Complexity of determining the most vital elements for the p-median and p-center location problems

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Abstract

We consider the k most vital edges (nodes) and min edge (node) blocker versions of the p-median and p-center location problems. Given a weighted connected graph with distances on edges and weights on nodes, the k most vital edges (nodes) p-median (respectively p-center) problem consists of finding a subset of k edges (nodes) whose removal from the graph leads to an optimal solution for the p-median (respectively p-center) problem with the largest total weighted distance (respectively maximum weighted distance). The complementary problem, min edge (node) blocker p-median (respectively p-center), consists of removing a subset of edges (nodes) of minimum cardinality such that an optimal solution for the p-median (respectively p-center) problem has a total weighted distance (respectively a maximum weighted distance) at least as large as a specified threshold. We show that k most vital edges p-median and k most vital edges p-center are NP-hard to approximate within a factor $\frac{7}{5} - \epsilon$ and $\frac{4}{3} - \epsilon$ respectively, for any $\epsilon > 0$, while k most vital nodes p-median and k most vital nodes p-center are NP-hard to approximate within a factor $\frac{3}{2} - \epsilon$, for any $\epsilon > 0$. We also show that the complementary versions of these four problems are NP-hard to approximate within a factor 1.36.

Keywords: most vital edges and nodes, min edge and node blocker, p-median, p-center, complexity, approximation.

1 Introduction

For problems of security or reliability, it is important to assess the ability of a system to resist to a destruction or a failure of a number of its entities. This amounts to identifying critical entities which can be determined with respect to a measure of performance or a cost associated to the system. In this paper we

focus on simple location problems. Consider for instance the following problem. We aim at locating p hospitals or p supermarkets in order to serve n areas. Each area is characterized by a population which represents a potential demand. The areas are connected by roads with a given distance. The objective for locating these hospitals or supermarkets is not the same. Indeed, for hospitals, we aim at finding the locations that minimize the maximum distance weighted by population from the closest hospital to all areas while for supermarkets we aim at finding the locations that minimize the total weighted distance from the closest supermarket to all areas. However, there may occur incidents such as works on road or floods that make some roads inaccessible. In this case several problems may arise. We can aim at detecting the critical roads whose failure causes the largest increase in the weighted distance. Alternatively, we can aim at determining the maximum number of damaged roads which still ensures a certain quality of service level. Modeling the considered network by a weighted connected graph with distances on edges and weights on nodes, where roads are edges and areas are nodes, these problems consist either of finding a subset of edges or nodes whose removal from the graph generates the largest increase in the total or maximum weighted distance or of determining a subset of edges or nodes of minimal cardinality such that, when we remove this subset from the graph, the total or maximum weighted distance is at least as large as a specified threshold. In the literature these problems are referred respectively to as the k *most vital edges/nodes* and the *min edge/node blocker* problems.

The k most vital edges/nodes and min edge/node blocker versions have been studied for several problems, including shortest path, minimum spanning tree, maximum flow, maximum matching and independent set. The k most vital edges problem with respect to shortest path was proved NP-hard [2]. Later, k most vital edges/nodes shortest path (and min edge/node blocker shortest path, respectively) were proved to be not 2-approximable (not 1.36-approximable, respectively) if $P \neq NP$ [8]. For minimum spanning tree, k most vital edges is NP-hard [6] and $O(\log k)$-approximable [6]. In [11] it is proved that k most vital edges maximum flow is NP-hard. For maximum matching, min edge blocker is NP-hard even for bipartite graphs [12], but polynomial for grids and trees [10]. In [3], the k most vital nodes and min node blocker versions with respect to independent set for bipartite graphs remain polynomial on the unweighted graphs and become NP-hard for weighted graphs. For bounded treewidth graphs and cographs these versions remain polynomial [3]. Concerning the approximation on bipartite weighted graphs, k most vital nodes with respect to independent set has no ptas [3].

In this paper the k most vital edges (nodes) and min edge (node) blocker versions for the p-median and p-center problems are studied.

After introducing some preliminaries in Section 2, we prove in Sections 3 and 4 that k Most Vital Edges (Nodes) p-median (p-center) and Min Edge (Node) Blocker p-median (p-center) are not constant approximable for some constants, unless $P=NP$. Final remarks are provided in Section 5.

2 Basic concepts and definitions

Consider $G = (V, E)$ a connected weighted graph with $|V| = n$ and $|E| = m$. Let $d_{v_i v_j}$ be the distance between v_i and v_j for $(v_i, v_j) \in E$ and w_{v_i} be the weight associated to node v_i for $i = 1, \ldots, n$ (w_{v_i} represents the demand occurring at node v_i). Denote by $d(v_i, v_j)$ the minimum distance between two nodes v_i and v_j of G and let $d(F, v_i) = \min_{f \in F} d(f, v_i)$, for any $F \subseteq V$. The p-median (respectively p-center) problem consists of finding a subset F of p nodes which minimizes the total weighted distance (respectively the maximum weighted distance) to all nodes of the graph given by $\sum_{v_i \in V} w_{v_i} d(F, v_i)$ (respectively $\max_{v_i \in V} w_{v_i} d(F, v_i)$).

Denote by $G - R$ the graph obtained from G by removing the subset R of edges or nodes.

We consider in this paper the k most vital edges (nodes) and min edge (node) blocker versions of the p-median and p-center problems. These problems are defined as follows:

k Most Vital Edges p-median (p-center)
Input: A connected graph $G = (V, E)$ weighted by two functions $d : E \to N$ and $w : V \to N$ and a positive integer k.
Output: A subset $S^* \subseteq E$, with $|S^*| = k$, whose removal generates an optimal solution for the p-median (p-center) problem in the graph $G - S^*$ of maximal value.

k Most Vital Nodes p-median (p-center)
Input: A connected graph $G = (V, E)$ weighted by two functions $d : E \to N$ and $w : V \to N$ and a positive integer k.
Output: A subset $N^* \subseteq V$, with $|N^*| = k$, whose removal generates an optimal solution for the p-median (p-center) problem in the graph $G - N^*$ of maximal value.

Min Edge Blocker p-median (p-center)
Input: A connected graph $G = (V, E)$ weighted by two functions $d : E \to N$ and $w : V \to N$ and a positive integer U.
Output: An *edge blocker* $S^* \subseteq E$ of minimal cardinality where an edge blocker is a subset of edges such that the value of an optimal solution for the p-median (p-center) problem in the graph $G - S^*$ is greater than or equal to U.

Min Node Blocker p-median (p-center)
Input: A connected graph $G = (V, E)$ weighted by two functions $d : E \to N$ and $w : V \to N$ and a positive integer U.
Output: A *node blocker* $N^* \subseteq V$ of minimal cardinality where a node blocker is a subset of nodes such that the value of an optimal solution for the p-median (p-center) problem in the graph $G - N^*$ is greater than or equal to U.

Given an NPO optimization problem and an instance I of this problem, we use $|I|$ to denote the size of I, $opt(I)$ to denote the optimum value of I, and $val(I, S)$ to denote the value of a feasible solution S of instance I. The *performance ratio* of S (or *approximation factor*) is $r(I, S) = \max \left\{ \frac{val(I,S)}{opt(I)}, \frac{opt(I)}{val(I,S)} \right\}$. The *error* of S, $\varepsilon(I, S)$, is defined by $\varepsilon(I, S) = r(I, S) - 1$.

For a function f, an algorithm is an $f(|I|)$-*approximation*, if for every instance I of the problem, it returns a solution S such that $r(I, S) \leq f(|I|)$.

The notion of a *gap*-reduction was introduced in [1] by Arora and Lund. In this paper we use *gap*-reductions from minimization problems to maximization problems. A minimization problem Π is called *gap-reducible* to a maximization problem Π' with parameters (c, ρ) and (c', ρ'), if there exists a polynomial time computable function f such that f maps an instance I of Π to an instance I' of Π', while satisfying the following properties.

- If $opt(I) \leq c$ then $opt(I') \geq c'$

- If $opt(I) > c\rho$ then $opt(I') < \frac{c'}{\rho'}$

Parameters c and ρ are function of $|I|$ and parameters c' and ρ' are function of $|I'|$. Also, we have $\rho, \rho' \geq 1$.

The interest of a *gap*-reduction is that if Π is not approximable within a factor ρ then Π' is not approximable within a factor ρ'.

The notion of an E-reduction (*error-preserving* reduction) was introduced in [9] by Khanna et al. A problem Π is called E-*reducible* to a problem Π', if there exist polynomial time computable functions f, g and a constant β such that

- f maps an instance I of Π to an instance I' of Π' such that $opt(I)$ and $opt(I')$ are related by a polynomial factor, i.e. there exists a polynomial p such that $opt(I') \leq p(|I|)opt(I)$,

- g maps any solution S' of I' to one solution S of I such that $\varepsilon(I, S) \leq \beta\varepsilon(I', S')$.

An important property of an E-reduction is that it can be applied uniformly to all levels of approximability; that is, if Π is E-reducible to Π' and Π' belongs to $\mathcal{C}$ then Π belongs to $\mathcal{C}$ as well, where $\mathcal{C}$ is a class of optimization problems with any kind of approximation guarantee (see also [9]).

3 Inapproximability results for the k Most Vital Edges (Nodes) p-median and p-center problems

We prove that k MOST VITAL EDGES (NODES) p-MEDIAN and k MOST VITAL EDGES (NODES) p-CENTER are not constant approximable for some constants,

unless $P=NP$. For this, we construct, in Theorems 1 to 4, *gap*-reductions from
MIN VERTEX COVER restricted to tripartite graphs. This problem is shown
NP-hard in [7] where Garey *et al.* prove that it is *NP*-hard to find a minimum
vertex cover in graphs with maximum degree 3, considering also that these
graphs, with the exception of the clique K_4, are 3-colorable [4].

Theorem 1 k MOST VITAL EDGES p-MEDIAN *is NP-hard to approximate within
a factor* $\frac{7}{5} - \epsilon$*, for any* $\epsilon > 0$ *and* $p \geq 1$.

Proof: We first prove the statement for $p = 1$. Let I be an instance of
MIN VERTEX COVER formed by a graph $G = (V, E)$ with a tripartition $V =
V_1 \cup V_2 \cup V_3$ and $|V| = n$. We construct an instance I' of k MOST VITAL
EDGES 1-MEDIAN consisting of a graph $G' = (V', E')$ with $k < n$ as follows
(see Figure 1). We associate for each node $v_\ell^i \in V_i$, two nodes $v_{\ell,1}^i$ and $v_{\ell,2}^i$ in
V' and connect them in E', for $i = 1, 2, 3$ and $\ell = 1, \ldots, |V_i|$. We add for each
edge $(v_\ell^i, v_r^j) \in E$, with $i < j$, the edge $(v_{\ell,2}^i, v_{r,1}^j)$ to E'. We also add four nodes
x_1, x_2, x_2', x_3 connected by the path $(x_1, x_2'), (x_2', x_2), (x_2, x_3)$. We connect x_1 to
$v_{\ell,1}^1$ for $\ell = 1, \ldots, |V_1|$, x_2' to $v_{\ell,1}^2$ and x_2 to $v_{\ell,2}^2$ for $\ell = 1, \ldots, |V_2|$ and x_3 to $v_{\ell,2}^3$
for $\ell = 1, \ldots, |V_3|$. We assign a distance 1 to edges $(x_1, x_2'), (x_1, v_{\ell,1}^1), (x_2', v_{j,1}^2),$
$(x_2, v_{j,2}^2)$ and (x_2', x_3) for $\ell = 1, \ldots, |V_1|$ and $j = 1, \ldots, |V_2|$, a distance 2 for the
edge (x_2', x_2) and a distance 0 for all the other edges in E'. We set $w_{x_1} = 8$,
$w_{x_2} = w_{x_3} = 1$ and assign a weight 0 to all other nodes in V'. We replace all
edges of E', except the edges $(v_{\ell,1}^i, v_{\ell,2}^i)$, for $i = 1, 2, 3$ and $\ell = 1, \ldots, |V_i|$, by the
gadget given in Figure 2. For each edge to be replaced, one chooses indifferently
the vertex playing the role of i in Figure 2, except for all edges incident to x_1
for which we take x_1 as i. We show in the following that:

1. $opt(I) \leq k \Rightarrow opt(I') \geq 7$

2. $opt(I) > k \Rightarrow opt(I') \leq 5$

which proves that k MOST VITAL EDGES 1-MEDIAN is *NP*-hard to approx-
imate within a factor $\frac{7}{5} - \epsilon$, for any $\epsilon > 0$.

First observe that there exists at least one optimal solution of k MOST
VITAL EDGES 1-MEDIAN containing only edges among the edges $(v_{\ell,1}^i, v_{\ell,2}^i)$, for
$i = 1, 2, 3$ and $\ell = 1, \ldots, |V_i|$. Indeed, if a solution contains edges from a gadget
corresponding to an initial edge (i, j), it must contain at least n edges from this
gadget in order to have a chance to increase the solution value by suppressing
communication between i and j. Therefore, since $k < n$, it is at least as good
to select k edges among those which do not belong to the gadgets.

Observe also that G' is designed so as to ensure that x_1 will always be the
optimal 1-median node. Indeed, since the weight of vertex x_1 is 8 and all edges
incident to x_1 have distance 1, any other node would have a total weighted
distance of at least 8. In the following, x_1 has always a total distance of at
most 7.
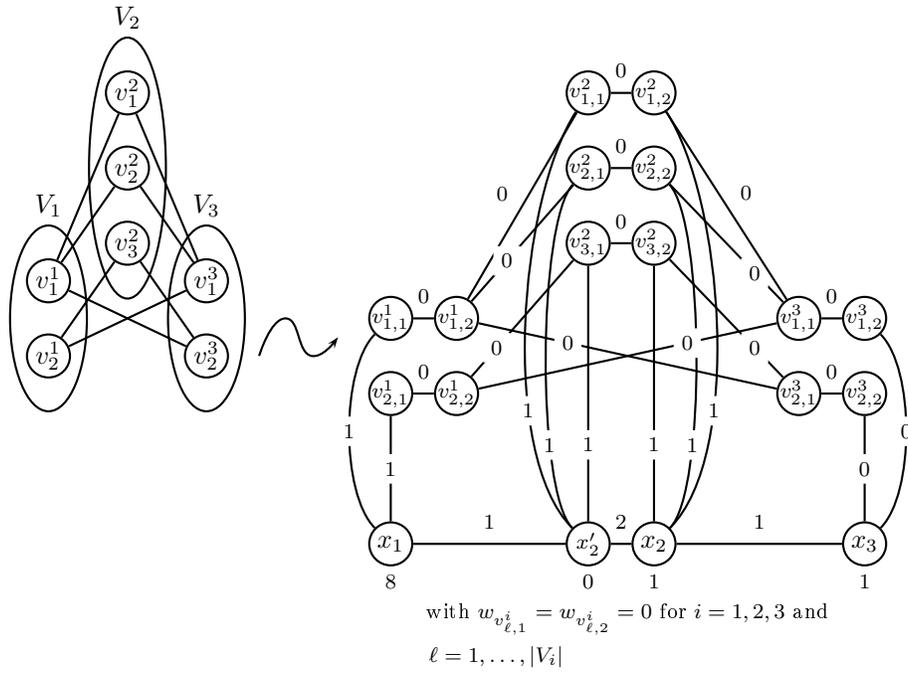
Figure 1: Construction of G' from G
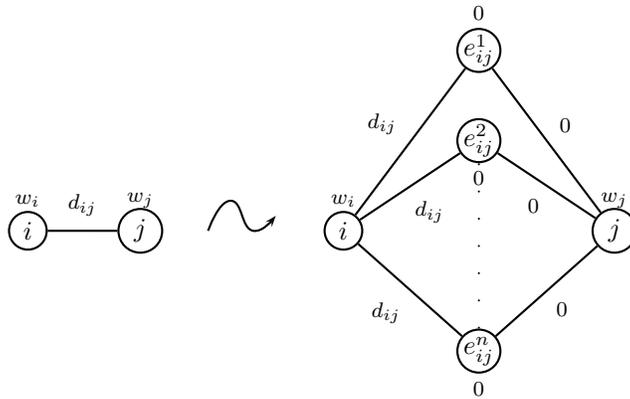
Figure 2: The replacement gadget of an edge $e = (i,j) \in E'$

1. If there exists a vertex cover $V' \subseteq V$ of cardinality less than k in G then consider any set of vertices $V'' \supset V'$ of cardinality k, and remove $S'' = \{(v^i_{\ell,1}, v^i_{\ell,2}) : v^i_\ell \in V''\}$ from G'. The optimal 1-median node in $G' - S''$ is x_1 with a total weighted distance $d(x_1, x_2) + d(x_1, x_3) = 3 + 4 = 7$. Hence, $opt(I') \geq 7$.

2. Let S^* be any solution of k MOST VITAL EDGES 1-MEDIAN which contains only edges $(v^i_{\ell,1}, v^i_{\ell,2})$, for $i = 1, 2, 3$ and $\ell = 1, \ldots, |V_i|$. The optimal 1-median node in $G' - S^*$ is x_1 with $opt(I') = d(x_1, x_2) + d(x_1, x_3)$. Each edge $(v^i_{\ell,1}, v^i_{\ell,2})$ of S^* corresponds to a node $v^i_\ell \in V_i$ in the graph G, for $i = 1, 2, 3$ and $\ell = 1, \ldots, |V_i|$. Let N^* be the subset of nodes in G that correspond to edges of S^*. Since $|N^*| = k$ and $opt(I) > k$, N^* is not a vertex cover in G. Thus, there exists at least one edge $(v^i_\ell, v^j_r) \in E$ which is not covered. This implies in G' the existence of a path from x_i (or x'_i) to x_j, with $i < j$, passing through the gadget corresponding to the edge $(v^i_{\ell,2}, v^j_{r,1})$, enabling a decrease of some shortest path distances. Hence,

   - if $i = 1$ and $j = 2$ then $opt(I') \leq 5$
   - if $i = 1$ and $j = 3$ then $opt(I') \leq 3$
   - if $i = 2$ and $j = 3$ then $opt(I') \leq 5$

   Therefore, $opt(I') \leq 5$.

We now prove the statement for $p \geq 2$. We use the same construction as above for $p = 1$, and we add $p - 1$ nodes $x^1_1, \ldots, x^{p-1}_1$. We connect x^r_1 to x'_2 and $v^1_{\ell,1}$, for $r = 1, \ldots, p - 1$ and $\ell = 1, \ldots, |V_1|$, and we assign a distance 1 to all these edges. We assign a weight 8 to each node x^r_1, for $r = 1, \ldots, p - 1$. Finally, we replace all edges (x^r_1, x'_2) and $(x^r_1, v^1_{\ell,1})$ for $\ell = 1, \ldots, |V_1|$ and $r = 1, \ldots, p - 1$, by the gadget given in Figure 2 where we take x^r_1 as i, for $r = 1, \ldots, p - 1$.

As previously, there exists at least one optimal solution of k MOST VITAL EDGES p-MEDIAN containing only edges among the edges $(v^i_{\ell,1}, v^i_{\ell,2})$, for $i = 1, 2, 3$ and $\ell = 1, \ldots, |V_i|$. Furthermore, observe that G' is designed so as to ensure that x_1 and x^r_1 for $r = 1, \ldots, p - 1$ will always be the optimal p-median nodes. Indeed, since the weight of these vertices is 8 and all edges incident to them have distance 1, any other node would have a total weighted distance of at least 8. Therefore, we can prove similarly as for $p = 1$, that:

1. $opt(I) \leq k \Rightarrow opt(I') \geq 7$

2. $opt(I) > k \Rightarrow opt(I') \leq 5$

which proves that k MOST VITAL EDGES p-MEDIAN is NP-hard to approximate within a factor $\frac{7}{5} - \epsilon$, for any $\epsilon > 0$. $\qquad \square$

Theorem 2 k Most Vital Edges p-center *is NP-hard to approximate within a factor* $\frac{4}{3} - \epsilon$, *for any* $\epsilon > 0$ *and* $p \geq 1$.

Proof: We first prove the statement for $p = 1$. We use the same construction as in Theorem 1 for $p = 1$. We show that:

1. $opt(I) \leq k \Rightarrow opt(I') \geq 4$

2. $opt(I) > k \Rightarrow opt(I') \leq 3$

which proves that k Most Vital Edges 1-center is *NP*-hard to approximate within a factor $\frac{4}{3} - \epsilon$, for any $\epsilon > 0$.

Similarly as above, there exists at least one optimal solution of k Most Vital Edges 1-center containing only edges among the edges $(v_{\ell,1}^i, v_{\ell,2}^i)$, for $i = 1, 2, 3$ and $\ell = 1, \ldots, |V_i|$. Moreover, as before, x_1 will always be the optimal 1-center node.

1. If there exists a vertex cover $V' \subseteq V$ of cardinality less than k in G then consider any set of vertices $V'' \supset V'$ of cardinality k, and remove $S'' = \{(v_{\ell,1}^i, v_{\ell,2}^i) : v_\ell^i \in V''\}$ from G'. The optimal 1-center node in $G' - S''$ is x_1 with a maximum weighted distance $\max\{d(x_1, x_2), d(x_1, x_3)\} = 4$. Hence, $opt(I') \geq 4$.

2. Let S^* be any solution of k Most Vital Edges 1-center which contains only edges $(v_{\ell,1}^i, v_{\ell,2}^i)$, for $i = 1, 2, 3$ and $\ell = 1, \ldots, |V_i|$. The optimal 1-center node in $G' - S^*$ is x_1 with $opt(I') = \max\{d(x_1, x_2), d(x_1, x_3)\}$. Each edge $(v_{\ell,1}^i, v_{\ell,2}^i)$ of S^* corresponds to a node $v_\ell^i \in V_i$ in the graph G, for $i = 1, 2, 3$ and $\ell = 1, \ldots, |V_i|$. Let N^* be the subset of nodes of G corresponding to edges in S^*. Since $|N^*| = k$ and $opt(I) > k$, N^* is not a vertex cover in G. Thus, there exists at least one edge $(v_\ell^i, v_r^j) \in E$ which is not covered. This implies in G' the existence of a path from x_i (or x_i') to x_j, with $i < j$, passing through the gadget corresponding to the edge $(v_{\ell,2}^i, v_{r,1}^j)$. Hence,

   - if $i = 1$ and $j = 2$ then $opt(I') \leq 3$
   - if $i = 1$ and $j = 3$ then $opt(I') \leq 3$
   - if $i = 2$ and $j = 3$ then $opt(I') \leq 3$

   Therefore, $opt(I') \leq 3$.

We now prove the statement for $p \geq 2$. We use the same construction as in Theorem 1 for $p \geq 2$.

Similarly as for k Most Vital Edges p-median, there exists at least one optimal solution of k Most Vital Edges p-center containing only edges among the edges $(v_{\ell,1}^i, v_{\ell,2}^i)$, for $i = 1, 2, 3$ and $\ell = 1, \ldots, |V_i|$. Moreover, as in Theorem 1, x_1 and x_1^r, for $r = 1, \ldots, p-1$, will always be the optimal p-center nodes. Therefore, we can prove similarly as for $p = 1$, that:

1. $opt(I) \leq k \Rightarrow opt(I') \geq 4$

2. $opt(I) > k \Rightarrow opt(I') \leq 3$

which proves that k MOST VITAL EDGES p-CENTER is NP-hard to approximate within a factor $\frac{4}{3} - \epsilon$, for any $\epsilon > 0$. □

Theorem 3 k MOST VITAL NODES p-MEDIAN *is NP-hard to approximate within a factor* $\frac{3}{2} - \epsilon$, *for any* $\epsilon > 0$ *and* $p \geq 1$.

Proof: We begin by proving the statement for $p = 1$. Let I be an instance of MIN VERTEX COVER formed by a graph $G = (V, E)$ with a tripartition $V = V_1 \cup V_2 \cup V_3$ and $|V| = n$. We construct an instance I' of k MOST VITAL NODES 1-MEDIAN consisting of a graph $G' = (V', E')$ with $k < n$ as follows (see Figure 3). G' is a copy of G to which we add complete graphs K_n^i with n nodes $x_i^1, \ldots, x_i^n$ for $i = 1, 2, 3$. We connect each node $v_\ell^i \in V_i$ with each node x_i^r, for $i = 1, 2, 3$, $\ell = 1, \ldots, |V_i|$ and $r = 1, \ldots, n$. We connect also each node x_i^r to each node x_{i+1}^r for $i = 1, 2$ and $r = 1, \ldots, n$. We assign a distance 2 to edges (x_i^r, x_{i+1}^r) for $i = 1, 2$ and $r = 1, \ldots, n$, a distance 1 to edges (x_1^r, v_ℓ^1) for $\ell = 1, \ldots, |V_1|$ and $r = 1, \ldots, n$ and a distance 0 to all other edges in E'. We set $w_{x_1^r} = 7$ and $w_{x_2^r} = w_{x_3^r} = 1$ for $r = 1, \ldots, n$, and $w_{v_\ell^i} = 0$ for $i = 1, 2, 3$, $\ell = 1, \ldots, |V_i|$. We show in the following that:

1. $opt(I) \leq k \Rightarrow opt(I') \geq 6n$

2. $opt(I) > k \Rightarrow opt(I') \leq 4n$

which proves that k MOST VITAL NODES 1-MEDIAN is NP-hard to approximate within a factor $\frac{3}{2} - \epsilon$, for any $\epsilon > 0$.

First observe that there exists at least one optimal solution of k MOST VITAL NODES 1-MEDIAN containing only nodes of V. Indeed, if a solution contains nodes from K_n^i for some i, it must contain all nodes of K_n^i in order to have a chance to increase the solution value by disconnecting these nodes from the graph. Therefore, since $k < n$, it is at least as good to select k nodes in V only.

Observe also that G' is designed so as to ensure that any node x_1^r for $r = 1, \ldots, n$ will always be an optimal 1-median node. Indeed, since the weight of a vertex x_1^r is 7 and all edges incident to x_1^r, except the edges (x_1^r, x_1^j) for $j = 1, \ldots, n$ and $j \neq r$ have distance at least 1, any other node would have a total weighted distance of at least 7, while any node x_1^r has always a total weighted distance of at most 6. We consider arbitrarily in the following that x_1^1 is the selected optimal 1-median node.

1. If there exists a vertex cover $V' \subseteq V$ of cardinality less than k in G then consider any set of vertices $V'' \supset V'$ of cardinality k, and remove V'' from G'. Taking x_1^1 as the optimal 1-median node in $G' - V''$, we get a total weighted distance $\sum_{j=1}^{n}(d(x_1^1, x_2^j) + d(x_1^1, x_3^j)) = \sum_{j=1}^{n}(2 + 4) = 6n$. Hence, $opt(I') \geq 6n$.

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with $w_{v_\ell} = 0$ for $v_\ell \in V_i$, $i = 1, 2, 3$
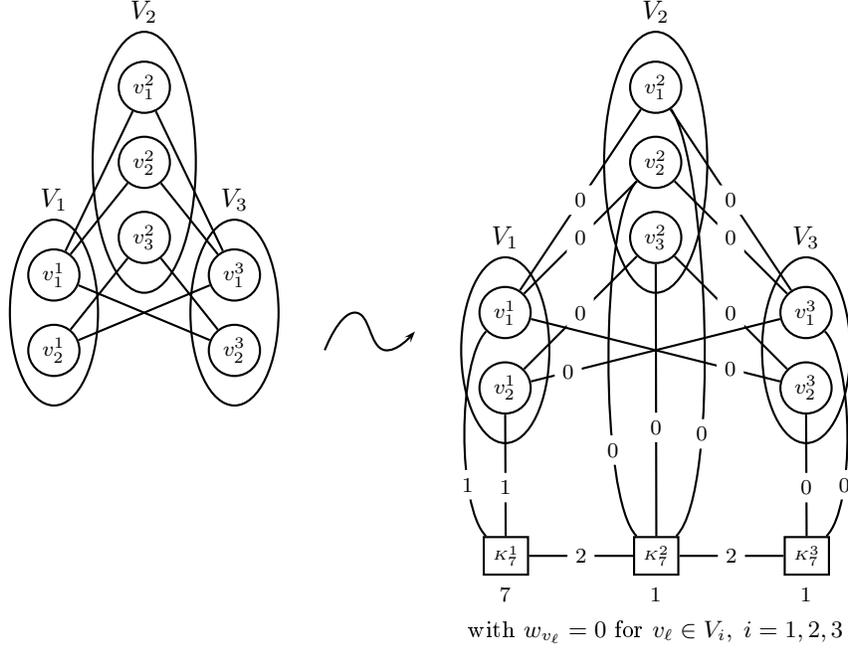
Figure 3: Construction of G' from G

2. Let $N^* \subseteq V$ be any solution of k MOST VITAL NODES 1-MEDIAN which contains only nodes of V. Taking x_1^1 as the optimal 1-median node in $G' - N^*$, we get $opt(I') = \sum_{\ell=1}^{n}(d(x_1^1, x_2^\ell) + d(x_1^1, x_3^\ell))$. Since $|N^*| = k$ and $opt(I) > k$, N^* is not a vertex cover in G. Thus, there exists at least one edge $(v_i, v_j) \in E$ which is not covered. This implies in G' the existence of a path from each x_i^r to each x_j^r for $r = 1, \ldots, n$, passing through the edge (v_i, v_j). Hence,

- if $i = 1$ and $j = 2$ then $opt(I') \leq \sum_{\ell=1}^{n}(1 + 3) = 4n$
- if $i = 1$ and $j = 3$ then $opt(I') \leq \sum_{\ell=1}^{n}(2 + 1) = 3n$
- if $i = 2$ and $j = 3$ then $opt(I') \leq \sum_{\ell=1}^{n}(2 + 2) = 4n$

Consequently, $opt(I') \leq 4n$.

We now prove the statement for $p \geq 2$. We use the same construction as above for $p = 1$, and we add $p - 1$ complete graphs $K_n^{1,h}$ with n nodes $x_{1,h}^1, \ldots, x_{1,h}^n$ for $h = 1, \ldots, p - 1$. We connect each node $v_\ell^1 \in V_1$ with each node $x_{1,h}^r$, for $\ell = 1, \ldots, |V_1|$, $h = 1, \ldots, p-1$ and $r = 1, \ldots, n$. We connect also each node $x_{1,h}^r$ to each node x_2^r for $h = 1, \ldots, p-1$ and $r = 1, \ldots, n$. We assign a distance 2 to edges $(x_{1,h}^r, x_2^r)$ for $h = 1, \ldots, p-1$ and $r = 1, \ldots, n$, a distance 1 to edges $(x_{1,h}^r, v_\ell^1)$ for $\ell = 1, \ldots, |V_1|$, $h = 1, \ldots, p-1$ and $r = 1, \ldots, n$ and a

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distance 0 to all edges in $K_n^{1,h}$ for $h = 1, \ldots, p-1$. Finally, we set $w_{x_{1,h}^r} = 7$ for $h = 1, \ldots, p-1$ and $r = 1, \ldots, n$.

As previously, there exists at least one optimal solution of k Most Vital Nodes p-median containing only nodes of V. Furthermore, observe that G' is designed so as to ensure that any node x_1^r for $r = 1, \ldots, n$ and any node $x_{1,h}^r$ for $r = 1, \ldots, n$ and $h = 1, \ldots, p-1$ will always be an optimal p-median nodes. Indeed, since the weight of these vertices is 7 and all edges incident to them, except the edges (x_1^r, x_1^j) for $j = 1, \ldots, n$ and $j \neq r$, have a distance at least 1, any other subset of p nodes would have a total weighted distance of at least 7. Therefore, we can prove similarly as for $p = 1$, that:

1. $opt(I) \leq k \Rightarrow opt(I') \geq 6n$

2. $opt(I) > k \Rightarrow opt(I') \leq 4n$

which proves that k Most Vital Nodes p-median is NP-hard to approximate within a factor $\frac{3}{2} - \epsilon$, for any $\epsilon > 0$. □

Theorem 4 k Most Vital Nodes p-center *is NP-hard to approximate within a factor $\frac{3}{2} - \epsilon$, for any $\epsilon > 0$ and $p \geq 1$.*

Proof: We first prove the statement for $p = 1$. We use the same construction as in Theorem 3 for $p = 1$, but we modify the distance associated to the edges (x_2^r, x_3^r) for $r = 1, \ldots, n$ for which we assign a distance 1. We show that:

1. $opt(I) \leq k \Rightarrow opt(I') \geq 3$

2. $opt(I) > k \Rightarrow opt(I') \leq 2$

which proves that k Most Vital Nodes 1-center is NP-hard to approximate within a factor $\frac{4}{3} - \epsilon$, for any $\epsilon > 0$.

As previously, we can show that only the nodes of V can be removed. We observe as above that any node x_1^r for $r = 1, \ldots, n$ will always be an optimal 1-center node. We consider arbitrarily in the following that x_1^1 is the selected optimal 1-center node.

1. If there exists a vertex cover $V' \subseteq V$ of cardinality less than k in G then consider any set of vertices $V'' \supset V'$ of cardinality k, and remove V'' from G'. Taking x_1^1 as the optimal 1-center node in $G' - V''$, we get a maximum weighted distance of value $\max \{ \max_{j=1,\ldots,n} d(x_1^1, x_2^j), \max_{j=1,\ldots,n} d(x_1^1, x_3^j) \} = 3$. Hence, $opt(I') \geq 3$.

2. Let $N^* \subseteq V$ be any solution of k Most Vital Nodes 1-center which contains only nodes of V. Taking x_1^1 as the optimal 1-center node in $G' - N^*$, we get $opt(I') = \max \{ \max_{\ell=1,\ldots,n} d(x_1^1, x_2^\ell), \max_{\ell=1,\ldots,n} d(x_1^1, x_3^\ell) \}$. Since

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$|N^*| = k$ and $opt(I) > k$, N^* is not a vertex cover in G. Thus, there exists at least one edge $(v_i, v_j) \in E$ which is not covered. This implies in G' the existence of a path from each x_i^r to each x_j^r for $r = 1, \ldots, n$, passing through the edge (v_i, v_j). Hence,

- if $i = 1$ and $j = 2$ then $opt(I') = \max\{d(x_1^1, x_2^1), d(x_1^1, x_3^1)\} \leq 2$
- if $i = 1$ and $j = 3$ then $opt(I') = d(x_1^1, x_2^1) \leq 2$
- if $i = 2$ and $j = 3$ then $opt(I\prime) = \max\{d(x_1^1, x_2^1), d(x_1^1, x_3^1)\} \leq 2$.

Therefore, $opt(I') \leq 2$.

We now prove the statement for $p \geq 2$. We use the same construction as in Theorem 3 for $p \geq 2$, but we modify the distance associated to the edges (x_2^r, x_3^r) for $r = 1, \ldots, n$ for which we assign a distance 1.

Similarly as for k Most Vital Edges p-median, we can show that only the nodes of V can be removed. Moreover, as in Theorem 3, any node x_1^r for $r = 1, \ldots, n$ and any node $x_{1,h}^r$ for $r = 1, \ldots, n$ and $h = 1, \ldots, p-1$, will always be the optimal p-center nodes. Therefore, we can prove similarly as for $p = 1$, that:

1. $opt(I) \leq k \Rightarrow opt(I') \geq 3$

2. $opt(I) > k \Rightarrow opt(I') \leq 2$

which proves that k Most Vital Nodes 1-center is NP-hard to approximate within a factor $\frac{4}{3} - \epsilon$, for any $\epsilon > 0$. $\qquad\square$

4 Inapproximability results for Min Edge (Node) Blocker p-median and p-center problems

We prove that the four problems Min Edge (Node) Blocker p-median and Min Edge (Node) Blocker p-center are not 1.36 approximable, unless $P{=}NP$. These results, stated in Theorems 5 to 8, are obtained by constructing E-reductions from Min Vertex Cover shown NP-hard to approximate within a factor 1.36 [5].

Theorem 5 Min Edge Blocker p-median *is NP-hard to approximate within a factor 1.36, for any $p \geq 1$.*

Proof: We first prove the statement for $p = 1$. Let I be an instance of Min Vertex Cover consisting of a graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$. We construct an instance I' of Min Edge Blocker 1-median formed by a graph $G' = (V', E')$ and a positive integer U as follows (see Figure 4). We associate for each node $v_i \in V$ two nodes v_i and v_i' in V' and connect them in

E' for $i = 1, \ldots, n$. We add for each edge $(v_i, v_j) \in E$, with $i < j$, an edge (v_i', v_j) to E'. We also add $2n$ nodes $x_1, x_1', x_2, x_2', \ldots, x_n, x_n'$ connected by the path (x_1, x_1'), (x_1', x_2), (x_2, x_2'), (x_2', x_3), $\ldots, (x_{n-1}', x_n)$, (x_n, x_n'). Finally, we connect x_i to v_i and x_i' to v_i' for $i = 1, \ldots, n$. We assign the following distances to the edges of E': $d_{v_i v_i'} = 0$, $d_{x_i v_i} = d_{x_i' v_i'} = 1$ and $d_{x_i x_i'} = 2$ for $i = 1, \ldots, n$, $d_{x_i' x_{i+1}} = 0$ for $i = 1, \ldots, n-1$ and $d_{v_i' v_j} = 2(j-i)-1$ for $(v_i, v_j) \in E$ and $i < j$. We set $w_{x_1} = 2n^2 + 1$, $w_{x_i} = 1$ for $i = 2, \ldots, n$, $w_{x_i'} = 1$ and $w_{v_i} = w_{v_i'} = 0$ for $i = 1, \ldots, n$ and we consider that $U = 2n^2$. We replace each edge of E', except the edges (v_i, v_i') for $i = 1, \ldots, n$, by the gadget given in Figure 2 where each edge is replaced by $n + 1$ instead of n disjoint paths of length 2 (for edges (x_1, v_1) and (x_1, x_1'), x_1 plays the role of i in Figure 2).

with $w_{v_\ell} = w_{v_\ell'} = 0$ for $\ell = 1, \ldots, 4$
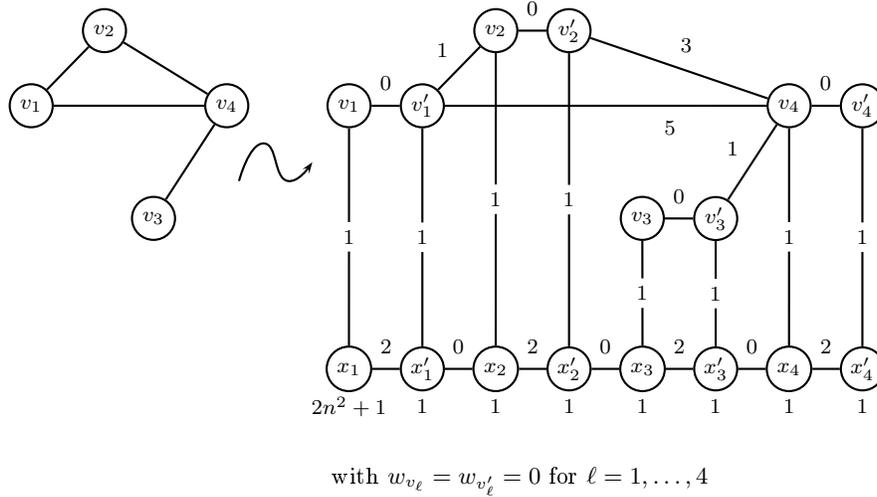
Figure 4: Construction of G' from G with $n = 4$ nodes

Observe that G' is designed so as to ensure that x_1 will always be the optimal 1-median node. Indeed, since the weight of vertex x_1 is $2n^2 + 1$ and all edges incident to x_1 have distance at least 1, any other node would have a total weighted distance of at least $2n^2 + 1$. In the following, x_1 has always a total distance of at most $2n^2$.

We prove first that $opt(I') \leq opt(I)$. Let $V^* \subseteq V$ be a minimum vertex cover of G. Let us consider $S^* = \{(v_i, v_i') : v_i \in V^*\}$. By removing the edges in S^* from G', the optimal 1-median node is x_1 with a total weighted distance $\sum_{i=1}^{n} w_{x_i} d(x_1, x_i) + \sum_{i=1}^{n} w_{x_i'} d(x_1, x_i') = 2(\sum_{i=1}^{n-1} i + \sum_{i=1}^{n} i) = 2n^2 = U$. Hence, $opt(I') \leq |S^*| = opt(I)$.

When we remove all edges (v_i, v_i'), for $i = 1, \ldots, n$ from G', the optimal 1-median node in the resulting graph is x_1 with value U. Hence, $opt(I') \leq n$. Let $S \subseteq E'$

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be an edge blocker for G'. If S contains an edge (i, e_{ij}^ℓ) or (e_{ij}^ℓ, j) from a gadget corresponding to an initial edge (i, j), it must contain at least $n + 1$ edges from this gadget in order to suppress the communication between i and j, otherwise the value of an optimal solution for the 1-median problem in $G' - S$ is the same as in $G' - (S \backslash \{(i, e_{ij}^\ell)\})$ or $G' - (S \backslash \{(e_{ij}^\ell, j)\})$. Therefore, since $opt(I') \leq n$, we can consider in the following that S contains only edges among the edges (v_i, v_i'), $i \in \{1, \ldots, n\}$.

Let us consider $N = \{v_i : (v_i, v_i') \in S\}$ where S is an edge blocker. We prove, by contradiction, that N is a vertex cover in G. Suppose that there exists an edge $(v_i, v_j) \in E$ such that $v_i \notin N$, $v_j \notin N$ and $i < j$. We show in the following that by removing S from G', the value of an optimal solution for the 1-median problem in the remaining graph is strictly less than $2n^2$. Indeed, x_1 is the optimal 1-median node in $G' - S$. Let $D(x_1)$ be the total weighted distance associated to x_1 in $G' - S$. We have $D(x_1) = \sum_{\ell=1}^{n} d(x_1, x_\ell') + \sum_{\ell=1}^{n} d(x_1, x_\ell) = \sum_{\ell=1}^{j-1} d(x_1, x_\ell') + d(x_1, x_j') + \sum_{\ell=j+1}^{n} d(x_1, x_\ell') + \sum_{\ell=1}^{n} d(x_1, x_\ell)$. Then, $D(x_1) \leq 2\sum_{\ell=1}^{j-1} \ell + d(x_1, x_j') + 2\sum_{\ell=j+1}^{n} \ell + 2\sum_{\ell=1}^{n-1} \ell = 2\sum_{\ell=1}^{n} \ell - 2j + d(x_1, x_j') + 2\sum_{\ell=1}^{n-1} \ell = 2n^2 - 2j + d(x_1, x_j')$. The edge (v_i, v_j) being not covered, this implies the existence of a path from x_1 to x_j' using a subpath from x_1 to x_i and joining x_i to x_j' by a subpath passing through the gadget associated to the edge (v_i, v_j). We have $d(x_1, x_j') \leq 2(i - 1) + 1 + 2(j - i) - 1 + 1 = 2j - 1$. Thus, we have $D(x_1) \leq 2n^2 - 1 < 2n^2$, contradicting the assumption that S is an edge blocker. Therefore, N is a vertex cover in G such that $val(I, N) = val(I', S)$. Consequently, $\varepsilon(I, N) = \frac{val(I,N)}{opt(I)} - 1 \leq \frac{val(I',S)}{opt(I')} - 1 = \varepsilon(I', S)$, which achieves the proof for $p = 1$.

We now prove the statement for $p \geq 2$. We use the same construction as above for $p = 1$, and we add $p - 1$ nodes $x_1^1, \ldots, x_1^{p-1}$. We connect x_1^r to x_1' and v_1, for $r = 1, \ldots, p - 1$. We assign a distance 1 to edges (x_1^r, v_1) and a distance 2 to edges (x_1^r, x_1'), for $r = 1, \ldots, p - 1$. We assign a weight $2n^2 + 1$ to each node x_1^r, for $r = 1, \ldots, p - 1$. Finally, we replace all edges (x_1^r, x_1') and (x_1^r, v_1) for $r = 1, \ldots, p - 1$, by the gadget given in Figure 2 where we take x_1^r as i, for $r = 1, \ldots, p - 1$. We set $U = 2n^2$.

Observe that G' is designed so as to ensure that x_1 and x_1^r, for $r = 1, \ldots, p-1$, will always be the optimal p-median nodes. Indeed, since the weight of these vertices is $2n^2 + 1$ and all edges incident to them have distance at least 1, any other subset of p nodes would have a total weighted distance of at least $2n^2 + 1$. As previously, we can assume that only edges among edges (v_i, v_i') can be removed. Therefore, similarly to $p = 1$, we prove that $opt(I') \leq opt(I)$ and that for an edge blocker S in G' we construct in a polynomial time a vertex cover N in G such that $val(I, N) = val(I', S)$. Consequently, $\varepsilon(I, N) \leq \varepsilon(I', S)$, which achieves the proof for $p \geq 2$. $\square$

Theorem 6 MIN EDGE BLOCKER p-CENTER *is NP-hard to approximate within a factor 1.36, for any $p \geq 1$.*

Proof: We first prove the statement for $p = 1$. We use the same construction as in Theorem 5 for $p = 1$ with $U = 2n$. As above in Theorem 5, G' is designed so as to ensure that x_1 will always be the optimal 1-center node.

We show first that $opt(I') \leq opt(I)$. Let $V^* \subseteq V$ be a minimum vertex cover in G. Let us consider $S^* = \{(v_i, v_i') : v_i \in V^*\}$. By removing the edges of S^* from the graph G', the optimal 1-center node is x_1 with a maximum weighted distance $d(x_1, x_n') = 2n = U$. Hence, $opt(I') \leq |S^*| = opt(I)$.

Let $S \subseteq E'$ be an edge blocker. We can assume, similarly to the 1-median problem, that S contains only edges among the edges (v_i, v_i'), $i \in 1, \ldots, n$. Let us consider $N = \{v_i : (v_i, v_i') \in S\}$. In the following, we show by contradiction that N is a vertex cover in G. Suppose that there exists an edge $(v_i, v_j) \in E$ such that $v_i \notin N$, $v_j \notin N$ and $i < j$. Then x_1 is the optimal 1-center node in $G' - S$ with a maximum weighted distance $D_{max}(x_1) = d(x_1, x_n)$. The edge (v_i, v_j) being not covered, this implies the existence of a path from x_1 to x_j' using a subpath from x_1 to x_i and joining x_i to x_j' by a subpath passing through the gadget associated to the edge (v_i, v_j). Then $D_{max}(x_1) \leq 2(i-1) + 1 + 2(j-i) - 1 + 1 + 2(n-j) = 2n - 1 < 2n$, contradicting the assumption that S is an edge blocker. Therefore N is a vertex cover in G such that $val(I, N) = val(I', S)$. Consequently, $\varepsilon(I, N) = \frac{val(I,N)}{opt(I)} - 1 \leq \frac{val(I',S)}{opt(I')} - 1 = \varepsilon(I', S)$, which achieves the proof for $p = 1$.

We now prove the statement for $p \geq 2$. We use the same construction as in Theorem 5 for $p \geq 2$ with $U = 2n$.

Similarly to Min Edge Blocker p-median, G' is designed so as to ensure that x_1 and x_1^r, for $r = 1, \ldots, p - 1$, will always be the optimal p-center nodes. Also, we can assume that only edges among edges (v_i, v_i') can be removed. Therefore, as for Min Edge Blocker 1-center, we prove that $opt(I') \leq opt(I)$ and that for an edge blocker S in G' we construct in a polynomial time a vertex cover N in G such that $val(I, N) = val(I', S)$. Consequently, $\varepsilon(I, N) \leq \varepsilon(I', S)$, which achieves the proof for $p \geq 2$. $\qquad\square$

Theorem 7 Min Node Blocker p-median *is NP-hard to approximate within a factor 1.36, for any $p \geq 1$.*

Proof: We first prove the statement for $p = 1$. Let I be an instance of Min Vertex Cover consisting of a graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$. We construct an instance I' of Min Node Blocker 1-median formed by a graph $G' = (V', E')$ and a positive integer U as follows (see Figure 5). G' is a copy of G to which we add one node x_1 and complete graphs K_{n+1}^i with $n + 1$ nodes $x_i^1, \ldots, x_i^{n+1}$ for $i = 2, \ldots, n$. We connect x_1 to v_1 and x_2^r for $r = 1, \ldots, n + 1$, and each node x_i^r to v_i for $i = 2, \ldots, n$ and $r = 1, \ldots, n + 1$. We also connect each node x_i^r to each node x_{i+1}^r for $r = 1, \ldots, n + 1$ and $i = 2, \ldots, n - 1$. We assign a distance 1 to the edge (x_1, v_1), a distance 2 to the edges (x_1, x_2^r) and (x_i^r, x_{i+1}^r) for $i = 2, \ldots, n - 1$ and $r = 1, \ldots, n + 1$, and a distance 0 to all other

edges in E'. Let us set $w_{x_1} = n^3$, $w_{x_i^r} = 1$ for $i = 2, \ldots, n$ and $r = 1, \ldots, n+1$ and $w_{v_i} = 0$ for $i = 1, \ldots, n$. Finally, we set $U = n(n^2 - 1)$.
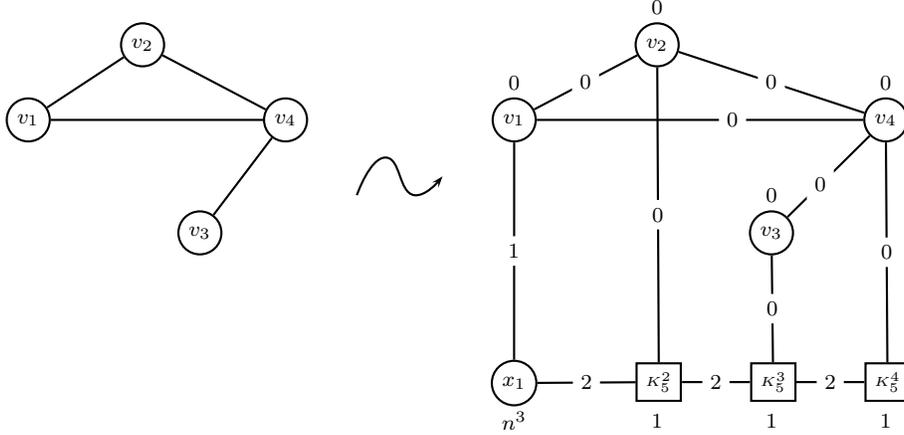
Figure 5: Construction of G' from G with $n = 4$ nodes

Observe that G' is designed so as to ensure that x_1 will always be the optimal 1-median node. Indeed, since the weight of vertex x_1 is n^3 and all edges incident to x_1 have distance at least 1, any other node would have a total weighted distance of at least n^3. In the following, x_1 has always a total distance of at most $n(n^2 - 1)$.

We show first that $opt(I') \leq opt(I)$. Let $V^* \subseteq V$ be a minimum vertex cover in G. By removing V^* from G', the optimal 1-median node is x_1 with a total weighted distance $\sum_{\ell=2}^{n} \sum_{r=1}^{n+1} d(x_1, x_\ell^r) = 2(n+1) \sum_{i=1}^{n-1} i = n(n-1)(n+1) = U$. Hence, $opt(I') \leq |V^*| = opt(I)$.

Let $N \subseteq V'$ be a node blocker. According to the construction of G', in order to obtain an optimal solution for the 1-median problem in the graph $G' - N$ of a value at least U, N must be included in V. We show, by contradiction, that N is a vertex cover in G. Suppose that there exists an edge $(v_i, v_j) \in E$ such that $v_i \notin N$, $v_j \notin N$ and $i < j$. The optimal 1-median node in $G' - N$ is x_1 with value strictly less than $n(n-1)(n+1)$. Indeed, let $D(x_1)$ be the total weighted distance associated to x_1 in $G' - N$. Hence, $D(x_1) = \sum_{\ell=2}^{n} \sum_{r=1}^{n+1} d(x_1, x_\ell^r) = \sum_{\ell=2}^{j-1} \sum_{r=1}^{n+1} d(x_1, x_\ell^r) + \sum_{r=1}^{n+1} d(x_1, x_j^r) + \sum_{r=1}^{n+1} \sum_{\ell=j+1}^{n} d(x_1, x_\ell^r))$. We distinguish two cases:

- If $v_i = v_1$ then $d(x_1, x_j^r) = d_{x_1 v_1} + d_{v_1 v_j} + d_{v_j x_j^r} = 1$ for $r = 1, \ldots, n+1$. Hence, we obtain $D(x_1) \leq 2(n+1) \sum_{\ell=1}^{j-2} \ell + (n+1) + 2(n+1) \sum_{\ell=j}^{n-1} \ell < 2(n+1) \sum_{\ell=1}^{j-2} \ell + 2(j-1)(n+1) + 2(n+1) \sum_{\ell=j}^{n-1} \ell = n(n-1)(n+1)$, contradiction.

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- If $v_i \neq v_1$ then $d(x_1, x_j^r) = d(x_1, x_i^1) + d_{x_i^1 v_i} + d_{v_i v_j} + d_{v_j x_j^r} = d(x_1, x_i^1)$ for $r = 1, \ldots, n+1$. Hence, we obtain $D(x_1) \leq 2(n+1)\sum_{\ell=1}^{j-2} \ell + 2(i-1)(n+1) + 2(n+1)\sum_{\ell=j}^{n-1} \ell < 2(n+1)\sum_{\ell=1}^{j-2} \ell + 2(j-1)(n+1) + 2(n+1)\sum_{\ell=j}^{n-1} \ell = n(n-1)(n+1)$, contradiction.

Therefore N is a vertex cover in G such that $val(I, N) = val(I', N)$. Consequently, $\varepsilon(I, N) = \frac{val(I,N)}{opt(I)} - 1 \leq \frac{val(I',N)}{opt(I')} - 1 = \varepsilon(I', N)$, which achieves the proof for $p = 1$.

We now prove the statement for $p \geq 2$. We use the same construction as above for $p = 1$, and we add $p-1$ nodes $x_1^1, \ldots, x_1^{p-1}$. We connect x_1^h to v_1 and x_2^r, for $r = 1, \ldots, n+1$ and $h = 1, \ldots, p-1$. We assign a distance 1 to edges (x_1^h, v_1) for $h = 1, \ldots, p-1$, and a distance 2 to edges (x_1^h, x_2^r) for $r = 1, \ldots, n+1$ and $h = 1, \ldots, p-1$. We assign a weight n^3 to each node x_1^h, for $h = 1, \ldots, p-1$. We set $U = n(n^2 - 1)$.

Observe that G' is designed so as to ensure that x_1 and x_1^h, for $h = 1, \ldots, p-1$, will always be the optimal p-median nodes. Indeed, since the weight of these vertices is n^3 and all edges incident to them have distance at least 1, any other subset of p nodes would have a total weighted distance of at least n^3. As previously, according to the construction of G', in order to obtain an optimal solution for the p-median problem in resulting graph of a value at least U, we can only remove nodes from V. Therefore, similarly to $p = 1$, we prove that $opt(I') \leq opt(I)$ and that for a node blocker N in G', N is a vertex cover in G such that $val(I, N) = val(I', N)$. Consequently, $\varepsilon(I, N) \leq \varepsilon(I', N)$, which achieves the proof for $p \geq 2$. $\qquad\square$

Theorem 8 MIN NODE BLOCKER p-CENTER *is NP-hard to approximate within a factor 1.36, for any $p \geq 1$.*

Proof: We first prove the statement for $p = 1$. We use the same construction as in Theorem 7 for $p = 1$ with $U = 2(n-1)$. Here again, we observe that G' is designed so as to ensure that x_1 will always be the optimal 1-center node.

We show first that $opt(I') \leq opt(I)$. Let $V^* \subseteq V$ be a minimum vertex cover in G. By deleting the nodes of V^* from G', the optimal 1-center node in the remaining graph is x_1 with a maximum weighted distance $d(x_1, x_n^r) = 2(n-1) = U$ for any $r = 1, \ldots, n+1$. Hence, $opt(I') \leq |V^*| = opt(I)$.

When we remove all nodes v_i, $i = 1, \ldots, n$ from G', the optimal 1-center node in the resulting graph is x_1 with value U. Hence, $opt(I') \leq n$. Let $N \subseteq V'$ be a node blocker. According to the construction of G', in order to obtain an optimal 1-center node in $G' - N$ of value at least U, N cannot contain x_1. If N contains nodes x_i^ℓ for a given i and ℓ, then N must contains all the $n+1$ nodes x_i^r for $r = 1, \ldots, n+1$, otherwise the value of an optimal solution for the 1-center problem in $G' - N$ is the same as in $G' - (N \backslash \{x_i^\ell\})$. Therefore, since $opt(I') \leq n$, we can consider in the following that N is included in V. In the

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following, we prove by contradiction that N forms a vertex cover in G. Suppose that there exists an edge $(v_i, v_j) \in E$ such that $v_i \notin N$, $v_j \notin N$ and $i < j$. By removing N from G', the optimal 1-center node is x_1 with a maximum weighted distance $D_{max}(x_1) = d(x_1, x_n^r)$ for any $r = 1, \ldots, n$. We distinguish two cases:

- if $v_i = v_1$ then $D_{max}(x_1) = d_{x_1 v_1} + d_{v_1 v_j} + d_{v_j x_j^1} + d(x_j^1, x_n^r) \leq 1 + 0 + 0 + 2(n-j) \leq 1 + 2n - 4 < 2(n-1)$, contradiction.

- if $v_i \neq v_1$ then $D_{max}(x_1) \leq d(x_1, x_i^1) + d_{x_i^1 v_i} + d_{v_i v_j} + d_{v_j x_j^1} + d(x_j^1, x_n^r) \leq 2(i-1) + 0 + 0 + 0 + 2(n-j) = 2(n-1) - 2(j-i) < 2(n-1)$, contradiction.

Therefore N is a vertex cover in G such that $val(I, N) = val(I', N)$. Consequently, $\varepsilon(I, N) = \frac{val(I,N)}{opt(I)} - 1 \leq \frac{val(I',N)}{opt(I')} - 1 = \varepsilon(I', N)$, which achieves the proof for $p = 1$.

We now prove the statement for $p \geq 2$. We use the same construction as above in Theorem 7 for $p \geq 2$ with $U = 2(n-1)$.

Similarly to MIN NODE BLOCKER p-MEDIAN, G' is designed so as to ensure that x_1 and x_1^h, for $h = 1, \ldots, p-1$, will always be the optimal p-center nodes. Also, when we remove all nodes v_i, $i = 1, \ldots, n$ from G', the optimal p-center nodes in the resulting graph is x_1 and x_1^h, for $h = 1, \ldots, p-1$, with value U. Hence, $opt(I') \leq n$. Let $N \subseteq V'$ be a node blocker. According to the construction of G', in order to obtain an optimal p-center nodes in $G' - N$ of value at least U, N cannot contain x_1 and x_1^h, for $h = 1, \ldots, p-1$. If N contains nodes x_i^ℓ for a given i and ℓ, then N must contains all the $n+1$ nodes x_i^r for $r = 1, \ldots, n+1$, otherwise the value of an optimal solution for the p-center problem in $G' - N$ is the same as in $G' - (N \setminus \{x_i^\ell\})$. Therefore, since $opt(I') \leq n$, we can consider in the following that N is included in V. Therefore, as for MIN NODE BLOCKER 1-CENTER, we prove that $opt(I') \leq opt(I)$ and that $val(I, N) = val(I', N)$. Consequently, $\varepsilon(I, N) \leq \varepsilon(I', N)$, which achieves the proof for $p \geq 2$. □

5 Conclusions

We established in this paper negative results concerning the approximation of k most vital edges (nodes) and min edge (node) blocker versions of the p-median and p-center location problems. An interesting open question would be to establish positive results concerning the approximability of these problems. Another interesting perspective is to find efficient exact algorithms to solve them.

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