

Complexity of the min-max and min-max regret assignment problems

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Abstract

This paper investigates the complexity of the min-max and min-max regret assignment problems both in the discrete scenario and interval data cases. We show the strong *NP*-hardness of the min-max and min-max regret versions of the assignment problem when the number of scenarios is not bounded by a constant. We also show that the interval data min-max regret assignment problem is strongly *NP*-hard.

Keywords: assignment problem, min-max, min-max regret, complexity, *NP*-hard.

1 Introduction

Min-max and min-max regret optimization deal with optimization problems where some numerical parameters occurring in the objective function are imprecise or uncertain. A possible assignment of values to these numerical parameters is called a *scenario*. Each scenario s can be represented as a vector in \mathbb{R}^m where m is the number of relevant numerical parameters. In min-max optimization, the aim is to find a solution having the best worst case value across all scenarios. In min-max regret problem, it is required to find a feasible solution minimizing, over all possible scenarios, the maximum deviation of the value of the solution from the optimal value of the corresponding scenario. Two natural ways of describing the set of all possible scenarios S have been considered in the literature. In the *discrete scenario case*, S is described explicitly by the list of all vectors $s \in S$. In the *interval data case*, each numerical parameter can take any value between a lower and upper bound, independently of the values of the other parameters. Thus, in this case, S is the cartesian product of the intervals of uncertainty for the parameters.

Min-max and min-max regret versions of various optimization problems have been studied both in the discrete scenario and interval data cases. Kouvelis and Yu, who mainly focused on the discrete scenario case, report in their book [8] motivations, various applications, complexity results and algorithms for min-max and min-max regret versions of various optimization problems. More recently, extensive research has been devoted to the interval data case for min-max regret versions of various optimization problems including shortest path [7, 4], minimum spanning tree [10, 4, 1], location [2] and resource allocation problems [3].

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In this paper, we study the complexity of the discrete scenario and the interval data min-max and min-max regret assignment problems. In the classical assignment problem, the goal is to find an assignment of minimum total cost of agents to tasks without assigning an agent more than once and ensuring that all tasks are completed. The classical assignment problem is polynomially solvable [9].

According to our knowledge, the only result known up to now concerning min-max and min-max regret assignment problems is by Kouvelis and Yu [8] who showed that in the discrete scenario case both problems are *NP*-hard even for instances with only two scenarios. We show in this paper that both problems are *strongly NP*-hard when the number of scenarios is not bounded by a constant. In the interval data case, the min-max assignment problem is easily shown to be polynomial, whereas the min-max regret assignment problem is proved to be strongly *NP*-hard in general, with some specific polynomial cases.

After introducing some preliminaries in the next section, we develop a general reduction and state some of its properties in Section 3. This central reduction is used to establish results both in the discrete scenarios case (Section 4) and in the interval data case (Section 5). Final remarks are provided in Section 6.

2 Preliminaries

Let us consider an instance of a minimization problem Q and let X be the set of its feasible solutions. We denote by $val(x, s) = \sum_{i=1}^m s_i x_i$ the value of solution $x \in X$ under scenario $s \in S$. In the interval data case, we assume that each coefficient s_i can take any value in the interval $[\underline{s}_i, \bar{s}_i]$.

The min-max optimization problem corresponding to Q consists of finding a solution x having the best worst case value across all scenarios, which can be stated as:

$$\min_{x \in X} \max_{s \in S} val(x, s)$$

The *regret* $R(x, s)$ of a solution $x \in X$ under a scenario $s \in S$ is defined as follows:

$$R(x, s) = \max_{y \in X} \{val(x, s) - val(y, s)\}$$

The *maximum regret* $R_{max}(x)$ of solution x is defined as

$$R_{max}(x) = \max_{s \in S} R(x, s)$$

The min-max regret optimization problem corresponding to Q consists of finding a solution x minimizing the maximum regret $R_{max}(x)$, or equivalently

$$\min_{x \in X} R_{max}(x) = \min_{x \in X} \max_{\substack{y \in X \\ s \in S}} \{val(x, s) - val(y, s)\}$$

Given a complete bipartite graph $G = (V, E)$ with a bipartition $V = V_1 \cup V_2$ and $|V_1| = |V_2| = n$ and costs c_{ij} associated with each edge $(i, j) \in E$, the assignment problem consists of determining a perfect matching of minimum total cost. Equivalently, the assignment problem can be formulated as follows:

$$\begin{aligned}
\min val(a, c) &= \sum_{i=1}^n \sum_{j=1}^n c_{ij} a_{ij} \\
\text{s.t. } a \in \mathcal{A} &= \{a_{ij} \in \{0, 1\} \mid (i = 1, \dots, n; j = 1, \dots, n) : \\
&\quad \sum_{i=1}^n a_{ij} = 1, \quad (j = 1, \dots, n) \\
&\quad \sum_{j=1}^n a_{ij} = 1, \quad (i = 1, \dots, n)\}
\end{aligned}$$

We consider in this paper the discrete scenario and interval data cases for min-max and min-max regret assignment problems. The decision versions of these problems are defined respectively as follows:

DISCRETE MIN-MAX ASSIGNMENT

Input: A complete bipartite graph $G = (V, E)$ with bipartition $V = V_1 \cup V_2$ and $|V_1| = |V_2| = n$, a set C of scenarios (all edge costs are assumed to be positive in all scenarios in C), and a nonnegative integer k .

Question: Is there an assignment a with $\max_{c \in C} val(a, c) \leq k$?

DISCRETE MIN-MAX REGRET ASSIGNMENT

Input: A complete bipartite graph $G = (V, E)$ with bipartition $V = V_1 \cup V_2$ and $|V_1| = |V_2| = n$, a set C of scenarios (all edge costs are assumed to be positive in all scenarios in C), and a nonnegative integer r .

Question: Is there an assignment a with $R_{max}(a) \leq r$?

INTERVAL MIN-MAX ASSIGNMENT

Input: A complete bipartite graph $G = (V, E)$ with bipartition $V = V_1 \cup V_2$ and $|V_1| = |V_2| = n$, lower and upper nonnegative integer bounds \underline{c}_{ij} , and \bar{c}_{ij} for each edge $(i, j) \in E$, and a nonnegative integer r .

Question: Is there an assignment a with $\max_{c \in C} val(a, c) \leq k$?

INTERVAL MIN-MAX REGRET ASSIGNMENT

Input: A complete bipartite graph $G = (V, E)$ with bipartition $V = V_1 \cup V_2$ and $|V_1| = |V_2| = n$, lower and upper nonnegative integer bounds \underline{c}_{ij} , and \bar{c}_{ij} for each edge $(i, j) \in E$, and a nonnegative integer r .

Question: Is there an assignment a with $R_{max}(a) \leq r$?

3 Reduction from the shortest path problem to the assignment problem

Hoffman and Markowitz [6] have described a polynomial reduction from the shortest path problem to the assignment problem. We extend this reduction taking into account scenarios. This extended reduction will be used in all the *NP*-hardness proofs.

Consider an instance of the shortest path problem: a directed graph $G = (V, A)$ with $|V| = n$ and two nodes $s, t \in V$ corresponding to the origin and destination nodes respectively. We obtain an instance $G' = (V', E')$ of the assignment problem with bipartition $V' = V'_1 \cup V'_2$ in the following way: for each vertex $i \in V \setminus \{s, t\}$ we associate two vertices $i \in V'_1$ and $i' \in V'_2$, add vertex s to V'_1 and vertex t to V'_2 . Hence, we have $|V'_1| = |V'_2| = n - 1$. For each arc $(i, j) \in A$, we add edge (i, j') in E' and for each vertex $i \in V \setminus \{s, t\}$ we add edge (i, i') in E' . Finally, in order to obtain a *complete* bipartite graph G' , we add *dummy* edges $(i, j') \in E'$ which correspond to the absence of arc (i, j) in G .

Let \mathcal{P} denote the set of all simple paths from s to t in G . We denote by \mathcal{A} the set of all feasible assignments in G' and by $\mathcal{A}' \subseteq \mathcal{A}$ the set of all feasible assignments in G' that do not include any dummy edge.

The following constructions describe a transformation from paths in \mathcal{P} to assignments in \mathcal{A}' and its converse transformation.

1. For each simple path p in \mathcal{P} we associate a unique assignment a^p in \mathcal{A}' in the following way: for each arc $(i, j) \in p$, include edge $(i, j') \in E'$ in a^p as well as edges $(i, i') \in E'$ if vertex i does not belong to path p .
2. Each assignment a in \mathcal{A}' contains a set of edges $P(a) \subseteq E'$ corresponding to a unique simple path p^a in \mathcal{P} , a set $I(a) = \{(i, i') : (i, i') \in a\}$, and possibly a set $F(a) \subseteq E'$ corresponding to arcs forming cycles in G . Observe that $F(a) = \emptyset$ if G is acyclic.

These two transformations define a one-to-one correspondence between simple paths of \mathcal{P} and assignments without dummy edges of \mathcal{A}' .

We show now how to preserve this correspondence for the values of solutions under all possible scenarios. Let L denote the set of scenarios for an instance of the shortest path problem. Each scenario $\ell \in L$ is defined by positive values ℓ_{ij} , for each arc $(i, j) \in A$, representing the length of arc (i, j) under scenario ℓ . In the following, ℓ_{max} denotes the maximum length under scenario ℓ . Let C denote the set of scenarios for the instance of the assignment problem corresponding to the instance of the shortest path problem. Each scenario $c \in C$ is defined by positive values $c_{ij'}$, for each edge $(i, j') \in E'$, representing the cost of edge (i, j') under scenario c . We create a one-to-one correspondence between L and C defining the two following constructions:

3. For each scenario $\ell \in L$, we associate a unique scenario $c^\ell \in C$ where:
 - $c_{ij'}^\ell = \ell_{ij}$, for each arc $(i, j) \in A$
 - $c_{ii'}^\ell = 0$, for $i \in V \setminus \{s, t\}$
 - $c_{ij'}^\ell = M$, for each dummy edge $(i, j') \in E'$ where M is a sufficiently large value that will be defined precisely when needed.
4. For each scenario $c \in C$, we associate a unique scenario $\ell^c \in L$ where

$$\ell_{ij}^c = c_{ij'}, \text{ for each arc } (i, j) \in A$$

We can now state the following results.

Proposition 1. *Considering an instance of the shortest path problem and its corresponding instance of the assignment problem, as well as constructions 1-4, we have:*

(i) *for any path $p \in \mathcal{P}$ and scenario $\ell \in L$, $val(p, \ell) = val(a^p, c^\ell)$*

(ii) *for any assignment $a \in \mathcal{A}'$ and scenario $c \in C$, $val(a, c) = val(p^a, \ell^c)$*

(iii) *if $M \geq (n-1)\ell_{max} + 1$, for any assignment $a \in \mathcal{A} \setminus \mathcal{A}'$, any assignment $b \in \mathcal{A}'$, and any scenario $c \in C$, $val(b, c) - val(a, c) < 0$*

Proof. (i) $val(p, \ell) = \sum_{(i,j) \in p} \ell_{ij} = \sum_{(i,j') \in P(a^p)} c_{ij'}^\ell = val(a^p, c^\ell)$

(ii) $val(a, c) = \sum_{(i,j') \in P(a)} c_{ij'} = \sum_{(i,j) \in p^a} \ell_{ij}^c = val(p^a, \ell^c)$

(iii) Clearly $val(b, c) \leq (n-1)\ell_{max}$ and $val(a, c) \geq (n-1)\ell_{max} + 1$ since a contains at least one dummy edge. □

Proposition 2. *Considering an instance of the shortest path problem and its corresponding instance of the assignment problem, as well as constructions 1-4, and $M \geq (n-1)\ell_{max} + 1$, we have:*

(i) *for any path $p \in \mathcal{P}$, $R_{max}(p) = R_{max}(a^p)$.*

(ii) *for any assignment $a \in \mathcal{A}'$, $R_{max}(a) = R_{max}(p^a)$.*

Proof. (i) We have, using definition of R_{max} and Proposition 1(iii):

$$\begin{aligned} R_{max}(a^p) &= \max_{\substack{a \in \mathcal{A} \\ c \in C}} \{val(a^p, c) - val(a, c)\} \\ &= \max_{\substack{a \in \mathcal{A}' \\ c \in C}} \{val(a^p, c) - val(a, c)\} \\ &= \max_{\substack{p^a \in \mathcal{P} \\ \ell^c \in L}} \{val(p, \ell^c) - val(p^a, \ell^c)\} \\ &= R_{max}(p) \end{aligned}$$

(ii) For any $a \in \mathcal{A}'$, there exists $p = p^a$ such that $a = a^p$. Thus, using (i), we have:

$$R_{max}(a) = R_{max}(a^p) = R_{max}(p) = R_{max}(p^a)$$

□

4 Discrete scenario case

Kouvelis and Yu [8] have proved that DISCRETE MIN-MAX ASSIGNMENT and DISCRETE MIN-MAX REGRET ASSIGNMENT are NP-hard for a constant number of scenarios, even when this constant is two. We investigate the case of an unbounded number of scenarios for the min-max (Section 4.1) and min-max regret (Section 4.2) assignment problems.

4.1 Complexity of DISCRETE MIN-MAX ASSIGNMENT

We prove here that DISCRETE MIN-MAX ASSIGNMENT is strongly *NP*-hard for an unbounded number of scenarios. For this, we construct a reduction from DISCRETE MIN-MAX SHORTEST PATH, described just below:

DISCRETE MIN-MAX SHORTEST PATH

Input: A directed graph $G = (V, A)$, two vertices $s, t \in V$ corresponding to the origin and destination nodes, a set L of scenarios (all arc lengths are assumed to be positive in all scenarios in L) and a nonnegative integer k .

Question: Is there a path p from s to t with $\max_{\ell \in L} \text{val}(p, \ell) \leq k$?

As shown in [11] by Yu and Yang, DISCRETE MIN-MAX SHORTEST PATH is strongly *NP*-hard even for acyclic directed graphs.

Theorem 1. DISCRETE MIN-MAX ASSIGNMENT *is strongly NP-hard.*

Proof. Consider an acyclic instance of DISCRETE MIN-MAX SHORTEST PATH. We use the reduction from Section 3 to obtain from G an instance $G' = (V', E')$ of DISCRETE MIN-MAX ASSIGNMENT with $|C| = |L|$ scenarios. Scenario set C is defined according to construction 3 with $M = k + 1$.

We claim that there exists a path $p \in \mathcal{P}$ with $\max_{\ell \in L} \text{val}(p, \ell) \leq k$ if and only if there exists an assignment $a \in \mathcal{A}$ with $\max_{c \in C} \text{val}(a, c) \leq k$.

\Rightarrow Consider a path $p \in \mathcal{P}$ with $\max_{\ell \in L} \text{val}(p, \ell) \leq k$. From Proposition 1(i) we have $\text{val}(a^p, c^\ell) = \text{val}(p, \ell)$ for all $\ell \in L$ and thus, we have $\max_{c \in C} \text{val}(a^p, c) = \max_{\ell \in L} \text{val}(p, \ell) \leq k$.
 \Leftarrow Consider an assignment $a \in \mathcal{A}$ with $\max_{c \in C} \text{val}(a, c) \leq k$. It is clear that assignment $a \in \mathcal{A}'$, i.e. does not include any dummy edges, since otherwise we would have $\text{val}(a, c) \geq k + 1$. By means of Proposition 1(ii) we have $\text{val}(a, c) = \text{val}(p^a, \ell^c)$ for all $c \in C$ and thus, we have $\max_{\ell \in L} \text{val}(p^a, \ell) = \max_{c \in C} \text{val}(a, c) \leq k$. \square

4.2 Complexity of DISCRETE MIN-MAX REGRET ASSIGNMENT

In this section we prove that DISCRETE MIN-MAX REGRET ASSIGNMENT is strongly *NP*-hard. For this, we construct a reduction from DISCRETE MIN-MAX REGRET SHORTEST PATH, described just below:

DISCRETE MIN-MAX REGRET SHORTEST PATH

Input: A directed graph $G = (V, A)$, two vertices $s, t \in V$ corresponding to the origin and destination nodes, a set L of scenarios (all arc lengths are assumed to be positive in all scenarios in L) and a nonnegative integer r .

Question: Is there a path p from s to t with $R_{\max}(p) \leq r$?

As shown in [11], DISCRETE MIN-MAX REGRET SHORTEST PATH is strongly *NP*-hard even for acyclic directed graphs.

Theorem 2. DISCRETE MIN-MAX REGRET ASSIGNMENT *is strongly NP-hard.*

Proof. Consider an acyclic instance of DISCRETE MIN-MAX REGRET SHORTEST PATH. We construct graph $G' = (V', E')$ as in Section 3. Scenario set C is defined according to construction 3 with $M = (n - 1)\ell_{\max} + 1 + r$.

We claim that there exists a path $p \in \mathcal{P}$ with $R_{max}(p) \leq r$ if and only if there exists an assignment $a \in \mathcal{A}$ with $R_{max}(a) \leq r$.

\Rightarrow Consider a path $p \in \mathcal{P}$ with $R_{max}(p) \leq r$. From Proposition 2(i) we have $R_{max}(a^p) \leq r$.
 \Leftarrow Let a be an assignment with $R_{max}(a) \leq r$. We first prove that assignment $a \in \mathcal{A}'$, i.e. does not include any dummy edge. In that case, we would have, for any scenario c , $val(a, c) \geq (n-1)\ell_{max} + 1 + r$. For any path $p \in \mathcal{P}$, and any scenario $c \in C$, $val(a^p, c) = \sum_{(i,j) \in p} \ell_{ij}^c \leq (n-1)\ell_{max}$. This would imply $R_{max}(a) \geq r + 1$.

From Proposition 2(ii) it follows that $R_{max}(p^a) \leq r$. □

5 Interval data case

We first state the polynomiality of the min-max assignment problem (Section 5.1), and establish the *NP*-hardness of the min-max regret version (Section 5.2).

5.1 Complexity of INTERVAL MIN-MAX ASSIGNMENT

In the interval data case, the min-max version of a minimization problem corresponds to solving this problem for the worst-case scenario defined by the upper bounds of all intervals. Therefore, the min-max version of a minimization problem has the same complexity as the classical version of the problem. INTERVAL MIN-MAX ASSIGNMENT is thus polynomial-time solvable.

5.2 Complexity of INTERVAL MIN-MAX REGRET ASSIGNMENT

We show that INTERVAL MIN-MAX REGRET ASSIGNMENT is strongly *NP*-hard. However, when the number $u \leq m$ of uncertain/imprecise parameters, corresponding to non-degenerate intervals, is small enough, then the problem becomes polynomial. More precisely, as shown by Averbakh and Lebedev [4] for general networks problems solvable in polynomial time, if u is fixed or bounded by the logarithm of a polynomial function of m , then the min-max regret version is also solvable in polynomial time (based on the fact that an optimal solution for the min-max regret version corresponds to one of the optimal solutions for the 2^u extreme scenarios). This clearly applies to the assignment problem.

In order to prove the strong *NP*-hardness of INTERVAL MIN-MAX REGRET ASSIGNMENT, we construct a reduction from INTERVAL MIN-MAX REGRET SHORTEST PATH, defined as follows:

INTERVAL MIN-MAX REGRET SHORTEST PATH

Input: A directed graph $G = (V, A)$, two vertices $s, t \in V$ corresponding to the origin and destination nodes, lower and upper nonnegative integer lengths $\underline{\ell}_{ij}$ and $\bar{\ell}_{ij}$ for each arc $(i, j) \in A$, and a nonnegative integer r .

Question: Is there a path p from s to t in G with $R_{max}(p) \leq r$?

As shown in [4], INTERVAL MIN-MAX REGRET SHORTEST PATH is strongly *NP*-hard, even when G is acyclic and $\underline{\ell}_{ij}, \bar{\ell}_{ij} \in \{0, 1\}$.

Theorem 3. INTERVAL MIN-MAX REGRET ASSIGNMENT is strongly *NP*-hard even when $\underline{c}_{ij}, \bar{c}_{ij} \in \{0, 1, n+r\}$.

Proof. Consider an acyclic instance of INTERVAL MIN-MAX REGRET SHORTEST PATH defined on n vertices with $\underline{\ell}_{ij}, \bar{\ell}_{ij} \in \{0, 1\}$. We use reduction of Section 3 to obtain the instance $G' = (V', E')$ of INTERVAL MIN-MAX REGRET ASSIGNMENT from G . Scenario set C is defined according to construction 3 with $M = (n - 1)\ell_{max} + 1 + r = n + r$.

We claim that there exists a path $p \in \mathcal{P}$ with $R_{max}(p) \leq r$ if and only if there exists an assignment $a \in \mathcal{A}$ with $R_{max}(a) \leq r$.

\Rightarrow Consider a path $p \in \mathcal{P}$ with $R_{max}(p) \leq r$. From Proposition 2(i) we have $R_{max}(a^p) \leq r$.
 \Leftarrow Let a be an assignment with $R_{max}(a) \leq r$. We first prove that assignment $a \in \mathcal{A}'$, i.e. does not include any dummy edge. In that case, we would have, for any scenario $c \in C$, $val(a, c) \geq n + r$. For any path $p \in \mathcal{P}$, and any scenario $c \in C$, $val(a^p, c) \leq n - 1$. This would imply $R_{max}(a) \geq r + 1$.

From Proposition 2(ii) it follows that $R_{max}(p^a) \leq r$. □

6 Conclusions

We reviewed in this paper positive and negative results concerning the complexity of min-max and min-max regret versions of the assignment problem. These results also apply to the weighted matching problem which contains as a particular case the assignment problem (even for positive results, where the polynomial algorithmic approaches described in Section 5.1 and at the beginning of Section 5.2 remain valid).

It is remarkable that all the proofs of *NP*-hardness previously used for min-max and min-max regret versions of classical optimization problems are based on reductions from standard problems like 3-partition, set cover or hamiltonian path. Our proofs are the first ones using reductions from a min-max or min-max regret version of a classical optimization problem, namely shortest path. A main advantage of our reductions is to preserve the value of solutions and therefore approximation properties between these versions of shortest path and assignment. Thus, a polynomial time approximation algorithm for min-max or min-max regret assignment would imply a polynomial time approximation algorithm with the same approximation ratio for the corresponding version of shortest path. Conversely, any inapproximability result for min-max or min-max regret shortest path, would give rise to the same result for the corresponding version of assignment.

All the negative results proved in this paper are *strong NP*-hardness results. An interesting case is the discrete case with a constant number of scenarios. For this case, the original proof of *NP*-hardness of min-max and min-max regret assignment [8] is obtained using a reduction from partition, which has a pseudo-polynomial time algorithm [5]. Our reductions would also apply, but, for this case, min-max or min-max regret shortest path have pseudo-polynomial time algorithms [8]. Therefore, it remains an open question whether min-max and min-max regret assignment have pseudo-polynomial time algorithms or are strongly *NP*-hard for a constant number of scenarios.

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