Partitioning vertices of 1-tough graphs into paths*

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Abstract

In this paper we prove that every 1-tough graph has a partition of its vertices into paths of length at least two.

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1 Introduction

We use Bondy and Murty's book for notation and terminology not defined here [2]. In addition, all the graphs considered in this paper are undirected and simple. Let $G = (V, E)$ be a graph. For each $u \in V$, we denote by $d(u)$ the degree of u in G and by $N(u)$ the set of neighbors of u in G. If X is a subset of V, let $N(X) = \cup_{v \in X} N(v)$.

A set $\mathcal{P} = \{P_1, \ldots, P_k\}$ of vertex-disjoint paths of G with length at least two (i.e., at least three vertices) is called a *long path system* in G. A graph G has a partition of its vertices into a long path system if there exists a long path system $\mathcal{P}$ in G such that $V(\mathcal{P}) = V(G)$, where $V(\mathcal{P}) = \cup_{P \in \mathcal{P}} V(P)$.

Let $S \subset V(G)$. We denote by $c(G - S)$ the number of connected components of the induced subgraph $G - S$. A graph G is said to be t*-tough* if for each subset S of vertices with $c(G - S) > 1$ we have $c(G - S) \leq \frac{|S|}{t}$. The toughness of G, denoted by $\tau(G)$, is the largest value of t such that G is t-tough.

The parameter "toughness" is strongly related to connectivity. It is clear that a 1-tough graph is 2-connected. Chvátal [3] proved that for a non-complete graph G with connectivity $\kappa(G)$, $\tau(G) \leq \frac{\kappa(G)}{2}$. Toughness conditions also imply many other properties of the graph, in particular properties related to cycles, paths and factors. The following conjecture due to Chvátal is well known.

Conjecture 1 ([3]) *There exists a constant t such that every t-tough graph is hamiltonian.*

Chvátal has also conjectured that every 2-tough graph is hamiltonian. Recently, Bauer, Broersma and Veldman [1] gave examples of non-hamiltonian graphs that are $(9/4 - \epsilon)$-tough for any $\epsilon > 0$. So if the above conjecture were true, t should be at least $9/4$.

The relation between the toughness of a graph and the possibility to partition its vertex set into paths has also been studied. Ota conjectured the following:

Conjecture 2 ([5]) *For $n \equiv 0$ (mod k), every $\frac{k}{2}$-tough graph on n vertices admits a partition of its vertex set into paths P_k.*

Saito [6] showed that the above conjecture is true for $k = 2, 4$.

In this paper, we consider toughness condition and long path systems of graphs. Our main result is the following:

Theorem 3 *If G is a 1-tough graph, then G has a partition of its vertices into a long path system.*

We will give a complete proof of this theorem in section 3.

2 Preliminaries

In this section we introduce some notation and we prove a lemma necessary for the proof of Theorem 3.

Let $P = c_1c_2...c_p$ be a path in G. For each $i \leq j$ we denote by $c_i\overrightarrow{P}c_j$, the path $c_ic_{i+1}...c_j$, and by $c_i\overleftarrow{P}c_j$ the path $c_jc_{j-1}...c_i$. We consider $c_i\overrightarrow{P}c_j$ and $c_i\overleftarrow{P}c_j$ both as paths and as vertex sets. For any i, we let $c_i^+ = c_{i+1}$, $c_i^- = c_{i-1}$, $c_i^{++} = c_{i+2}$ and $c_i^{--} = c_{i-2}$. We shall denote the paths P of G by $P[u,v]$ where u and v are the end-vertices of P.

Let H_1 and H_2 be two subgraphs of G. H_1 and H_2 are said to be *remote* if $V(H_1) \cap V(H_2) = \emptyset$ and there is no edge between $V(H_1)$ and $V(H_2)$.

Lemma 1 *Suppose G is a graph. Let $\mathcal{P}$ be a long path system which contains a maximum number of vertices of G. Let $P[u,v]$ be a path of $\mathcal{P}$ and let $H = V(G) - V(\mathcal{P})$. Then*

a) The vertices u and v are not adjacent to H.

b) If a vertex $w \in V(P)$ is adjacent to a vertex $x \in V(H)$ then the length of the paths $u\overrightarrow{P}w$ and $w\overrightarrow{P}v$ is at most two.

c) P contains at most one vertex of $N(H)$.

Proof: a) Suppose that u is adjacent to a vertex $x \in V(H)$. Replacing P by the path $xu\overrightarrow{P}v$ in $\mathcal{P}$, we obtain a long path system containing more vertices than $\mathcal{P}$, which contradicts the choice of $\mathcal{P}$. Similarly, $N(v) \cap V(H) = \emptyset$.

b) Let $w \in V(P)$ be a vertex which is adjacent to $x \in V(H)$ such that the path $u\overrightarrow{P}w$ or the path $w\overrightarrow{P}v$ is of length at least three. Suppose that $u\overrightarrow{P}w$ is of length at least three. So, the path $u\overrightarrow{P}w^-$ has the length at least two. Replacing in $\mathcal{P}$ the path P by the paths $xw\overrightarrow{P}v$ and $u\overrightarrow{P}w^-$, we obtain a long path system containing more vertices than $\mathcal{P}$, a contradiction.

c) By a) and b), it follows that if $N(H) \cap V(P)$ contains at least two vertices w_1 and w_2, then w_1 and w_2 are consecutive on P, say $w_2 = w_1^+$. If they have a common neighbor x in H, replacing the path P by the path $u\overrightarrow{P}w_1xw_2\overrightarrow{P}v$ yields a contradiction. If there exist $x' \in N(w_1) \cap H$ and $x'' \in N(w_2) \cap H$, replacing the path P by the paths $u\overrightarrow{P}w_1x'$ and $x''w_2\overrightarrow{P}v$ in $\mathcal{P}$ results in a contradiction. □

3 Proof of Theorem 3

Suppose that G is a 1-tough graph which does not have a partition of its vertices into a long path system. Let $\mathcal{P}$ be a long path system such that:

1) $|V(\mathcal{P})|$ is as large as possible;

2) Subject to 1, the number of paths of $\mathcal{P}$ is as small as possible.

Obviously there is no edge connecting the end-vertices of two paths of $\mathcal{P}$ since otherwise condition 2) of the definition of $\mathcal{P}$ would not be satisfied. Let $H = V(G) - V(\mathcal{P})$.

In the following, we give a procedure to construct two sets A and B where A is a set of vertices and B a set of induced subgraphs.

First, we initialize $A = \emptyset$ and $B = \emptyset$. Let B_0 be the subgraph induced by H. Add the subgraph B_0 to B.

Step 1. Let P_1 be a path joined to B_0 by an edge ax where $a \in V(P_1)$ and $x \in V(B_0)$. Let us set $A_1 = N(B_0) \cap V(P_1)$ and B_1 the subgraph induced by $V(P_1) - A_1$. From Lemma 1, we deduce that the length of P_1 is at most four and $|A_1| = 1$.

If B_0 is not joined to some path of $\mathcal{P}$ different from P_1, then the number of connected components of $G - A_1$ is at least two. So $c(G - A_1) \geq |A_1| + 1$ which contradicts the fact that G is 1-tough.

So B_0 is joined to a path of $\mathcal{P}$ which is different from P_1. Add the subgraph B_1 to B. We now describe the second step of the procedure.

Step 2. Let $P_2[u_2, v_2]$ be a path of $\mathcal{P}$ which is joined to B_0 by an edge. Let $A_2 = N(B) \cap V(P_2)$ and let B_2 be the subgraph induced by $V(P_2) - A_2$. Add the subgraph B_2 to B.

Fact 1 For each vertex $u \in A_2$ the length of the paths $u_2\overrightarrow{P_2}u$ and $u\overrightarrow{P_2}v_2$ is at most two.

Proof of Fact 1: Suppose that there exists a vertex $u \in A_2$ such that $|V(u_2\overrightarrow{P_2}u)| > 3$. The proof is similar for $|V(u\overrightarrow{P_2}v_2)| > 3$.

From Lemma 1b), we deduce that u is not adjacent to a vertex of B_0. So, u is adjacent to a vertex of B_1. Let u' be a vertex in B_1 which is adjacent to u. Without loss of generality suppose that $u' \in a^+\overrightarrow{P_1}v_1$.

By Lemma 1b) we know $|V(a^+\overrightarrow{P_1}v_1)| \leq 2$. If $|V(a^+\overrightarrow{P_1}v_1)| = 2$ we have $u' = a^+$ or $u' = v_1$. If $u' = a^+$ ($u' = v_1$, resp.), then let $\mathcal{P}'$ be the long path system obtained from $\mathcal{P}$ by replacing P_1 and P_2 by the paths $u_1\overrightarrow{P_1}ax$, $u_2\overrightarrow{P_2}u^-$ and $v_2\overleftarrow{P_2}uu'v_1$ ($v_2\overleftarrow{P_2}uu'a^+$, resp.). If $|V(a^+\overrightarrow{P_1}v_1)| = 1$ then $u' = v_1$. Let $\mathcal{P}'$ be the long path system obtained from $\mathcal{P}$ by replacing P_1 and P_2 by the paths $u_1\overrightarrow{P_1}ax$, $u_2\overrightarrow{P_2}u^-$ and $v_2\overleftarrow{P_2}uu'$. Clearly, $\mathcal{P}'$ contains more vertices than $\mathcal{P}$, a contradiction, which completes the proof of Fact 1. □

From Fact 1, we deduce the following:

Remark 1 *The length of P_2 is at most four and $|A_2| \leq 2$.*

Fact 2 If $|A_2| = 2$, then the subgraph B_2 is not connected.

Proof of Fact 2: Assume that $|A_2| = 2$ and that B_2 is connected. From Fact 1 and since the length of P_2 is at most four, we deduce that the length of P_2 is at most three and $u_2v_2 \in E$.

Let $u \in A_2$. Replace the path P_2 by the path $u\overrightarrow{P_2}v_2u_2\overrightarrow{P_2}u^-$. Then we get a path system which contradicts Fact 1. □

Finally, if there is no path different from P_1 and P_2 joined to B, then we add $A_1 \cup A_2$ to A. According to the construction of the sets A and B, we deduce that the subgraphs B_0, B_1 and B_2 are not connected by an edge. From Fact 2 it follows that $c(B_2) \geq |A_2|$. Since $|A_1| = 1$ and $|A_2| \leq 2$, we find $c(B) \geq c(B_0) + c(B_1) + c(B_2) \geq 2 + |A_2| = 1 + |A|$. We obtain that $c(G - A) \geq c(B) \geq |A| + 1$, a contradiction.

So there exists a path of $\mathcal{P}$, different from P_1 and P_2, and joined to B. More generally, we define step $i+1$ of the procedure. Let $P_i[u_i, v_i]$ be the path defined in step i. Let B_i be the corresponding subgraph and A_i the corresponding set of vertices. Assume that for each $u \in A_i$, the length of the paths $u_i\overrightarrow{P_i}u$ and $u\overrightarrow{P_i}v_i$ is at most two. Let B be the set of subgraphs obtained at the end of step i. If there exists a path different from the paths P_j, $j \leq i$, then we define step $i+1$ as follows:

Step i+1. Let $P_{i+1}[u_{i+1}, v_{i+1}]$ be a path of $\mathcal{P}$ joined to B, such that P_{i+1} is different from the paths P_j, with $j \leq i$. Let $A_{i+1} = N(B) \cap V(P_{i+1})$ and let B_{i+1} be the subgraph induced by $V(P_{i+1}) - A_{i+1}$. Add the subgraph B_{i+1} to B.

Claim 1 *At each step i of the procedure and for each $u \in A_i$,*
1) There exists a long path system $\mathcal{P}'$ such that $V(\mathcal{P}') = (V(\mathcal{P}) \cup V(H')) - V(u^+\overrightarrow{P_i}v_i)$, with $H' \neq \emptyset$, $H' \subseteq H$ and u_i is an end-vertex of a path of $\mathcal{P}'$. Also the length of the path $u\overrightarrow{P_i}v_i$ is at most two.
2) There exists a long path system $\mathcal{P}''$ such that $V(\mathcal{P}'') = (V(\mathcal{P}) \cup V(H'')) - V(u_i\overrightarrow{P_i}u^-)$, with $H'' \neq \emptyset$, $H'' \subseteq H$ and v_i is an end-vertex of a path of $\mathcal{P}''$. Also the length of the path $u_i\overrightarrow{P_i}u$ is at most two.

Proof : We will prove assertions 1) and 2) of Claim 1 simultaneously. We proceed by induction on the index of the steps.

Suppose that Claim 1 is true for each step j with $j < i$. We prove the claim for step i. If $i = 1$, clearly the long path system $\mathcal{P}'$ obtained from $\mathcal{P}$ by replacing P_1 by $u_1\overrightarrow{P_1}ax$ is such that $V(\mathcal{P}') = (V(\mathcal{P}) \cup \{x\}) - V(a^+\overrightarrow{P_1}v_1)$ which proves assertion 1) of Claim 1. The long path system $\mathcal{P}''$ obtained from $\mathcal{P}$ by replacing P_1 by $xa\overrightarrow{P_1}v_1$ is such that $V(\mathcal{P}'') = (V(\mathcal{P}) \cup \{x\}) - V(u_1\overrightarrow{P_1}a^-)$ which proves assertion 2) of Claim 1. From Lemma 1b) the lengths of the paths $u\overrightarrow{P_i}v_i$, $u_i\overrightarrow{P_i}u$ are at most two.

Since i is a step of the procedure, $A_i \neq \emptyset$. Let $u \in A_i$. Clearly u is adjacent to B. If u is adjacent to B_0, then Claim 1 follows as in case $i = 1$.

If u is not adjacent to B_0, then let $P_r[u_r, v_r]$ be a path of $\mathcal{P}$ with $r < i$ and such that u is adjacent to B_r by an edge uu'. We distinguish two main cases:

Case 1. $V(u_r\overrightarrow{P_r}u') \cap A_r = \emptyset$.

Let b be the vertex of A_r such that $V(u'^+\overrightarrow{P_r}b^-) \cap A_r = \emptyset$. By the inductive hypothesis, there exists a long path system $\mathcal{P}'$ such that $V(\mathcal{P}') = (V(\mathcal{P}) \cup V(H')) - V(u_r\overrightarrow{P_r}b^-)$, where $H' \neq \emptyset$, $H' \subseteq H$ and the length of the path $u_r\overrightarrow{P_r}b$ is at most two.

The long path system $\mathcal{P}''$ obtained from $\mathcal{P}'$ by replacing the path P_i by the path obtained by joining $u_r\overrightarrow{P_r}b^-$, uu' and $u_i\overrightarrow{P_i}u$ would satisfy assertion 1) of Claim 1. Assume that $|V(u^+\overrightarrow{P_i}v_i)| \geq 3$. Then the long path system obtained from $\mathcal{P}''$ by adding the path $u^+\overrightarrow{P_i}v_i$ contains more vertices than $\mathcal{P}$, a contradiction, which implies that the length of the path $u\overrightarrow{P_i}v_i$ is at most two.

The long path system $\mathcal{P}'''$ obtained from $\mathcal{P}'$ by replacing the path P_i by the path obtained by joining $u_r\overrightarrow{P_r}b^-$, uu' and $u\overrightarrow{P_i}v_i$ would satisfy assertion 2) of Claim 1. Assume that $|V(u_i\overrightarrow{P_i}u^-)| \geq 3$. Then the long path system obtained from $\mathcal{P}'''$ by adding the path $u_i\overrightarrow{P_i}u^-$ contains more vertices than $\mathcal{P}$, a contradiction, which implies that the length of the path $u_i\overrightarrow{P_i}u$ is at most two.

Case 2. $V(u_r\overrightarrow{P_r}u') \cap A_r \neq \emptyset$.

Let b be a vertex of A_r such that $V(b^+\overrightarrow{P_r}u'^-) \cap A_r = \emptyset$. By the inductive hypothesis, there exists a long path system $\mathcal{P}'$ such that $V(\mathcal{P}') = (V(\mathcal{P}) \cup V(H')) - V(b^+\overrightarrow{P_r}v_r)$, where $H' \neq \emptyset$, $H' \subseteq H$ and the length of the path $b\overrightarrow{P_r}v_r$ is at most two.

The long path system $\mathcal{P}''$ obtained from $\mathcal{P}'$ by replacing the path P_i by the path obtained by joining $b^+\overrightarrow{P_r}v_r$, uu' and $u_i\overrightarrow{P_i}u$ would satisfy assertion 1) of Claim 1. Assume that $|V(u^+\overrightarrow{P_i}v_i)| \geq 3$. Then the long path system obtained from $\mathcal{P}''$ by adding the path $u^+\overrightarrow{P_i}v_i$ contains more vertices than $\mathcal{P}$, a contradiction, which implies that the length of the path $u\overrightarrow{P_i}v_i$ is at most two.

The long path system $\mathcal{P}'''$ obtained from $\mathcal{P}'$ by replacing the path P_i by the path obtained by joining $b^+\overrightarrow{P_r}v_r$, uu' and $u\overrightarrow{P_i}v_i$ would satisfy assertion 2) of Claim 1. Assume that $|V(u_i\overrightarrow{P_i}u^-)| \geq 3$. Then the long path system obtained from $\mathcal{P}''$ by adding the path $u_i\overrightarrow{P_i}u^-$ contains more vertices than $\mathcal{P}$, a contradiction, which implies that the length of the path $u_i\overrightarrow{P_i}u$ is at most two.

□

From Claim 1, we deduce the following:

Remark 2 *At each step i of the procedure, if $|A_i| = 2$ then the length of the path P_i is at most three.*

Claim 2 *At each step i of the procedure, if $|A_i| = 2$ then the subgraph B_i is not connected.*

Proof: Assume that there exists a step i such that $|A_i| = 2$, and B_i is connected. Let $P_i[u_i, v_i]$ be the path obtained at step i. Since B_i is connected, using Remark 2, we deduce that $u_iv_i \in E$. The vertices u_i^+ and u_i^{++} belong to A_i. From Claim 1, there exists a long path system $\mathcal{P}'$ such that $V(\mathcal{P}') = (V(\mathcal{P}) \cup V(H')) - V(u_i^{++}\overrightarrow{P_i}v_i)$, with $H' \neq \emptyset$, $H' \subseteq H$ and u_i is an end-vertex of a path of $\mathcal{P}'$. The long path system obtained from $\mathcal{P}'$ by adding the path $u_i^{++}\overrightarrow{P_i}v_iu_i$, contains more vertices than $\mathcal{P}$, a contradiction. □

According to the construction of the set B, the subgraphs B_j are mutually remote, where j is a step of the procedure.

In the following, we prove that if two subgraphs B_i and B_j are connected by a path $P = u_0u_1...u_p$ internally disjoint from B_i and B_j, with u_0 in B_i and u_p in B_j, then the vertices u_1 and u_{p-1} belong to A. Remark that u_1 and u_{p-1} can be the same vertex. The vertices u_1 and u_{p-1} do not belong to H, because otherwise if $u_1 \in V(H)$ then u_0 belongs to A_i, a contradiction. We obtain a similar contradiction, if $u_{p-1} \in V(H)$. So u_1 and u_{p-1} belong to $V(\mathcal{P})$. Since the subgraphs of B are mutually remote, u_1 and u_{p-1} belong to A, which concludes the proof of the assertion.

We deduce that the number of connected components of the subgraph $G - A$ is the number of components of the subgraphs of B. From Claim 2, we deduce that the number of connected components of $G - A$ is at least $|A| + 1$ which contradicts the fact that the graph G is 1-tough and achieves the proof of Theorem 3.

Remark 3 *Using the ideas of the proof of Theorem 3 we can define a polynomial time algorithm to construct a partition into long path system in* 1*-tough graphs.*

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