Critical edges for the assignment problem: complexity and exact resolution

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Abstract

This paper investigates two problems related to the determination of critical edges for the minimum cost assignment problem. Given a complete bipartite balanced graph with n vertices on each part and with costs on its edges, k Most Vital Edges Assignment consists of determining a set of k edges whose removal results in the largest increase in the cost of a minimum cost assignment. A dual problem, Min Edge Blocker Assignment, consists of removing a subset of edges of minimum cardinality such that the cost of a minimum cost assignment in the remaining graph is larger than or equal to a specified threshold. We show that k Most Vital Edges Assignment is NP-hard to approximate within a factor $c < 2$ and Min Edge Blocker Assignment is NP-hard to approximate within a factor 1.36. We also provide an exact algorithm for k Most Vital Edges Assignment that runs in $O(n^{k+2})$. This algorithm can also be used to solve exactly Min Edge Blocker Assignment.

Keywords: most vital edges, min edge blocker, assignment problem, complexity, approximation, exact algorithm.

1 Introduction

In many applications involving the use of communication or transportation networks, we often need to identify critical infrastructures. By critical infrastructure we mean a set of lines/nodes whose damage causes the largest inconvenience within the network. Modeling the network by a weighted graph, where weights represent costs, identifying a vulnerable infrastructure amounts to finding a subset of edges/nodes whose removal from the graph causes the largest cost increase. In the literature this problem is referred to as the *k most vital edges/nodes* problem. A dual problem consists of determining a set of edges/nodes of minimum cardinality whose removal causes the cost within the residual network to become larger than a given threshold. In the literature this problem is referred to as the *min edge/node blocker* problem. In this paper the k most vital edges and min edge blocker versions for the assignment problem are investigated.

The k most vital edges/nodes and min edge/node blocker versions have been studied for various problems including shortest path, spanning tree, maximum flow, independent set, vertex cover, p-median, p-center and maximum matching. The k most vital arcs problem with respect to shortest path was proved NP-hard in [2]. Later, k most vital arcs/nodes shortest path and min arc/node blocker shortest path were proved to be not 2-approximable and not 1.36-approximable, respectively, if $P \neq NP$ [8]. No positive result is known about the approximation of these problems. For minimum spanning tree, k most vital edges is NP-hard and $O(\log k)$-approximable [6] while several efficient exact algorithms have been proposed [10, 4]. It is proved in [15] that k most vital arcs maximum flow is NP-hard. It is shown in [3] that k most vital nodes and min node blocker with respect to independent set and vertex cover for bipartite graphs remain polynomial time solvable on unweighted graphs and become *NP*-hard for weighted graphs. It is shown in [5] that k most vital edges p-median and k most vital edges p-center are *NP*-hard to approximate within a factor $\frac{7}{5} - \epsilon$ and $\frac{4}{3} - \epsilon$ respectively, for any $\epsilon > 0$, while k most vital nodes p-median and k most vital nodes p-center are *NP*-hard to approximate within a factor $\frac{3}{2} - \epsilon$, for any $\epsilon > 0$. The blocker versions of these four problems are *NP*-hard to approximate within a factor 1.36 [5]. For maximum matching, k most vital nodes was shown polynomial time solvable for unweighted bipartite graphs and *NP*-hard for bipartite graphs when edge weights are bounded by a constant [16]. Moreover, min edge blocker maximum matching is *NP*-hard even for unweighted bipartite graphs [17], but polynomial for grids and trees [14].

After introducing some preliminaries in Section 2, we prove in Section 3 that k Most Vital Edges Assignment and Min Edge Blocker Assignment are *NP*-hard to approximate within a constant factor. An exact algorithm is presented in Section 4 for both problems. Conclusions are provided in Section 5.

2 Basic concepts and preliminary results

Given a directed or an undirected graph $G = (V, E)$, we denote by $G - E'$ the graph obtained from G by removing a subset $E' \subseteq E$ of arcs or edges. Moreover, for any $V' \subseteq V$, $\Gamma(V')$ denotes the set of vertices which are adjacent to V'.

Given a complete bipartite graph $G = (V, E)$ with a bipartition $V = V_1 \cup V_2$ where $|V_1| = |V_2| = n$ and costs c_{ij} associated with each edge $(i, j) \in E$, the assignment problem consists of determining a perfect matching of minimum total cost. Let a^* denote a minimum cost assignment in G.

We consider in this paper the k most vital edges and min edge blocker versions of the assignment problem. These problems are defined respectively as follows.

k Most Vital Edges Assignment
Input: A complete bipartite graph $G = (V, E)$ with bipartition $V = V_1 \cup V_2$ and $|V_1| = |V_2| = n$, where each edge $(i, j) \in E$ has a cost c_{ij}, and an integer k.
Output: A subset $S^* \subseteq E$, with $|S^*| = k$, such that the minimum cost of an assignment in $G - S^*$ is maximum.

Min Edge Blocker Assignment
Input: A complete bipartite graph $G = (V, E)$ with bipartition $V = V_1 \cup V_2$ and $|V_1| = |V_2| = n$, where each edge $(i, j) \in E$ has a cost c_{ij}, and an integer U.
Output: A subset $S^* \subseteq E$ of minimum cardinality such that the minimum cost of an assignment in $G - S^*$ is at least U.

Given an optimization problem and an instance I of this problem, we denote by $|I|$ the size of I, by $opt(I)$ the optimum value of I and by $val(I, S)$ the value of a feasible solution S of I. The *performance ratio* of S (or *approximation factor*) is $r(I, S) = \max\left\{\frac{val(I,S)}{opt(I)}, \frac{opt(I)}{val(I,S)}\right\}$. The *error* of S, $\varepsilon(I, S)$, is defined by $\varepsilon(I, S) = r(I, S) - 1$.

For a function f, an algorithm is an *$f(n)$-approximation*, if for every instance I of the problem, it returns a solution S such that $r(I, S) \leq f(|I|)$.

The notion of a *gap*-reduction was introduced in [1] by Arora and Lund. In this paper we use a *gap*-reduction between two minimization problems. A minimization problem Π is called *gap-reducible* to a minimization problem Π' with parameters (c, ρ) and (c', ρ'), if there exists a polynomial time computable function f such that f maps an instance I of Π to an instance I' of Π', while satisfying the following properties.

- If $opt(I) \leq c$ then $opt(I') \leq c'$

- If $opt(I) > c\rho$ then $opt(I') > c'\rho'$

Parameters c and ρ are function of $|I|$ and parameters c' and ρ' are function of $|I'|$. Also, we have $\rho, \rho' \geq 1$.

The interest of a *gap*-reduction is that if Π is not approximable within a factor ρ then Π' is not approximable within a factor ρ'.

The notion of an E-reduction (*error-preserving* reduction) was introduced by Khanna *et al.* [9]. A problem Π is called *E-reducible* to a problem Π', if there exist polynomial time computable functions f, g and a constant β such that

- f maps an instance I of Π to an instance I' of Π' such that $opt(I)$ and $opt(I')$ are related by a polynomial factor, i.e. there exists a polynomial $p(n)$ such that $opt(I') \leq p(|I|)opt(I)$,
- g maps solutions S' of I' to solutions S of I such that $\varepsilon(I, S) \leq \beta\varepsilon(I', S')$.

An important property of an E-reduction is that it can be applied uniformly to all levels of approximability; that is, if Π is E-reducible to Π' and Π' belongs to $\mathcal{C}$ then Π belongs to $\mathcal{C}$ as well, where $\mathcal{C}$ is a class of optimization problems with any kind of approximation guarantee (see [9] for more details).

To conclude this section, we give a preliminary result concerning our problems.

Lemma 1 *Given a complete bipartite graph $G = (V_1 \cup V_2, E)$ with $|V_1| = |V_2| = n$, for any subset $S \subset E$ with $|S| \leq n-1$, $G - S$ contains an assignment.*

Proof: We show that the sufficient condition of Hall's theorem is satisfied, i.e. that $|\Gamma(A)| \geq |A|$ for all $A \subset V_1$, which means that we can match V_1 in V_2, thus obtaining an assignment. In order to reduce $|\Gamma(A)|$ by one unit, S must contain $|A|$ edges incident to the same node of V_2. Thus, after removing edges of S, A loses at most $\lfloor \frac{|S|}{|A|} \rfloor$ neighbors in V_2. Then, we have $|\Gamma(A)| \geq n - \lfloor \frac{|S|}{|A|} \rfloor$. If $|A| = n$, we have $|\Gamma(A)| \geq n$ and then $|\Gamma(A)| \geq |A|$. If $|A| \leq n-1$, we have $|\Gamma(A)| \geq n - \frac{|S|}{|A|} \geq n - \frac{n-1}{|A|} = \frac{(|A|-1)\, n+1}{|A|} \geq \frac{(|A|-1)(|A|+1)+1}{|A|} = |A|$. □

Observe that there exists a subset S of edges, with $|S| \geq n$, such that no assignment exists in $G - S$. Indeed, if we select in S n edges incident to the same node v, then in $G - S$ node v becomes isolated and cannot be assigned.

Therefore, we suppose in the following that $k \leq n-1$ for k Most Vital Edges Assignment and that $|S^*| \leq n$ for any optimal solution S^* for Min Edge Blocker Assignment.

Observe finally that in order to have a chance to increase the value of a minimum cost assignment in $G - S^*$, S^* must contain at least one edge of a^* so as to eliminate a^* as an optimal solution.

3 Complexity

We study in this section the complexity of k Most Vital Edges Assignment and Min Edge Blocker Assignment. We show that each of these two problems is not approximable within a ratio that is better than a certain constant, unless *P=NP*.

Hoffman and Markowitz [7] describe a polynomial reduction from the shortest path problem to the assignment problem. We extend this reduction in order to prove our inapproximability results. For this, we propose reductions from k Most Vital Arcs Shortest Path and Min Arc Blocker Shortest Path defined as follows:

k Most Vital Arcs Shortest Path
Input: A directed graph $G = (V, A)$, two vertices $s, t \in V$, the length ℓ_{ij} for each arc $(i, j) \in A$,

and an integer k.
Output: A subset $A' \subseteq A$, with $|A'| = k$, such that the minimum length of a path from s to t in $G - A'$ is maximum.

For an instance of k Most Vital Arcs Shortest Path formed by a graph G, we consider that $k \leq \lambda_{s,t}(G) - 1$, where $\lambda_{s,t}(G)$ is the cardinality of an $s - t$ minimum cut in G. Otherwise, taking all arcs of an $s - t$ minimum cut among the k arcs to be removed would lead to a solution with infinite value.

Min Arc Blocker Shortest Path
Input: A directed graph $G = (V, A)$, two vertices $s, t \in V$, the length ℓ_{ij} for each arc $(i, j) \in A$, and an integer U.
Output: A subset $A' \subseteq A$ of minimum cardinality such that the minimum length of a path from s to t in $G - A'$ is at least U.

An optimal solution A' of an instance of Min Arc Blocker Shortest Path formed by a graph G is such that $|A'| \leq \lambda_{s,t}(G)$.

We define in the following the construction used in our reductions.

Consider an instance of the shortest path problem: a directed graph $G = (V, A)$ with $|V| = n$ including two vertices $s, t \in V$ corresponding to the origin and destination nodes respectively, and the length ℓ_{ij} for each arc $(i, j) \in A$. We construct an instance $\widetilde{G} = (W, E)$ of the assignment problem with bipartition $W = V' \cup V''$ (see Figure 1). For each vertex $i \in V \setminus \{s, t\}$ we associate two vertices $i' \in V'$ and $i'' \in V''$, and we add vertex s' to V' and vertex t'' to V''. We create, for each arc $(i, j) \in A$, an edge (i', j'') in E of cost ℓ_{ij} and, for each vertex $i \in V \setminus \{s, t\}$, an edge (i', i'') in E of cost 0. To complete the construction of $\widetilde{G}$, we consider a complete bipartite graph $K^i = (X_i, Y_i)$ for each $i \in V \setminus \{s, t\}$ with $X_i = X'_i \cup X''_i$, where $X'_i = \{x'_{i1}, \ldots, x'_{i(n-1)}\}$ and $X''_i = \{x''_{i1}, \ldots, x''_{i(n-1)}\}$, and a cost 0 associated to each edge of Y_i. We add the edges $(i', x''_{i\ell})$ and $(x'_{i\ell}, i'')$ of cost 0 for each $i \in V \setminus \{s, t\}$ and $\ell = 1, \ldots, n-1$. Hence, we have $|V'| = |V''| = 1 + n(n-2)$. Finally, in order to obtain a *complete* bipartite graph $\widetilde{G}$, we add *dummy* edges of cost $M = \sum_{(i,j) \in A} \ell_{ij} + 1$.

We denote by $\mathcal{P}$ the set of all simple paths from s to t in G, by $\mathcal{A}$ the set of all feasible assignments in $\widetilde{G}$ and by $\mathcal{A}' \subseteq \mathcal{A}$ the set of all feasible assignments in $\widetilde{G}$ that do not include any dummy edge of cost M.

The following constructions describe a transformation from a path in $\mathcal{P}$ to an assignment in $\mathcal{A}'$ and its converse transformation.

1. For each simple path p in $\mathcal{P}$ we associate a unique assignment a^p in $\mathcal{A}'$ in the following way: we include in a^p, the edge $(i', j'') \in E$ for each arc $(i, j) \in p$, the edges $(i', i'') \in E$ if vertex i does not belong to path p and the edges $(x'_{i\ell}, x''_{i\ell})$ for $\ell = 1, \ldots, n-1$, $i \in V \setminus \{s, t\}$. Clearly, the cost of a^p is the same as the length of p.

2. Each assignment a in $\mathcal{A}'$ contains a subset of edges $(s', i''_1), (i'_1, i''_2), \ldots, (i'_{b-1}, i''_b), (i'_b, t'')$ corresponding to a unique simple path $p^a = (s, i_1, i_2, \ldots, i_b, t)$ in $\mathcal{P}$. Indeed, each a in $\mathcal{A}'$ necessarily contains an edge of type (s', i''). Moreover, if edges $(s', i''_1), (i'_1, i''_2), \ldots, (i'_{c-1}, i''_c)$ belong to a then there exists $k \in V \setminus \{i_1, i_2, \ldots, i_c\}$ such that (i'_c, k'') belongs to a. Clearly $k \in \{i_1, i_2, \ldots, i_c\}$ is impossible, but also $(i'_c, x''_{i_c\ell})$ since otherwise a must contain a dummy edge incident to one vertex of X'_{i_c}. Assignment a can also contain a set of edges of type (i', i'') or $(i', x''_{i\ell})$ or $(x'_{i\ell}, i'')$ or $(x'_{i\ell}, x''_{ij})$ and possibly a set of edges corresponding to arcs forming circuits in G.

   In general, the cost of a is equal to the length of p^a plus the lengths of the circuits corresponding to the cycles described by a. However, when a is a minimum cost assignment, the cost of a coincides with the length of p^a, since the cycles described by a can only have a cost 0 (otherwise all vertices i of these cycles could be replaced by edges (i', i'') with cost 0).

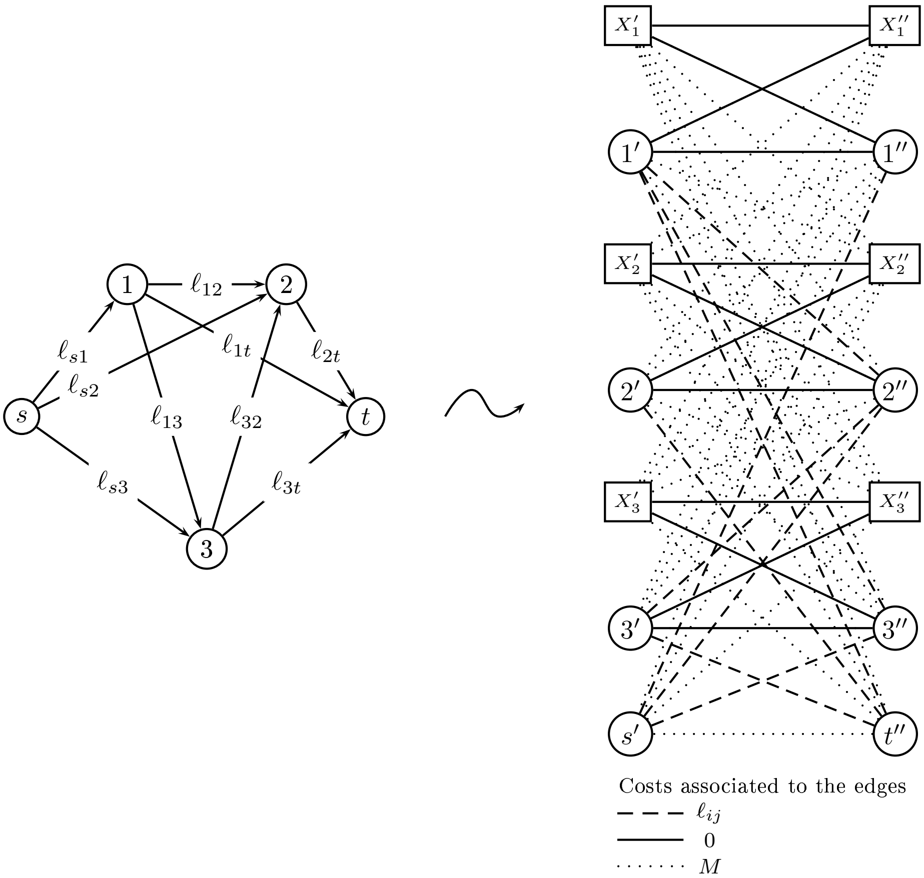

Figure 1: Construction of $\widetilde{G}$ from G

Given a subset S of arcs from G, the subset of edges associated to S in $\widetilde{G}$, denoted by $\widetilde{Im}(S)$, is defined by $\widetilde{Im}(S) = \{(i', j'') \in E : (i,j) \in S\}$. We have $|\widetilde{Im}(S)| = |S|$.

Given a subset $\widetilde{S}$ of edges from $\widetilde{G}$, the subset of arcs associated to $\widetilde{S}$ in G, denoted by $Im(\widetilde{S})$, is defined by $Im(\widetilde{S}) = \{(i,j) \in A : (i', j'') \in \widetilde{S}, i \neq j, c_{i'j''} \neq M\}$. We have $|Im(\widetilde{S})| \leq |\widetilde{S}|$.

Observe that for any subset S of arcs we have $Im(\widetilde{Im}(S)) = S$.

In the following, we present two preliminary results. The first one characterizes a minimum cost assignment generated by deleting a subset of edges and the second one allows us to establish the non-approximability results for k Most Vital Edges Assignment and Min Edge Blocker Assignment.

Lemma 2 *For any subset $\widetilde{S} \subset E$ of cardinality k, with $k \leq \lambda_{s,t}(G) - 1$, any minimum cost assignment in $\widetilde{G} - \widetilde{S}$ does not contain any dummy edge of cost M.*

Proof: By removing the subset of edges $\widetilde{S}$ of E of cardinality k, the subset of arcs $Im(\widetilde{S})$ contains at most k arcs of G. Since $k \leq \lambda_{s,t}(G) - 1$ then there exists at least one path from s to t in $G - Im(\widetilde{S})$. Denote by p a shortest path from s to t in $G - Im(\widetilde{S})$. If no edge of a^p belongs to $\widetilde{S}$, then the result is established since a^p is an assignment in $\widetilde{G} - \widetilde{S}$ of cost less than M. Otherwise, consider the nonempty set of edges $a^p \cap \widetilde{S}$. These edges belong either to complete bipartite subgraphs K_i' induced by $X_i' \cup X_i''$ when $i \in V(p) \setminus \{s,t\}$ or to complete bipartite subgraphs K_i'' induced by $X_i' \cup X_i'' \cup \{i', i''\}$ when $i \in V \setminus V(p)$. All these subgraphs contain only edges of cost 0. Moreover, subgraphs K_i' contain $n-1$ vertices on each part while subgraphs K_i'' contain n vertices on each part. Since $|\widetilde{S}| \leq n-2$, we can apply Lemma 1 to all relevant subgraphs K'_i and K''_i and derive an assignment a' with the same cost as a^p (and thus without dummy edges) but without edges belonging to $\widetilde{S}$. Since a' has a cost less than M, it is also the case for any minimum cost assignment in $\widetilde{G} - \widetilde{S}$ which thus does not contain dummy edges. □

Lemma 3 *(i) Let S be a subset of k arcs of G, with $k \leq \lambda_{s,t}(G) - 1$, and p be a shortest path from s to t in $G - S$. There exists a subset $\bar{S} = \widetilde{Im}(S)$ of k edges of $\widetilde{G}$ such that the assignment a^p is a minimum cost assignment in $\widetilde{G} - \bar{S}$ and the cost of a^p is the same as the length of p.*

(ii) Let $\widetilde{S}$ be a subset of k edges of $\widetilde{G}$, with $k \leq \lambda_{s,t}(G) - 1$, and a be a minimum cost assignment in $\widetilde{G} - \widetilde{S}$. There exists a subset $S' \supseteq Im(\widetilde{S})$ of k arcs such that the path p^a is a shortest path from s to t in $G - S'$ and its length is the same as the cost of a.

Proof: *(i)* The existence of an assignment a of cost lower than that of a^p in $\widetilde{G} - \widetilde{Im}(S)$ would imply that there exists in $G - Im(\widetilde{Im}(S)) = G - S$ a path p^a of length strictly less than that of p. Hence, a^p is a minimum cost assignment in $\widetilde{G} - \bar{S}$ and its cost is the same as the length of p.

(ii) According to Lemma 2, a contains no dummy edge of cost M. Let $S' = Im(\widetilde{S}) \cup S''$, where S'' is any subset of $k - |Im(\widetilde{S})|$ arcs not belonging to p^a. The length of p^a is the same as the cost of a. We show in the following that p^a is a shortest path from s to t in $G - S'$.

Suppose that there exists a path p from s to t in $G - S'$ of length strictly less than that of p^a. Let a^p be the assignment corresponding to p in $\widetilde{G} - \widetilde{Im}(S')$. By construction, a^p contains no dummy edge. If a^p contains no edge of $\widetilde{S}$ then a^p is an assignment in $\widetilde{G} - \widetilde{S}$ of cost strictly less than that of a, which contradicts the optimality of a in $\widetilde{G} - \widetilde{S}$. Otherwise, a^p can contain only edges of $\widetilde{S}$ of type (i', i''), $i = 1, \ldots, n-2$, or $(x'_{i\ell}, x''_{i\ell})$, $\ell = 1, \ldots, n-1$. Then, we can exhibit an assignment a' from a^p in $\widetilde{G} - \widetilde{Im}(S')$ which contains no edge of $\widetilde{S}$ and with the same cost as that of a^p, as shown in the proof of Lemma 2. Hence, a' is an assignment in $\widetilde{G} - \widetilde{S}$ of cost strictly less than that of a, contradicting again the optimality of a in $\widetilde{G} - \widetilde{S}$. Therefore, p^a is a shortest path from s to t in $G - S'$. □

We are now in a position to give our two main inapproximability results.

Theorem 1 *k Most Vital Edges Assignment is NP-hard to approximate within a factor $2 - \epsilon$, for any $\epsilon > 0$.*

Proof: We construct an E-reduction from k Most Vital Arcs Shortest Path which is shown to be *NP*-hard to approximate within a factor $2 - \epsilon$, for any $\epsilon > 0$ [8]. This establishes that k Most Vital Edges Assignment is also *NP*-hard to approximate within a factor $2 - \epsilon$, for any $\epsilon > 0$.

Let I be an instance of k Most Vital Arcs Shortest Path consisting of a graph $G = (V, A)$. We use the previous construction to define from I an instance $\widetilde{I}$ of k Most Vital Edges Assignment formed by the graph $\widetilde{G} = (W, E)$.

Consider an optimal solution $S \subset A$ for I, with $|S| = k$, and denote by p a path of minimum length from s to t in $G - S$. When removing from $\widetilde{G}$ the subset of edges $\widetilde{Im}(S)$, the assignment a^p is, according to Lemma 3(i), a minimum cost assignment in $\widetilde{G} - \widetilde{Im}(S)$. Thus, $opt(\widetilde{I}) \geq opt(I)$.

Consider now a solution $\widetilde{S} \subset E$ of $\widetilde{I}$, with $|\widetilde{S}| = k$, and denote by a a minimum cost assignment in $\widetilde{G} - \widetilde{S}$. Consider the subset of arcs $Im(\widetilde{S})$ and let p^a be the path from s to t in $G - Im(\widetilde{S})$ corresponding to a. Let S be a subset of k arcs consisting of $Im(\widetilde{S})$ possibly completed by any subset of $k - |Im(\widetilde{S})|$ arcs not belonging to p^a. According to Lemma 3(ii), p^a is a path of minimum length in $G - S$ whose length is equal to the cost of a. Hence, $val(I, S) = val(\widetilde{I}, \widetilde{S})$. In particular, if $\widetilde{S}$ is an optimal solution of $\widetilde{I}$, then $opt(\widetilde{I}) = val(I, S) \leq opt(I)$.

Therefore, we have $opt(I) = opt(\widetilde{I})$ and the error of the two solutions S and $\widetilde{S}$ are equal $\varepsilon(I, S) = \varepsilon(\widetilde{I}, \widetilde{S})$. □

We prove now an inapproximability result for MIN ARC BLOCKER ASSIGNMENT. Unlike for k MOST VITAL EDGES ASSIGNMENT, using our construction, it seems difficult to build an E-reduction which imposes conditions on all feasible solutions (in particular for those in $\widetilde{G}$ of size more than $\lambda_{s,t}(G)$ that do not give necessarily a feasible solution in G). Thus, we resort to a gap-reduction which imposes conditions on optimal solutions only.

Theorem 2 MIN EDGE BLOCKER ASSIGNMENT *is* NP*-hard to approximate within a factor 1.36.*

Proof: We construct a gap-reduction from MIN ARC BLOCKER SHORTEST PATH which is known to be *NP*-hard to approximate within a factor 1.36 even for graphs G such that the optimum value is less than $\lambda_{s,t}(G)$ [8].

Let I be an instance of MIN ARC BLOCKER SHORTEST PATH consisting of a graph $G = (V, A)$ and a positive integer U. We use the previous construction to define from I an instance $\widetilde{I}$ of MIN EDGE BLOCKER ASSIGNMENT formed by the graph $\widetilde{G} = (W, E)$ and U.

Consider an optimal solution $S \subset A$ for I, and denote by p a path of minimum length in $G-S$ from s to t. Since $|S| \leq \lambda_{s,t}(G) - 1$, according to Lemma 3(i), the assignment a^p is a minimum cost assignment in $\widetilde{G} - \widetilde{Im}(S)$ of cost equal to the length of p, which is at least U. Thus, we have $opt(\widetilde{I}) \leq opt(I) \leq \lambda_{s,t}(G) - 1$.

Let $\widetilde{S} \subset E$ be an optimal solution of $\widetilde{I}$, and denote by a an assignment of minimum cost in $\widetilde{G}-\widetilde{S}$. Assignment a is such that its cost is at least U. According to Lemma 3(ii), there exists a subset S' of $|\widetilde{S}|$ arcs such that the path p^a is a shortest path in $G - S'$ and its length is the same as the cost of a. The length of p^a is then greater than or equal to U. Hence, $opt(I) \leq |S'| = opt(\widetilde{I})$. Thus $opt(\widetilde{I}) = opt(I)$, showing that $opt(I) \leq c$ implies $opt(\widetilde{I}) \leq c$ and $opt(I) > c\rho$ implies $opt(\widetilde{I}) > c\rho$ which establishes that MIN EDGE BLOCKER ASSIGNMENT is also *NP*-hard to approximate within a factor 1.36. □

4 Exact resolution

We propose in this section an exact algorithm for solving k MOST VITAL EDGES ASSIGNMENT and MIN EDGE BLOCKER ASSIGNMENT. Consider $G = (V_1 \cup V_2, E)$ a complete bipartite graph with $|V_1| = |V_2| = n$ and a cost is associated to each edge of E. Denote by a^* a minimum cost assignment in G.

An approach to solve 1 MOST VITAL EDGE ASSIGNMENT is to delete one by one each of the n edges belonging to a^*, determine the minimum cost assignments on the n resulting partial graphs, and retain the deleted edge which leads to a largest minimum cost assignment. This approach is very similar to the scheme developed by Murty [12] for ranking the assignments in increasing cost order, except that in Murty's approach a minimum cost assignment is selected among the n candidate assignments. In this context, Miller *et al.* [11] and Pedersen *et al.* [13] showed that the n assignments can be found efficiently using reoptimization. Indeed, given an edge $e = (y, z) \in a^*$, a minimum cost assignment a_e in $G - \{e\}$ can be found using Dijktra's algorithm in $O(n^2)$ by solving a single shortest path problem between y and z where arcs are valued by (nonnegative) reduced costs. Therefore, the time complexity for finding all assignments a_e for all edges $e \in a^*$ is $O(n^3)$. Thus, we obtain the following result.

Theorem 3 1 MOST VITAL EDGE ASSIGNMENT *can be solved in $O(n^3)$ for complete bipartite graphs with n vertices in each part.*

In the following, we are interested in the exact resolution of k MOST VITAL EDGES ASSIGNMENT. Taking advantage of the fact that optimal solutions must contain at least one edge of a^*, a naive approach would be to remove each edge $e \in a^*$, consider all possible combinations of $k-1$ edges to delete from the $n^2 - 1$ remaining edges and determine a minimum cost assignment in the resulting partial graphs. An optimal solution is a subset of removed edges which leads to the largest minimum cost assignment. Hence, a naive approach for solving k MOST VITAL EDGES

ASSIGNMENT would require $n\binom{n^2-1}{k-1}O(n^3) = O(n^{2k+2})$ time. A more efficient algorithm can be obtained through the following result.

Theorem 4 *k MOST VITAL EDGES ASSIGNMENT can be solved in $O(n^{k+2})$ time for complete bipartite graphs with n nodes in each part and for general k.*

Proof: Consider a minimum cost assignment a^* in G. Obviously, a set S^* of k most vital edges must contain at least one edge e in a^*. Consider now a minimum cost assignment b^* in $G - \{e\}$. If $k \geq 2$, then S^* must contain at least one edge of b^*, and so on. Hence, by simply enumerating all possibilities to choose an edge in a^*, then one in b^* and so on, one can find an optimal solution by looking at $O(n^k)$ possible subsets of removed edges. At each step, we compute a minimum cost assignment in time $O(n^2)$ as for example when determining b^* in $G - \{e\}$ starting from a^*. Therefore, we compute in this way $n + n^2 + \ldots + n^k$ minimum cost assignments, resulting in a time $O(n^{k+2})$. □

This algorithm can be implemented by developing a search tree with $k + 1$ levels. The root node at level 0 corresponds to the optimal assignment a^* and each node at level i ($i = 1, \ldots, k$) represents a tentative selection of i edges which could be part of the k most vital edges. A refined implementation, avoiding the repetition of tentative selections but still in $O(n^{k+2})$, can be obtained using a branching scheme similar to the one used by Murty [12]. Moreover, observe that solving k MOST VITAL EDGES ASSIGNMENT in this way (developing a complete or reduced search tree) allows the determination of an optimal solution for i MOST VITAL EDGES ASSIGNMENT by simply scanning all nodes of level i and retaining a node corresponding to the largest minimum cost assignment ($i = 1, \ldots, k$).

We show now how to solve MIN EDGE BLOCKER ASSIGNMENT. If the minimum cost of an assignment is at least U then the optimal cardinality is 0. Otherwise, we search for the smallest level i, $1 \leq i \leq n - 1$ such that the optimum value of i MOST VITAL EDGES ASSIGNMENT is at least U. If such an i does not exists, then any subset of n edges incident to a vertex is optimal. Thus, considering that we need to develop our search tree until level $n - 1$ at most, we can solve MIN EDGE BLOCKER ASSIGNMENT in $O(n^{n+1})$.

5 Conclusions

We established in this paper negative results concerning the approximation of k most vital edges and min edge blocker versions of the assignment problem.

It is remarkable that all the proofs of *NP*-hardness or inapproximability previously used up to now for k most vital edges and min edge blocker versions of classical optimization problems are based on reductions from standard problems like vertex cover, clique, independent set, or min k cut. Our proofs are the first ones using reductions from a k most vital edges and min edge blocker version of a classical optimization problem, namely shortest path. A main advantage of our E-reduction is to preserve the value of solutions and therefore approximation properties between these versions of shortest path and assignment. Thus, a polynomial time approximation algorithm for k MOST VITAL EDGES ASSIGNMENT would imply a polynomial time approximation algorithm with the same approximation ratio for the corresponding versions of shortest path. A gap-reduction only preserves inapproximability results. Thus, any stronger inapproximability result for k most vital edges and min edge blocker shortest path, would give rise to the same result for the corresponding versions of assignment.

Concerning positive results, we proposed exact algorithms, in $O(n^{k+2})$ for k MOST VITAL EDGES ASSIGNMENT and in $O(n^{n+1})$ for MIN EDGE BLOCKER ASSIGNMENT. An interesting open question is to try to establish approximation algorithms or better exact algorithms for these problems.

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