to 1) [31, 15, 25, 21, 2] and its approximation was studied in [31, 15]. HEAVIEST SUBGRAPH is also a special case of Π. The problem satisfies Property 4 for $\rho(t,p) = \frac{t(t-1)}{q(q-1)}$ [31]. Hence k-objective HEAVIEST SUBGRAPH admits a $\frac{\lfloor q/k \rfloor(\lfloor q/k \rfloor - 1)}{q(q-1)}$-approximate Pareto set with one solution.

5.3 Max Coloring of interval graphs

The input of the maximum coloring problem in interval graphs (MAX COLORING for short) is a set of intervals $\mathcal{U}$ from the real line, each interval $u \in \mathcal{U}$ has a non negative weight $w(u)$ and q colors. The goal is to find a coloring of the intervals of maximum total weight such that two intersecting intervals must receive distinct colors. The problem is solvable in polynomial time [9]. Clearly, MAX COLORING is a special case of Π and it is not difficult to prove that it satisfies Property 4 polynomially for $\rho(t,q) = \frac{t}{q}$. Then, k-objective MAX COLORING admits a $\frac{\lfloor q/k \rfloor}{q}$-approximate Pareto set with one solution which can be computed in polynomial time.

6 Conclusion

In this paper, we have established some sufficient conditions that allow to conclude on the existence of constant approximations of the Pareto set with an explicitly given number of solutions for several biobjective maximization problems. The results we obtained establish a *polynomial time* approximation when we ask for a single solution in the approximation set except for HEAVIEST SUBGRAPH. A possible future work would be to give a polynomial time approximation for any explicitly given number of solutions. A necessary and sufficient condition is given for the construction of (polynomial time) constant approximation with one solution for biobjective maximization problems. It would be interesting to generalize this result to maximization problems with more than two objectives. Another interesting future work would be to establish lower bounds for any explicitly given number of solutions for multiobjective maximization problems. We also established in this paper a sufficient condition that allows to conclude on the existence of approximations of the Pareto set with one solution for multiobjective selection problems. A possible future work would be to establish polynomial time approximation for any explicitly given number of solutions.

Our approaches deal with maximization problems and they do not seem to apply to minimization problems. A possible explanation is that, in the maximization framework, adding elements to a partial solution rarely deteriorates it. Minimization problems rarely satisfy this property. Establishing constant approximation of the Pareto set with a given number of solutions or show that this is not possible for minimization problems is an interesting open question.

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