Single approximation for Biobjective Max TSP*

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Abstract

We mainly study Max TSP with two objective functions. We propose an algorithm which returns a single Hamiltonian cycle with performance guarantee on both objectives. The algorithm is analysed in three cases. When both (resp. at least one) objective function(s) fulfill(s) the triangle inequality, the approximation ratio is $\frac{5}{12} - \varepsilon \approx 0.41$ (resp. $\frac{3}{8} - \varepsilon$). When the triangle inequality is not assumed on any objective function, the algorithm is $\frac{1+2\sqrt{2}}{14} - \varepsilon \approx 0.27$-approximate.

1 Introduction

The traveling salesman problem (TSP) is one of the most studied problems in combinatorial optimization. Given an undirected complete graph with weights on the edges, the problem consists of finding a Hamiltonian cycle (also called tour) of maximum or minimum total weight, defined as the sum of its edges' weight. In this paper we mainly study the approximation of the biobjective maximization version, Biobjective Max TSP. In this case every edge has two weights and the total weight of a tour is a couple defined as the componentwise sum of its edges' weights. We are interested in the existence and the computation in polynomial time of a *single* tour with simultaneous performance guarantees on the two objectives. Our work falls into a recent stream of research on the approximability of multiobjective optimization problems [21, 20, 18, 10, 5, 11, 3, 1, 6] where multiobjective TSP takes a prominent place [2, 4, 16, 7, 13, 14].

In many real optimization problems not only one objective function is considered but several ones (see [9] about multiobjective combinatorial optimization). This is also the case for TSP where we might want to minimize the travel time, the cost or to maximize the profit, the number of viewpoints along the way etc. This gives rise to Multiobjective TSP. Unfortunately it is unlikely that optimality is met simultaneously by a single feasible solution on all objectives. However there always exists a set of efficient (also called Pareto optimal)

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solutions for which any improvement on an objective induces a deterioration of (at least) another one.

Generating the whole set of efficient solutions is a major challenge in multiobjective combinatorial optimization. However, even for moderately-sized problems, it is usually computationally prohibitive to identify the efficient set for two major reasons. First, the number of efficient solutions can be very large. Second, the associated decision version is often NP-complete, even if the underlying single objective problem is polynomial time solvable. To handle these two difficulties, researchers have been interested in developing approximation algorithms with a priori provable performance guarantees.

Given a positive real $\rho \leq 1$, and considering that all objectives have to be maximized, a ρ-approximation of the set of efficient solutions is a set of solutions that includes, for each efficient solution, a solution that approximates it within a factor ρ on *all* objectives. The ρ-approximation typically contains several incomparable solutions and it is assumed that one solution is selected with the help of a, yet unkown, *a posteriori* decision process.

One of the most important results concerning the approximation of multiobjective problems was given by Papadimitriou and Yannakakis [18]: under certain general assumptions, multiobjective optimization problems always have at least one $(1-\varepsilon)$-approximation of size polynomial in the size of the instance and $1/\varepsilon$, for any given accuracy $\varepsilon > 0$. This result makes the computation of approximate efficient sets of multiobjective problems accessible to polynomial time algorithms.

Nevertheless the efficient set is not the unique object that one can approximate. A popular approach in multiobjective optimization consists in optimizing only one objective while the others are turned into budget constraints [21, 20, 11, 6]. Budget constraints come from an *a priori* decision process which restricts the set of desired solutions. It is noteworthy that the efficient set approach and the budget approach are essentially the same [18].

In another popular approach, no decision process is sought. The goal is to compute a single solution which approximates a vector composed of the optimal values on every objective taken separately [22, 19, 3, 1]. Contrasting with the previous approaches, this framework aims at approximating an ideal point which is the image of a not necessarily feasible solution. Hence no ρ-approximation for every ρ is guaranteed to exist. Note that the ideal point approach and the efficient set approach restricted to sets of size 1 coincide. The former is a particular case of the latter. Since generating several solutions allows better approximations than what a single solution can achieve, approximation ratios under the respective approaches are not directly comparable.

Previous results for the multiobjective TSP are known; most of them follow the efficient set approach, approximating the Pareto set with two or more solutions, but some of them use the ideal point approach. In this article we exclusively follow the ideal point approach and provide deterministic approximation algorithms whose performance guarantees improve on previous results.

Previous results. Multiobjective TSP is well studied from the approximation point of view. Manthey and Ram [16] follow the efficient set approach for several variants of multiobjective Min TSP. In particular they generalize the well known tree doubling algorithm to provide a $(2+\epsilon)$-approximation of the efficient set. The other results of [16] deal with multiobjective Min TSP with the sharpened triangle inequality and multiobjective Min TSP with distance 1 or 2. This latter problem is investigated in [2, 4] under the efficient set approach.

More rencently Bläser et al. [7] study the multiobjective Max TSP with k objective func-

tions. Using the efficient set approach they devise randomized approximation algorithms with ratios $\frac{1}{k}-\epsilon$ and $\frac{1}{k+1}-\epsilon$ for the symmetric and asymmetric versions respectively. Subsequently these results were significantly improved by Manthey [14] who provides randomized approximation algorithms, using the efficient set approach, with ratios $\frac{2}{3}-\epsilon$ and $\frac{1}{2}-\epsilon$ for the symmetric and asymmetric versions respectively. These algorithms use as a black box the randomized PTAS for min-weight matching given by Papadimitriou and Yannakakis [18]. Recently, Manthey [15] establishes deterministic approximation algorithms, using the efficient set approach, with ratios $\frac{1}{2k}-\epsilon$ and $\frac{1}{4k-2}-\epsilon$ for the symmetric and asymmetric versions respectively that can be improved for the biobjective case to ratios $\frac{3}{8}-\epsilon$ and $\frac{1}{4}-\epsilon$ respectively.

Manthey also investigates the approximation of Biobjective Max TSP under the ideal point approach [14, 15], i.e. approximate efficient sets of size one. If the single objective Max TSP problem is ρ-approximable then Biobjective Max TSP is $\frac{\rho}{3}$-approximable with one solution [14]. Taking the best polynomial time approximation algorithms known so far for the symmetric Max TSP, he derives a $\frac{61}{243}$-approximate (resp. $\frac{7}{24}$-approximate) tour without (resp. with) the triangle inequality. The ratios come from a $\frac{61}{81}$-approximation and a 7/8-approximation given in [8] and [12] respectively. As mentioned very recently in [15], using a new $\frac{7}{9}$-approximation [17], the first ratio becomes $\frac{7}{27}$ instead of $\frac{61}{243}$. Another positive consequence of the general technique is that every biobjective instance admits a single $\frac{1}{3}$-approximate tour. From the negative side, Manthey [14] gives a 5 node non metric instance in which no single tour can be $(1/3+\epsilon)$-approximate ($\epsilon > 0$), thus meeting the previous bound. To our best knowledge, no such upper bound is known for metric instances so it is still possible that a single ρ-approximate tour exists in biobjective Max TSP for some $\rho > 1/3$. Finally one can observe that known inapproximability results on the single objective Max TSP imply that the general technique is limited to provide biobjective $(1/3-\epsilon)$-approximation in polynomial time ($\epsilon > 0$).

New results. In this paper, we establish a general algorithm which computes a maximum value matching on each objective taken separately and combines them into a single Hamiltonian cycle having a performance guarantee on both objectives. The algorithm is analyzed in three cases. When both objective functions fulfill the triangle inequality, we obtain a $\frac{5}{12}-\epsilon \approx 0.41$-approximate algorithm which improves the aforementioned $\frac{7}{24}-\epsilon \approx 0.291$-approximation. In this case, we also propose a 4-node instance without any single $(\frac{1}{2}+\epsilon)$-approximate solution and a family of instances without any single $(\frac{3}{4}+\epsilon)$-approximate solution when the number of nodes tends to infinity. If only one objective function fulfills the triangle inequality, we obtain a $(\frac{3}{8}-\epsilon)$-approximate algorithm. In the case where no objective function satisfies the triangle inequality, a quick analysis gives a ratio $1/4-\epsilon$ but in a more accurate case analysis, we can show that the algorithm is $\frac{1+2\sqrt{2}}{14}-\epsilon \approx 0.27$-approximate, improving the aforementioned $\frac{7}{27} \approx 0.259$-approximation. An extension of Manthey's instance to any number of vertices precludes any $(\frac{1}{3}+\epsilon)$-approximate algorithm returning one solution.

We conclude our work by considering the case of an unbounded number of objectives. We denote by n and k the number of vertices and objectives respectively. If all objective functions satisfy the triangle inequality, we show that every tour is $\frac{2}{n}$-approximate, and this ratio is tight for k sufficiently large.

The following table gives a summary of mentioned results on the biobjective Max TSP ($k = 2$). Approximations achieved with several solutions follow the Pareto set approach while those limited to one solution follow the ideal point approach.

Biobjective Max TSP			
	randomized algo.	deterministic algo.	this paper (deterministic)
general case	$2/3-\epsilon$ [14] several solutions	$7/27 \approx 0.259$ one solution [14, 17] $3/8-\epsilon$ several solutions [15]	$\frac{1+2\sqrt{2}}{14}-\epsilon \approx 0.27$ one solution
metric case	$2/3-\epsilon$ [14] several solutions	$7/24 \approx 0.291$ one solution [14, 12]	$5/12-\epsilon \approx 0.41$ one solution

Organization of the article. In Section 2 we give definitions on the problems and concepts used throughout the article. In Section 3 we establish some non existence results which give upper bounds on possible approximation ratios under the ideal point approach. Section 4 presents a general algorithm for Biobjective Max TSP and its analysis in three cases depending on the (non) metric nature of the objective functions. In Section 5 we improve the analysis of the previous algorithm in the non metric case. In Section 6 we consider the case of an unbounded number of objective functions. Future works are provided in a final section.

2 Preliminaries

Let $G = (V, E)$ be a complete undirected graph with a nonnegative weight $w(e)$ on every edge $e \in E$ and $n = |V|$ vertices. The *weight* of a set of edges $E' \subseteq E$ is the sum of the weights of the edges in E' and is denoted by $w(E')$. An instance is metric if its weights satisfy the triangle inequality, namely $w(x, z) \leq w(x, y) + w(y, z)$ for all distinct vertices $x, y, z \in V$.

Max TSP is to find a Hamiltonian cycle or tour (i.e. a cycle that visits every vertex of the graph exactly once) of maximum weight in a complete graph. In the multiobjective Maximum Traveling Salesman Problem every edge is endowed with k nonnegative values. For the biobjective case ($k = 2$), each edge $e \in E$ has a nonnegative weight $w(e)$ and a nonnegative length $\ell(e)$. Similarly the length of a set of edges E', denoted by $\ell(E')$, is the sum of the lengths of its elements.

Each feasible tour T is represented in the objective space by its corresponding objective vector $(w(T), \ell(T))$. A tour T *dominates* a tour T' if and only if $w(T) \geq w(T')$ and $\ell(T) \geq \ell(T')$ with at least one strict inequality. A tour T is *efficient* if and only if no other tour T' dominates T, and $(w(T), \ell(T))$ is said to be *non-dominated*. An *efficient set* contains, for each non-dominated vector, a corresponding efficient solution (no need to keep two tours having the same objective vector).

Unfortunately computing the *efficient set* of multiobjective Max TSP cannot be done in polynomial time, unless $P = NP$, so we are interested in its polynomial time computable approximations. For any $0 < \rho \leq 1$, a tour T ρ-approximates another tour T^* if and only if $w(T) \geq \rho w(T^*)$ and $\ell(T) \geq \rho \ell(T^*)$. A set of feasible tours $\mathcal{A}$ is a ρ-approximation of the efficient set $\mathcal{P}$ if for every $T^* \in \mathcal{P}$, there exists $T \in \mathcal{A}$ such that T ρ-approximates T^*. If $\mathcal{A}$ is reduced to a single tour, we say that we follow the ideal point approach.

Define opt_w (resp. opt_ℓ) as $\max_{T \in \mathcal{F}} w(T)$ (resp. $\max_{T \in \mathcal{F}} \ell(T)$) where $\mathcal{F}$ denotes the set of feasible tours. Under the ideal point approach, a tour T is a ρ-approximation if and only if $w(T) \geq \rho\ opt_w$ and $\ell(T) \geq \rho\ opt_\ell$.

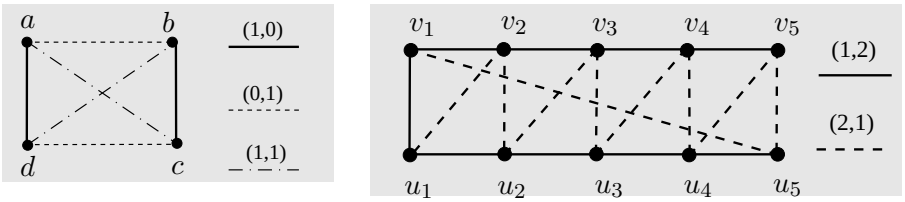

Figure 1: (Left) There is no $(0.5+\epsilon)$-approximate solution in this instance where every objective function satisfies the triangle inequality. (Right) Instance with $r = 5$ where non represented edges have value $(1, 1)$.

3 Non existence of a single ρ-approximate solution

It is unlikely that every instance admits a single solution which is nearly optimal for w and ℓ at the same time. Thus instances without any ρ-approximate solution imply that no deterministic ρ-approximate algorithm (even exponential) exists.

If the triangle inequality is satisfied on both objectives, the example given in Figure 1 (left) shows that there does not *always* exist a $(\frac{1}{2}+\epsilon)$-approximate solution, for all $\epsilon > 0$. The three possible tours in this instance are indeed (a, b, c, d, a), (a, c, d, b, a), and (a, c, b, d, a) whose values are $(2, 2)$, $(2, 4)$, and $(4, 2)$. However this instance only contains 4 nodes so it does not prevent an algorithm to provide a $(0.5+\epsilon)$-approximate solution for 5 nodes and more.

However one can build an instance which does not contain any $(\frac{3}{4}+\epsilon)$-approximate solution for n sufficiently large. The instance contains $2r$ nodes $\{v_1, \cdots, v_r\} \cup \{u_1, \cdots, u_r\}$. Edges (u_i, v_i) have value $(2, 1)$ for $i = 2, \cdots r$, see Figure 1 (right). Edges (u_i, v_{i+1}) have value $(2, 1)$ for $i = 1, \cdots r-1$. Edges (u_i, u_{i+1}) and (v_i, v_{i+1}) have value $(1, 2)$ for $i = 1, \cdots r-1$. Edges (u_1, v_1) and (u_r, v_1) have value $(1, 2)$ and $(2, 1)$ respectively. Any other edge has value $(1, 1)$. The coordinates being 1 or 2, the triangle inequality is satisfied. The tour containing all edges of value $(2, 1)$ (resp. $(1, 2)$) has value $(4r-1, 2r)$ (resp. $(2r, 4r-1)$) so the optimal weight/length is $4r-1$. Any given tour uses α edges with value $(2, 1)$, β edges with value $(1, 2)$ whereas $\alpha+\beta \leq 2r$. Its value is then $(2\alpha+\beta, 2\beta+\alpha) \leq (2r+\alpha, 2r+\beta)$. Observe that $\min\{2r+\alpha, 2r+\beta\} = 2r + \min\{\alpha, \beta\} \leq 3r$. Hence any tour is at most $\frac{3r}{4r-1}$-approximate.

If the objective functions do not necessarily fulfill the triangle inequality, Manthey [14] proved that for a K_5 there does not exist a $(\frac{1}{3}+\epsilon)$-approximate algorithm, for all $\epsilon > 0$. We can easily generalize his result to K_n with $n \geq 5$ in order to obtain an asymptotic result. For every $n \geq 5$, consider K_n where a fixed K_4 is decomposable into 2 Hamiltonian paths P_w and P_ℓ. For every edge $e \in E(K_n)$, set $w(e) = 1$ and $\ell(e) = 0$ if $e \in P_w$, $w(e) = 0$ and $\ell(e) = 1$ if $e \in P_\ell$ and $w(e) = 0$ and $\ell(e) = 0$ if $e \notin P_w \cup P_\ell$. We can check that there are four non-dominated tours T_i, $i = 1, \ldots, 4$ with $w(T_1) = w(P_w) = 3$, $\ell(T_1) = \ell(P_w) = 0$, $w(T_2) = w(P_\ell) = 0$, $\ell(T_2) = \ell(P_\ell) = 3$, $w(T_3) = 2$, $\ell(T_3) = 1$ and $w(T_4) = 1$, $\ell(T_4) = 2$. In conclusion, a single solution never approximates the Pareto set of the biobjective Max TSP with ratio better than $1/3$ for K_n with $n \geq 5$.

4 A generic algorithm for Biobjective Max TSP

In this section, we present an algorithm for the Biobjective Max TSP. This algorithm is based on the combination of the edges of a maximum weight matching for the objective w and a maximum weight matching for the objective ℓ. The algorithm is as follows :

1. Build a maximum weight (resp. length) matching of G and denote it by M_w (resp. M_ℓ). The set of edges $M_w \cup M_\ell$ is made of p connected components $C_1, \ldots, C_p$. Each C_i is a cycle of even size, or a path of length at least one. Note that there is at most one path of length at least two in $M_w \cup M_\ell$ (because the graph is complete and we can assume that M_w are M_ℓ are of maximum size). Likewise, each path of length one is in $M_w \cap M_\ell$.

2. For each component C_i which is a cycle, remove the edge in $C_i \cap M_w$ which has a minimum weight.
   We thus obtain a set of paths, which is called a *partial tour*.

3. Add edges in order to connect these paths and obtain an Hamiltonian cycle of K_n (edges are added arbitrarily unless otherwise noted. This step is detailled inside the proofs when needed).

Let us now show that the Hamiltonian cycle obtained with this algorithm has a weight larger than or equal to $\alpha w(M_w)$ and a length larger than or equal to $\alpha \ell(M_\ell)$, where $0 < \alpha \leq 1$. We will determine the value of α in a general graph (cf. Lemma 1), in a graph where one objective function (w.l.o.g. w) fulfills the triangle inequality (cf. Lemma 2), and in a graph where both objective functions fulfill the triangle inequality (cf. Lemma 3).

Lemma 1 *Step 1 and 2 of the algorithm build in polynomial time a partial tour on K_n with weight at least $\frac{1}{2}w(M_w)$ and length at least $\frac{1}{2}\ell(M_\ell)$.*

Proof: For each component C_i which is a cycle, step 2 of the algorithm removes the edge in $C_i \cap M_w$ with minimum weight. Since $|C_i \cap M_w| \geq 2$ the loss in weight is at most $w(C_i \cap M_w)/2$. The resulting set of edges is a partial tour of weight at least $\frac{1}{2}\sum_{i=1}^{p} w(C_i \cap M_w) = \frac{1}{2}w(M_w)$ and length $\sum_{i=1}^{p} \ell(C_i \cap M_\ell) = \ell(M_\ell)$. □

In the following Lemmas we consider two cases:

- Case 1: at the end of Step 1 of the algorithm, every component C_i is a cycle
- Case 2: at the end of Step 1 of the algorithm, at least one component C_i is a cycle and at least one component $C_{i'}$ is not a cycle.

If no component is a cycle then we are already done since the set of edges is then a partial tour of weight $w(M_w)$ and length $\ell(M_\ell)$.

Lemma 2 *Assuming that w satisfies the triangle inequality, we can build in polynomial time a partial tour on K_n with weight at least $\frac{3}{4}w(M_w)$ and length at least $\frac{3}{4}\ell(M_\ell)$.*

Proof: We distinguish two cases depending on the value of p that is the number of connected components of $M_w \cup M_\ell$. If $p = 1$ then C_1 is either a tour or a cycle on $n-1$ nodes (in this case n is odd) with weight at least $w(M_w)$ and length at least $\ell(M_\ell)$. If C_1 is a cycle on $n-1$ nodes, let x be the isolated node. Then by replacing any edge $(u, v) \in M_w$ by $(u, x), (x, v)$,

we get a tour C' of K_n satisfying $w(C') \geq w(C_1) \geq w(M_w)$ due to the triangle inequality and $\ell(C') \geq \ell(M_\ell)$.

Let us now consider the case where $p \geq 2$. Assume that case 1 occurs, that is each component C_i is a cycle and thus it contains at least four edges. Since $p \geq 2$ and $|M_\ell \cap C_i| \geq 2$ for each C_i we have $|M_\ell| \geq 4$. It follows that if $e \in M_\ell$ is an edge of minimum length among the edges of M_ℓ, then $\ell(e) \leq \ell(M_\ell)/4$. Thus, by deleting e, we are in case 2 since $\cup_{i=1}^{p} C_i \setminus \{e\}$ contain at least one cycle and at least one path with $w(\cup_{i=1}^{p} C_i \setminus \{e\}) \geq w(M_w)$ and

$$\ell(\cup_{i=1}^{p} C_i \setminus \{e\}) \geq 3\ell(M_\ell)/4 \tag{1}$$

Now, assume that case 2 occurs. By renaming the connected components, we can assume that there is an integer $r \in \{1, \ldots, p\}$ such that C_i for $i \geq r$ is not a cycle whereas C_i for $1 \leq i < r$ is a cycle. Let x and y be the two extremities of C_r. Proceed repeatedly as follows, for $i = r-1$ down to 1. Remove an edge of minimum weight in $M_w \cap C_i$ and call it (v_1^i, v_2^i). Add the edge with maximum weight between (v_1^i, x) and (v_2^i, x). If $w(v_1^i, x) \geq w(v_2^i, x)$ then $x := v_2^i$, otherwise $x := v_1^i$. By this way the procedure maintains a path with extremities x and y, while reducing the number of cycles. At the end of the procedure we get a partial tour that is the union between a path and $\cup_{i=r}^{p} C_i$. Using the triangle inequality we know that $\max\{w(v_1^i, x), w(v_2^i, x)\} \geq (w(v_1^i, x) + w(v_2^i, x))/2 \geq w(v_1^i, v_2^i)/2$, meaning that each time an edge (v_1^i, v_2^i) is removed ($i \in \{1, \ldots, r-1\}$), another one with at least half its weight is added so, in total, the loss in weight is bounded by $\frac{1}{2}\sum_{i=1}^{r-1} w(v_1^i, v_2^i)$. Since $|M_w \cap C_i| \geq 2$ we deduce that $w(v_1^i, v_2^i) \leq w(M_w \cap C_i)/2$. Summing up the previous inequality, we deduce that $\sum_{i=1}^{r-1} w(v_1^i, v_2^i) \leq w(\cup_{i=1}^{r-1} C_i \cap M_w)/2 \leq w(M_w)/2$. Thus the total loss in weight is bounded by $w(M_w)/4$.

In conclusion the partial tour has weight at least $3w(M_w)/4$ and length at least $3\ell(M_\ell)/4$ by inequality (1). □

Lemma 3 *Assuming that w and ℓ satisfy the triangle inequality, we can build in polynomial time a partial tour on K_n with weight at least $\frac{5}{6}w(M_w)$ and length at least $(\frac{5}{6} - \varepsilon(n))\ell(M_\ell)$. Here $\varepsilon(n) = 2/(n-1)$ and then tends to 0 when n tends to ∞.*

Proof: As it is done in Lemma 2, we transform case 1 into case 2. Thus, suppose that we are in case 1 that is each component C_i is a cycle and w.l.o.g. that the edge of M_ℓ with minimum length is e. Remove this edge e to create a path with endpoints denoted by x and y. When n is even (resp. odd) this deletion induces a loss of at most $2\ell(M_\ell)/n = \varepsilon(n)\ell(M_\ell)$ (resp. $2\ell(M_\ell)/(n-1) = \varepsilon(n)\ell(M_\ell)$). Note that $\varepsilon(n)$ tends to 0 when n tends to ∞.

Suppose now that we are in the case 2. As it is done in Lemma 2 we can assume that there is an integer $r \in \{1, \ldots, p\}$ such that C_i for $i \geq r$ is not a cycle whereas C_i for $1 \leq i < r$ is a cycle. We are going to patch the cycles to C_r, one by one. We explain how to patch C_1, and the procedure is repeated for the cycles $C_2, \cdots, C_{r-1}$. Let x and y be the two extremities of C_r.

If $|C_1 \cap M_w| \geq 3$ then delete an edge of minimum weight and call it (v_1^1, v_2^1). We get that $w(v_1^1, v_2^1) \leq \frac{1}{3}w(C_1 \cap M_w)$. Add the edge with maximum weight between (v_1^1, x) and (v_2^1, x). By the triangle inequality, $\max\{w(v_1^1, x), w(v_2^1, x)\} \geq w(v_1^1, v_2^1)/2$. If $w(v_1^1, x) \geq w(v_2^1, x)$ then $x := v_2^1$, otherwise $x := v_1^1$. Disregarding the weight of the edges in $C_1 \cap M_\ell$, the modification causes a loss in weight of at most $w(v_1^1, v_2^1) - w(v_1^1, v_2^1)/2 = w(v_1^1, v_2^1)/2 \leq \frac{1}{6}w(C_1 \cap M_w)$. Since no edge from M_ℓ was removed, and disregarding the length of the edges in $C_1 \cap M_w$, the

modification does not cause any loss in length. Hence the patching guarantees that the new path P satisfies $w(P) \geq w(C_r) + 5w(C_1 \cap M_w)/6$ and $\ell(P) \geq \ell(C_r) + \ell(C_1 \cap M_\ell)$.

Now suppose that C_1 is a cycle on 4 nodes and contains four edges (a,b), (b,c), (c,d), (d,a) such that $C_1 \cap M_w = \{(a,b),(c,d)\}$ and $C_1 \cap M_\ell = \{(b,c),(a,d)\}$. Using the triangle inequality we get that

$$w(a,c) + w(b,d) + w(C_1 \cap M_\ell) \geq w(C_1 \cap M_w) \quad (2)$$

$$\ell(a,c) + \ell(b,d) + \ell(C_1 \cap M_w) \geq \ell(C_1 \cap M_\ell) \quad (3)$$

- Suppose that $\ell(C_1 \cap M_w) \geq \ell(C_1 \cap M_\ell)/8$. W.l.o.g., assume $\ell(a,d) \geq \ell(b,c)$. Remove (b,c) and add the edge with maximum length between (b,x) and (x,c). Since $\max\{\ell(b,x),\ell(x,c)\} \geq \ell(b,c)/2$ by the triangle inequality, we get that the new path P satisfies $\ell(P) \geq \ell(C_r) + \ell(C_1 \cap M_w) + \ell(a,d) + \ell(b,c)/2 \geq \ell(C_r) + \ell(C_1 \cap M_\ell)/8 + \ell(C_1 \cap M_\ell)/2 + \ell(a,d)/2 \geq \ell(C_r) + \ell(C_1 \cap M_\ell)/8 + \ell(C_1 \cap M_\ell)/2 + \ell(C_1 \cap M_\ell)/4 = \ell(C_r) + 7\ell(C_1 \cap M_\ell)/8$.

- Suppose that $w(C_1 \cap M_\ell) \geq w(C_1 \cap M_w)/8$. W.l.o.g., assume $w(a,b) \geq w(c,d)$. Remove (c,d) and add the edge with maximum length between (c,x) and (x,d). Since $\max\{w(c,x),w(x,d)\} \geq w(c,d)/2$ by the triangle inequality, we get as in the previous case that $w(P) \geq w(C_r) + w(C_1 \cap M_\ell) + w(a,b) + w(c,d)/2 \geq w(C_r) + 7w(C_1 \cap M_w)/8$.

- Now suppose that $\ell(C_1 \cap M_w) < \ell(C_1 \cap M_\ell)/8$ and $w(C_1 \cap M_\ell) < w(C_1 \cap M_w)/8$. Using Inequalities (2) and (3) we get that $w(a,c) + w(b,d) > 7w(C_1 \cap M_w)/8$ and $\ell(a,c) + \ell(b,d) > 7\ell(C_1 \cap M_\ell)/8$. In this case the new path P obtained by adding any two edges to $(a,c),(b,d)$ and C_r satisfies $w(P) \geq w(C_r) + 7w(C_1 \cap M_w)/8$ and $\ell(P) \geq \ell(C_r) + 7\ell(C_1 \cap M_\ell)/8$.

In conclusion, when C_1 contains four nodes, we can always patch it to C_r so that the loss in weight (resp. length) is at most $w(C_1 \cap M_w)/8$ (resp. $\ell(C_1 \cap M_\ell)/8$).

We have seen that this loss is of (at most) $1/6$ on both objective functions when C_1 contains at least six nodes. We deduce that after the patching of all cycles C_i for $i < r$, the current solution is a path P and its weight (resp. length) is at least $w(C_r) + \frac{5}{6}w\big(\bigcup_{i=1}^{r-1} C_i \cap M_w\big)$ (resp. $\ell(C_r) + \frac{5}{6}\ell\big(\bigcup_{i=1}^{r-1} C_i \cap M_\ell\big)$). Adding $\cup_{i=r+1}^{p} C_i$ to P, we get a partial tour P'. Using $w(C_r) \geq w(C_r \cap M_w)$ and $\ell(C_r) \geq \ell(C_r \cap M_\ell) - \varepsilon(n)\ell(M_\ell)$ we get that the solution P' has weight (resp. length) at least $\frac{5}{6}w(M_w)$ (resp. $(\frac{5}{6} - \varepsilon(n))\ell(M_\ell)$). □

Theorem 1 *We can build in polynomial time a single tour on K_n which constitutes a $(\rho - \xi(n))$-approximate Pareto set for the biobjective Max TSP where $\rho = 5/12$ when w and ℓ satisfy the triangle inequality, $\rho = 3/8$ when only w satisfies the triangle inequality and $\rho = 1/4$ when neither w nor ℓ satisfies the triangle inequality. Here $\xi(n) = \Theta(1/n)$ and then tends to 0 when n tends to ∞.*

Proof: Consider first the case when x and ℓ satisfy the triangle inequality. Lemma 3 states that we can build a partial tour with weight (resp. length) at least $5w(M_w)/6$ (resp. $(\frac{5}{6} - \varepsilon(n))\ell(M_\ell)$) where $\varepsilon(n) = \frac{2}{n-1}$. If the partial tour is not a tour then connect its components to create a tour. Using the fact that every edge weight (resp. length) is nonnegative, the weight (resp. length) cannot decrease. Denote by opt_w (resp. opt_ℓ) the optimal weight (resp. length) of a tour. It is well known that $w(M_w) \geq (\frac{1}{2} - \varepsilon'(n))opt_w$ and $\ell(M_\ell) \geq (\frac{1}{2} - \varepsilon'(n))opt_\ell$ where

$\varepsilon'(n) = 0$ when n is even, otherwise $\varepsilon'(n) = \frac{1}{2p}$. Let $\xi(n) = \frac{\varepsilon(n)}{2} + \frac{5\varepsilon'(n)}{6} - \varepsilon'(n)\varepsilon(n)$. We get that the tour constructed has weight at least $\frac{5}{6}w(M_w) \geq \frac{5}{6}(\frac{1}{2} - \varepsilon'(n))opt_w > (\frac{5}{12} - \xi(n))opt_w$. The length is at least $(\frac{5}{6} - \varepsilon(n))\ell(M_\ell) \geq (\frac{5}{6} - \varepsilon(n))(\frac{1}{2} - \varepsilon'(n))opt_\ell = (\frac{5}{12} - \xi(n))opt_\ell$. Use Lemmas 2 and 3 and similar arguments for the other cases. □

5 An improved analysis

In this section, we refine the analysis of our approximation algorithm when the triangle inequality is not assumed on any objective function. We show that the tour returned by our algorithm is an asymptotic $\frac{1+2\sqrt{2}}{14} \approx 0.273$ approximation of the ideal point. Recall that some instances of the problem do not admit any $(\frac{1}{3} + \epsilon)$-approximate solution, for all $\epsilon > 0$ [14].

The intuition behind the improved analysis is the following. The ratio 1/4 of Theorem 1 follows from two observations: the tour returned by the approximation algorithm is a 1/2-approximation of the maximum weight/length matching, and this latter is an asymptotic 1/2-approximation of the maximum weight/length tour. Taken separately both observations are tight but we exploit the fact that they cannot occur simultaneously.

Theorem 2 *We can build in polynomial time a $(\frac{1+2\sqrt{2}}{14} - \xi(n))$-approximate Pareto set containing a single tour on K_n for Biobjective Max TSP. Here $\xi(n) = \Theta(1/n)$ and then, tends to 0 when n tends to ∞.*

Proof: Define δ as $\frac{4\sqrt{2}-5}{14} \approx 0.0469$. Actually, δ is the positive root of equation $-1 + 20x + 28x^2 = 0$. We can show that every instance K_n of the problem satisfies one of the following statements:

- (i) a partial tour P' on K_n with weight at least $(\frac{1}{2}+\delta)w(M_w)$ and, at the same time, length at least $(\frac{1}{2} + \delta)\ell(M_\ell)$ exists and can be computed in polynomial time.
- (ii) every Hamiltonian cycle has weight at most $(\frac{3}{2} + 7\delta)w(M_w)$ and, at the same time, its length is at most $(\frac{3}{2} + 7\delta)\ell(M_\ell)$.

Recall that $w(M_w) \geq (\frac{1}{2} - \varepsilon'(n))opt_w$, $\ell(M_\ell) \geq (\frac{1}{2} - \varepsilon'(n))opt_\ell$ where $\varepsilon'(n) = 0$ when n is even, otherwise $\varepsilon'(n) = 1/2n$. If K_n satisfies (i), then by hypothesis the partial tour P' has weight (resp. length) at least $(1/4 + \delta/2 - \xi(n))opt_w$ (resp. $(1/4 + \delta/2 - \xi(n))opt_\ell$) with $\xi(n) = \varepsilon'(n)(1/2 + \delta)$. If K_n satisfies (ii), then starting from $M_w \cup M_\ell$ as it is done in previous section and using Lemma 1, a partial solution P with weight (resp. length) at least $w(M_w)/2$ (resp. $\ell(M_\ell)/2$) can be built in polynomial time. Now, since by hypothesis $opt_w \leq (\frac{3}{2} + 7\delta)w(M_w)$, and $opt_\ell \leq (\frac{3}{2} + 7\delta)\ell(M_\ell)$, the partial solution P has a weight (resp. length) at least $\frac{1}{2}\frac{opt_w}{(\frac{3}{2}+7\delta)}$ (resp. $\frac{1}{2}\frac{opt_\ell}{(\frac{3}{2}+7\delta)}$).

Finally remark that on the one hand, a tour can be obtained by connecting the components of a partial tour without decreasing the weight/length since every edge weight/length is nonnegative and on the other hand, $\frac{1}{2}\frac{1}{(\frac{3}{2}+7\delta)} = 1/4 + \delta/2 = \frac{1+2\sqrt{2}}{14}$ because δ is the positive root of equation $-1 + 20x + 28x^2 = 0$.

We assume $n \geq 5$, since otherwise the partial solution P given in Lemma 1 has weight (resp. length) at least $opt_w/2$ (resp. $opt_\ell)/2$).

We consider three distinct cases which can be distinguished in polynomial time.

Figure 2: The cycle C_1. Bold edges belong to M_ℓ and dashed edges belong to M_w; the remaining edges form a third matching $M_r = \{(a,c),(b,d)\}$.

Case 1. Let us suppose that there exists a cycle, say C_1 w.l.o.g., such that the edge with minimum weight in $C_1 \cap M_w$ has weight at least $(\frac{1}{2}-\delta)w(M_w)$ and, at the same time, the edge with minimum length in $C_1 \cap M_\ell$ has length at least $(\frac{1}{2}-\delta)\ell(M_\ell)$. Since $1/2-\delta > 1/3$, C_1 must be a cycle on four nodes, i.e. $C_1 \cap M_w = \{(a,b),(c,d)\}$ and $C_1 \cap M_\ell = \{(b,c),(a,d)\}$ (see Figure 2 for an illustration).

We deduce that $\max\{w(a,b),w(c,d)\} = w(M_w) - w(\bigcup_{i=2}^{p} C_i \cap M_w) - \min\{w(a,b),w(c,d)\}$ and $\max\{\ell(a,d),\ell(b,c)\} = \ell(M_\ell) - \ell(\bigcup_{i=2}^{p} C_i \cap M_\ell) - \min\{\ell(a,d),\ell(b,c)\}$. Using $\min\{w(a,b),w(c,d)\} \geq (1/2-\delta)w(M_w)$ and $\min\{\ell(b,c),\ell(a,d)\} \geq (1/2-\delta)\ell(M_\ell)$ in the previous equalities gives

$$\max\{w(a,b),w(c,d)\} \leq (1/2+\delta)w(M_w) - w(\bigcup_{i=2}^{p} C_i \cap M_w) \quad (4)$$

$$\max\{\ell(b,c),\ell(a,d)\} \leq (1/2+\delta)\ell(M_\ell) - \ell(\bigcup_{i=2}^{p} C_i \cap M_\ell) \quad (5)$$

In addition we deduce that

$$w(\bigcup_{i=2}^{p} C_i \cap M_w) \leq 2\delta w(M_w) \text{ and } \ell(\bigcup_{i=2}^{p} C_i \cap M_\ell) \leq 2\delta\ell(M_\ell) \quad (6)$$

We conduct a subcase analysis depending on the weight or the length of the edges having at least one endpoint in $V(C_1)$: case (1.1.w) $\max\{w(e) : e \in C_1 \cap M_\ell\} > 2\delta w(M_w)$, case (1.1.$\ell$) $\max\{\ell(e) : e \in C_1 \cap M_w\} > 2\delta\ell(M_\ell)$, case (1.2.$w$) $\max\{w(a,c),w(b,d)\} > (\frac{1}{2}+\delta)w(M_w)$, case (1.2.$\ell$) $\max\{\ell(a,c),\ell(b,d)\} > (\frac{1}{2}+\delta)\ell(M_\ell)$, case (1.3.$w$) $\max\{w(i,j) : i \in V(C_1), j \notin V(C_1)\} > 2\delta w(M_w)$, case (1.3.$\ell$) $\max\{\ell(i,j) : i \in V(C_1), j \notin V(C_1)\} > 2\delta\ell(M_\ell)$ and case (1.4) $\max\{w(e) : e \in C_1 \cap M_\ell\} \leq 2\delta w(M_w)$, $\max\{\ell(e) : e \in C_1 \cap M_w\} \leq 2\delta\ell(M_\ell)$, $\max\{w(a,c),w(b,d)\} \leq (\frac{1}{2}+\delta)w(M_w)$, $\max\{\ell(a,c),\ell(b,d)\} \leq (\frac{1}{2}+\delta)\ell(M_\ell)$, $\max\{w(i,j) : i \in V(C_1), j \notin V(C_1)\} \leq 2\delta w(M_w)$ and $\max\{\ell(i,j) : i \in V(C_1), j \notin V(C_1)\} \leq 2\delta\ell(M_\ell)$.

We can prove that in case (1.4) the instance K_n satisfies (ii) whereas in other cases the instance K_n satisfies (i).

(1.1.w) If $w(a,d) > 2\delta w(M_w)$ or $w(b,c) > 2\delta w(M_w)$ then remove (c,d). We get that $w(a,b) + w(b,c) + w(a,d) > (\frac{1}{2}+\delta)w(M_w)$ and $\ell(a,b) + \ell(b,c) + \ell(a,d) \geq (1-2\delta)\ell(M_\ell) \geq (1/2+\delta)\ell(M_\ell)$.

(1.1.ℓ) If $\ell(a,b) > 2\delta\ell(M_\ell)$ or $\ell(c,d) > 2\delta\ell(M_\ell)$ then remove (b,c). We get that $\ell(a,d)+\ell(a,b)+\ell(c,d) > (\frac{1}{2}+\delta)\ell(M_\ell)$ and $w(a,d)+w(a,b)+w(c,d) \geq (1-2\delta)w(M_w) \geq (1/2+\delta)w(M_w)$.

(1.2.w) If $\max\{w(a,c), w(b,d)\} > (\frac{1}{2}+\delta)w(M_w)$ then remove $\{(a,b),(c,d)\}$ and add the edge with maximum weight between (a,c) and (b,d), say (a,c) without loss of generality. We get that $w(a,c)+w(b,c)+w(a,d) > (\frac{1}{2}+\delta)w(M_w)$ and $\ell(a,d)+\ell(a,c)+\ell(b,c) \geq (1-2\delta)\ell(M_\ell) \geq (1/2+\delta)\ell(M_\ell)$.

(1.2.ℓ) If $\max\{\ell(a,c), \ell(b,d)\} > (\frac{1}{2}+\delta)\ell(M_\ell)$ then remove $\{(a,d),(b,c)\}$ and add the edge with maximum length between (a,c) and (b,d), say (a,c) without loss of generality. We get that $w(a,c)+w(a,b)+w(c,d) > (1-2\delta)w(M_w) > (1/2+\delta)w(M_w)$ and $\ell(a,c)+\ell(a,b)+\ell(c,d) \geq (\frac{1}{2}+\delta)\ell(M_\ell)$.

(1.3.w) Suppose there exists an edge (i,j) such that $i \in \{a,b,c,d\}$, $j \in V \setminus \{a,b,c,d\}$ and $w(i,j) > 2\delta w(M_w)$. If $i \in \{a,b\}$ (resp. $i \in \{c,d\}$) then only keep the edges $\{(i,j),(b,c),(c,d),(a,d)\}$ (resp. $\{(i,j),(b,c),(a,b),(a,d)\}$) while any other edge is deleted. Suppose w.l.o.g. that $i \in \{a,b\}$, the case $i \in \{c,d\}$ being treated similarly. Using $w(c,d) \geq (\frac{1}{2}-\delta)w(M_w)$ and $\ell(C_1 \cap M_\ell) \geq 2(1/2-\delta)\ell(M_\ell)$ by hypothesis, we get that $w(i,j)+w(b,c)+w(c,d)+w(a,d) \geq w(i,j)+w(c,d) > (1/2+\delta)w(M_w)$. At the same time $\ell(i,j)+\ell(b,c)+\ell(c,d)+\ell(a,d) \geq \ell(a,d)+\ell(b,c) \geq 2(1/2-\delta)\ell(M_\ell) \geq (1/2+\delta)\ell(M_\ell)$.

(1.3.ℓ) Suppose there exists an edge (i,j) such that $i \in \{a,b,c,d\}$, $j \in V\setminus\{a,b,c,d\}$ and $\ell(i,j) > 2\delta\ell(M_\ell)$. If $i \in \{a,d\}$ (resp. $i \in \{b,c\}$) then only keep the edges $\{(i,j),(a,b),(b,c),(c,d)\}$ (resp. $\{(i,j),(a,b),(c,d),(a,d)\}$) while any other edge is deleted. Suppose w.l.o.g. that $i \in \{a,d\}$, the case $i \in \{b,c\}$ being treated similarly. Using $\ell(b,c) \geq (\frac{1}{2}-\delta)\ell(M_\ell)$ and $w(C_1 \cap M_w) \geq 2(1/2-\delta)w(M_w)$ by hypothesis, we get that $\ell(i,j)+\ell(a,b)+\ell(b,c)+\ell(c,d) \geq \ell(i,j)+\ell(b,c) > (1/2+\delta)\ell(M_\ell)$. At the same time $w(i,j)+w(a,b)+w(b,c)+w(c,d) \geq w(a,b)+w(c,d) \geq 2(1/2-\delta)w(M_w) \geq (1/2+\delta)w(M_w)$.

(1.4) Suppose that $w(a,d) \leq 2\delta w(M_w)$, $w(b,c) \leq 2\delta w(M_w)$, $\ell(a,b) \leq 2\delta\ell(M_\ell)$, $\ell(c,d) \leq 2\delta\ell(M_\ell)$, $\max\{w(a,c), w(b,d)\} \leq (\frac{1}{2}+\delta)w(M_w)$ and $\max\{\ell(a,c), \ell(b,d)\} \leq (\frac{1}{2}+\delta)\ell(M_\ell)$. In addition suppose that for all (i,j) such that $i \in \{a,b,c,d\}$ and $j \in V \setminus \{a,b,c,d\}$, we have $w(i,j) \leq 2\delta w(M_w)$ and $\ell(i,j) \leq 2\delta\ell(M_\ell)$. We claim that the weight of any tour is bounded above by $(\frac{3}{2}+7\delta)w(M_w)$ while its length is at most $(\frac{3}{2}+7\delta)\ell(M_\ell)$. The edge set of the graph is partitioned into three sets $E_1 = \{(i,j) : i,j \in \{a,b,c,d\}\}$, $E_2 = \{(i,j) : i \in \{a,b,c,d\} \text{ and } j \notin \{a,b,c,d\}\}$ and $E_3 = \{(i,j) : i,j \notin \{a,b,c,d\}\}$. A tour T is a set of edges partitioned in three sets $T_i = T \cap E_i$ for $i = 1,2,3$.

First observe that T_3 is a set of paths which can be decomposed into two matchings M and M' (alternate edges in M and edges in M'). If $w(M) \geq w(M')$ and $w(M) > w(\bigcup_{i=2}^{p} C_i \cap M_w)$ then $M \cup \{(a,b),(c,d)\}$ is a matching on the whole graph with larger weight than $w(M_w)$, contradiction. Using this argument and a similar one for the length we get that

$$w(T_3) \leq 2w(\bigcup_{i=2}^{p} C_i \cap M_w) \text{ and } \ell(T_3) \leq 2\ell(\bigcup_{i=2}^{p} C_i \cap M_\ell) \tag{7}$$

We get that $w(C_1 \cap M_w) = w(M_w) - w(\bigcup_{i=2}^{p} C_i \cap M_w)$ and $\ell(C_1 \cap M_\ell) = \ell(M_\ell) - \ell(\bigcup_{i=2}^{p} C_i \cap M_\ell)$. Thus, using inequality (7), we deduce that $w(C_1 \cap M_w) + w(T_3) \leq w(M_w) + w(\bigcup_{i=2}^{p} C_i \cap M_w)$ and $\ell(C_1 \cap M_\ell) + \ell(T_3) \leq \ell(M_\ell) + \ell(\bigcup_{i=2}^{p} C_i \cap M_\ell)$; using inequality (6), we obtain

$$w(C_1 \cap M_w) + w(T_3) \leq (1+2\delta)w(M_w) \text{ and } \ell(C_1 \cap M_\ell) + \ell(T_3) \leq (1+2\delta)\ell(M_\ell) \quad (8)$$

Because $\min\{w(a,b), w(c,d)\} \geq (\frac{1}{2}-\delta)w(M_w)$ and $\min\{\ell(b,c), \ell(a,d)\} \geq (\frac{1}{2}-\delta)\ell(M_\ell)$, we also deduce from inequality (8) that:

$$\max\{w(a,b), w(c,d)\} + w(T_3) \leq (\frac{1}{2}+3\delta)w(M_w) \quad (9)$$

$$\max\{\ell(b,c), \ell(a,d)\} + \ell(T_3) \leq (\frac{1}{2}+3\delta)\ell(M_\ell) \quad (10)$$

Since $M_r = \{(a,c),(b,d)\}$ is a matching on $V(C_1)$ (see Figure 2), we get that $w(M_r) \leq w(C_1 \cap M_w)$ and $\ell(M_r) \leq \ell(C_1 \cap M_\ell)$. Thus, using inequality (8) we get that:

$$w(M_r) + w(T_3) \leq (1+2\delta)w(M_w) \text{ and } \ell(M_r) + \ell(T_3) \leq (1+2\delta)\ell(M_\ell) \quad (11)$$

We also get that $w(M_r) \leq w(C_1 \cap M_w) = w(M_w) - w(\bigcup_{i=2}^{p} C_i \cap M_w)$ and $\ell(M_r) \leq \ell(C_1 \cap M_\ell) = \ell(M_\ell) - \ell(\bigcup_{i=2}^{p} C_i \cap M_\ell)$. Thus, on the one hand, using inequalities (4) (resp., (5)) and (7), we deduce:

$$w(M_r) + \max\{w(a,b), w(c,d)\} + w(T_3) \leq (\frac{3}{2}+\delta)w(M_w) \quad (12)$$

$$\ell(M_r) + \max\{\ell(b,c), \ell(a,d)\} + \ell(T_3) \leq (\frac{3}{2}+\delta)\ell(M_\ell) \quad (13)$$

Inequalities (7) and (6) also give

$$w(T_3) \leq 4\delta w(M_w) \text{ and } \ell(T_3) \leq 4\delta\ell(M_\ell) \quad (14)$$

By hypothesis every edge in E_2 has weight (resp. length) at most $2\delta w(M_w)$ (resp. $2\delta\ell(M_\ell)$). It follows that

$$w(T_2) \leq 2|T_2|\delta w(M_w) \text{ and } \ell(T_2) \leq 2|T_2|\delta\ell(M_\ell) \quad (15)$$

Now we argue on $T \cap E_1$. Note that $|T \cap E_1| \leq 3$ since $n \geq 5$. Then, if

- $T \cap E_1 = \{(a,c),(b,d),(a,b)\}$. The tour must contain 2 edges in E_2. Thus, using inequalities (12) and (15) with $|T_2| = 2$, we get $w(T) = w(M_r) + w(a,b) + w(T_3) + w(T_2) \leq \left(\frac{3}{2}+\delta+4\delta\right) w(M_w) \leq (\frac{3}{2}+7\delta)w(M_w)$ and using inequality (11) $\ell(T) = \ell(M_r) + \ell(T_3) + \ell(a,b) + \ell(T_2) \leq (1+2\delta+2\delta+4\delta)\,\ell(M_\ell) = (1+8\delta)\ell(M_\ell) < (\frac{3}{2}+7\delta)\ell(M_\ell)$.
- $T \cap E_1 = \{(a,c),(a,b),(c,d)\}$. The tour must contain 2 edges in E_2. Thus, using inequalities (8) and (15) with $|T_2| = 2$, we get that $w(T) = w(a,c) + w(C_1 \cap M_w) + w(T_3) + w(T_2) \leq (1/2+\delta+1+2\delta+4\delta)w(M_w) = (\frac{3}{2}+7\delta)w(M_w)$ and using inequality (14) $\ell(T) = \ell(a,c) + \ell(a,b) + \ell(c,d) + \ell(T_2) + \ell(T_3) \leq (1/2+\delta+2\delta+2\delta+4\delta+4\delta)\ell(M_\ell) = (1/2+13\delta)\ell(M_\ell) < (\frac{3}{2}+7\delta)\ell(M_\ell)$.

- $T \cap E_1 = \{(a,b),(b,c),(c,d)\}$. The tour must contain 2 edges in E_2. Thus, using inequalities (8) and (15) with $|T_2| = 2$, we get that $w(T) = w(C_1 \cap M_w) + w(T_3) + w(b,c) + w(T_2) \leq (1 + 2\delta + 2\delta + 4\delta)w(M_w) = (1 + 8\delta)w(M_w) < (\frac{3}{2} + 7\delta)w(M_w)$. Using inequality (10) we get that $\ell(T) = \ell(a,b) + \ell(c,d) + \ell(b,c) + \ell(T_2) + \ell(T_3) \leq (2\delta + 2\delta + 1/2 + 3\delta + 4\delta)\ell(M_\ell) = (1/2 + 11\delta) < (\frac{3}{2} + 7\delta)\ell(M_\ell)$
- $T \cap E_1 = \{(a,c),(b,d)\}$. The tour must contain 4 edges in E_2. Thus, using inequalities (11) and (15) with $|T_2| = 4$, $w(T) = w(M_r) + w(T_3) + w(T_2) \leq (1 + 2\delta + 8\delta)w(M_w) = (1 + 10\delta)w(M_w) < (\frac{3}{2} + 7\delta)w(M_w)$ and $\ell(T) = \ell(M_r) + \ell(T_3) + \ell(T_2) = (1 + 10\delta)\ell(M_\ell) \leq (\frac{3}{2} + 7\delta)\ell(M_\ell)$.
- $T \cap E_1 = \{(a,b),(a,c)\}$. The tour must contain 4 edges in E_2. Thus, using inequalities (9) and (15) with $|T_2| = 4$, we get that $w(T) = w(a,b) + w(T_3) + w(a,c) + w(T_2) \leq (1/2 + 3\delta + 1/2 + \delta + 8\delta)w(M_w) = (1 + 12\delta)w(M_w) \leq (\frac{3}{2} + 7\delta)w(M_w)$ and using inequality (14) $\ell(T) = \ell(a,b) + \ell(a,c) + \ell(T_2) + \ell(T_3) \leq (2\delta + 1/2 + \delta + 8\delta + 4\delta)\ell(M_\ell) = (1/2 + 15\delta)\ell(M_\ell) \leq (\frac{3}{2} + 7\delta)\ell(M_\ell)$.
- $T \cap E_1 = \{(a,b),(c,d)\}$. The tour must contain 4 edges in E_2. Thus, using inequalities (8) and (15) with $|T_2| = 4$, $w(T) = w(C_1 \cap M_w) + w(T_3) + w(T_2) \leq (1 + 2\delta + 8\delta)w(M_w) = (1 + 10\delta)w(M_w) \leq (\frac{3}{2} + 7\delta)w(M_w)$ and using inequality (14) $\ell(T) = \ell(a,b) + \ell(c,d) + \ell(T_2) + \ell(T_3) \leq (2\delta + 2\delta + 8\delta + 4\delta)\ell(M_\ell) = 16\delta\ell(M_\ell) \leq (\frac{3}{2} + 7\delta)\ell(M_\ell)$.
- $T \cap E_1 = \{(a,c)\}$. The tour must contain 6 edges in E_2. Thus, using inequalities (14) and (15) with $|T_2| = 6$, $w(T) = w(a,c) + w(T_2) + w(T_3) \leq (1/2 + \delta + 12\delta + 4\delta)w(M_w) = (1/2 + 17\delta)w(M_w) \leq (\frac{3}{2} + 7\delta)w(M_w)$ and $\ell(T) = \ell(a,c) + \ell(T_2) + \ell(T_3) \leq (1/2 + \delta + 12\delta + 4\delta)\ell(M_\ell) = (1/2 + 17\delta)\ell(M_\ell) \leq (\frac{3}{2} + 7\delta)\ell(M_\ell)$.
- $T \cap E_1 = \{(a,b)\}$. The tour must contain 6 edges in E_2. Thus, using inequalities (9) and (15) with $|T_2| = 6$, $w(T) = w(a,b) + w(T_3) + w(T_2) \leq (1/2 + 3\delta + 12\delta)w(M_w) = (1/2 + 15\delta)w(M_w) \leq (\frac{3}{2} + 7\delta)w(M_w)$ and using inequality (14) $\ell(T) = \ell(a,b) + \ell(T_2) + \ell(T_3) \leq (2\delta + 12\delta + 4\delta)\ell(M_\ell) = 18\delta\ell(M_\ell) \leq (\frac{3}{2} + 7\delta)\ell(M_\ell)$.
- $T \cap E_1 = \emptyset$. To cover a, b, c and d, the tour must contain 8 edges in E_2. Thus, using inequalities (14) and (15) with $|T_2| = 8$, $w(T) = w(T_2) + w(T_3) \leq (16\delta + 4\delta)w(M_w) = 20\delta w(M_w) \leq (\frac{3}{2} + 7\delta)w(M_w)$ and $\ell(T) = \ell(T_2) + \ell(T_3) \leq (16\delta + 4\delta)\ell(M_\ell) = 20\delta\ell(M_\ell) \leq (\frac{3}{2} + 7\delta)\ell(M_\ell)$.
- Any other subcase is isomorphic to a previously analyzed subcase by flipping w and ℓ.

The conclusion is that every tour T is such that $w(T) \leq (\frac{3}{2} + 7\delta)w(M_w)$ and $\ell(T) \leq (\frac{3}{2} + 7\delta)\ell(M_\ell)$.

Case 2. Suppose that there exists a cycle, say C_1 w.l.o.g., such that the edge with minimum weight in $C_1 \cap M_w$ has weight *at most* $(\frac{1}{2} - \delta)w(M_w)$ and, at the same time, the edge with minimum length in $C_1 \cap M_\ell$ has length *at least* $(\frac{1}{2} - \delta)\ell(M_\ell)$. We will prove that the instance K_n satisfies (i). Again, since $1/2 - \delta > 1/3$, C_1 must be a cycle on four nodes. Again we suppose that $C_1 \cap M_w = \{(a,b),(c,d)\}$ and $C_1 \cap M_\ell = \{(b,c),(a,d)\}$.

Remove the edge in $C_1 \cap M_w$ with minimum weight and for any other cycle C_i remove one edge in $C_i \cap M_\ell$ arbitrarily. We get a partial tour. Since $w(M_w) - \min\{w(a,b), w(c,d)\} \geq (\frac{1}{2} + \delta)w(M_w)$ and $\ell(C_1 \cap M_\ell) = \ell(a,d) + \ell(b,c) \geq 2(\frac{1}{2} - \delta)\ell(M_\ell) \geq (\frac{1}{2} + \delta)\ell(M_\ell)$, the partial tour has weight (resp. length) at least $(\frac{1}{2} + \delta)w(M_w)$ (resp. $(\frac{1}{2} + \delta)\ell(M_\ell)$).

The case where there exists a cycle C_1 such that the edge with minimum weight in $C_1 \cap M_w$ has weight *at least* $(\frac{1}{2} - \delta)w(M_w)$ and, at the same time, the edge with minimum length in $C_1 \cap M_\ell$ has length *at most* $(\frac{1}{2} - \delta)\ell(M_\ell)$ is dealt with similar arguments by flipping w and ℓ.

Case 3. Denote by e_i^w (resp. e_i^ℓ) the edge in $C_i \cap M_w$ (resp. $C_i \cap M_\ell$) with minimum weight (resp. length). We deal with the remaining case where $w(e_i^w) \leq (\frac{1}{2} - \delta)w(M_w)$ and $\ell(e_i^\ell) \leq (\frac{1}{2} - \delta)\ell(M_\ell)$ for all $i \in \{1, \ldots, p\}$. We will prove that the instance K_n satisfies (i). Since every cycle contains at least two edges of M_w and also two edges of M_ℓ we deduce that

$$\sum_{i=1}^{p} w(e_i^w) \leq w(M_w)/2 \text{ and } \sum_{i=1}^{p} \ell(e_i^\ell) \leq \ell(M_\ell)/2 \tag{16}$$

- Suppose there is an index i^* such that $w(e_{i^*}^w) \geq \delta w(M_w)$. Then for every cycle C_i except C_{i^*} remove e_i^w. Remove $e_{i^*}^\ell$. Using the first part of inequality (16) we get a partial tour with weight at least $w(M_w) - \sum_{i=1}^{p} w(e_i^w) + w(e_{i^*}^w) \geq (1/2 + \delta)w(M_w)$ and length at least $\ell(M_\ell) - \ell(e_{i^*}^\ell) \geq (1/2 + \delta)\ell(M_\ell)$.

- Suppose there is an index i^* such that $\ell(e_{i^*}^\ell) \geq \delta \ell(M_\ell)$. With similar arguments we can build a partial tour with weight at least $(1/2 + \delta)w(M_w)$ and length at least $(1/2 + \delta)\ell(M_\ell)$.

- Suppose that $w(e_i^w) < \delta w(M_w)$ and $\ell(e_i^\ell) < \delta \ell(M_\ell)$ for all i. If $\sum_{i=1}^{p} w(e_i^w) \leq (\frac{1}{2} - \delta)w(M_w)$, then by removing e_i^w for $i = 1, \ldots, p$ we get a partial tour P with weight at least $(1/2 + \delta)w(M_w)$ and length at least $\ell(M_\ell)$. Otherwise, there exists an index $i^* < p$ such that

$$\sum_{i=1}^{i^*} w(e_i^w) \leq (\frac{1}{2} - \delta)w(M_w) \text{ and } \sum_{i=1}^{i^*+1} w(e_i^w) > (\frac{1}{2} - \delta)w(M_w) \tag{17}$$

  Using inequalities (16), (17) and $w(e_{i^*+1}^w) < \delta w(M_w)$ we get that

$$\begin{aligned} \sum_{i=1}^{i^*+1} w(e_i^w) + \sum_{i=i^*+2}^{p} w(e_i^w) &\leq w(M_w)/2 \\ \sum_{i=i^*+2}^{p} w(e_i^w) &< \delta w(M_w) \\ \sum_{i=i^*+1}^{p} w(e_i^w) &< 2\delta w(M_w) \leq (\frac{1}{2} - \delta)w(M_w) \end{aligned} \tag{18}$$

  Now remark that

$$\min\{\sum_{i=1}^{i^*} \ell(e_i^\ell), \sum_{i=i^*+1}^{p} \ell(e_i^\ell)\} \leq \frac{1}{2} \sum_{i=1}^{p} \ell(e_i^\ell) \leq \frac{1}{4}\ell(M_\ell) \tag{19}$$

  where the right part of inequality (16) is used. If $\sum_{i=1}^{i^*} \ell(e_i^\ell) \leq \sum_{i=i^*+1}^{p} \ell(e_i^\ell)$ then remove e_i^ℓ for $i = 1, \ldots, i^*$ and remove e_i^w for $i = i^* + 1, \ldots, p$. We get a partial tour with weight at

least $(1/2+\delta)w(M_w)$ by inequality (18) and length at least $3\ell(M_\ell)/4 \geq (1/2+\delta)\ell(M_\ell)$ by inequality (19). If $\sum_{i=1}^{i^*} \ell(e_i^\ell) > \sum_{i=i^*+1}^{p} \ell(e_i^\ell)$ then remove e_i^w for $i = 1, \ldots, i^*$ and remove e_i^ℓ for $i = i^*+1, \ldots, p$. We get a partial tour with weight at least $(1/2+\delta)w(M_w)$ by inequality (17) and length at least $3\ell(M_\ell)/4 \geq (1/2+\delta)\ell(M_\ell)$ by inequality (19). □

6 Dealing with many objectives

As mentioned by Manthey [14], when we consider more than two objectives, and the triangle inequality is not assumed, there is no ρ-approximate tour, for any $\rho > 0$. This can be seen on a 3-objective instance where a given node is adjacent to three edges of weight $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ respectively while any other edge has weight $(0,0,0)$. Any tour must be of weight 0 on a coordinate. Then we consider now k objectives $w_1, \ldots, w_k$, which all fulfill the triangle inequality.

Theorem 3 *Suppose that every objective function satisfies the triangle inequality. Then, any tour is a $\frac{2}{n}$-approximate Pareto set for k-objective Max TSP. Moreover, there exists a family of instances for which no $(\frac{2}{n}+\varepsilon)$-approximate Pareto set containing a single tour exists, if k is not fixed.*

Proof: We show that each possible tour is at least a $\frac{2}{n}$-approximation of the Pareto set. Let T_i^* be an optimal tour for the objective w_i, and let $w_i(T_i^*)$ be the weight of T_i^* on objective w_i. Let (u,v) be the maximum weight edge of T_i^* with respect to objective w_i. The weight of this edge is at least $\frac{w_i(T_i^*)}{n}$. Let T be any tour. There are two paths between vertices u and v in T, and the weight of each of these paths on objective w_i is at least $\frac{w_i(T_i^*)}{n}$ because w_i fulfills the triangle inequality. This tour is thus a $\frac{2}{n}$-approximation for objective w_i.

Let us show the tightness of this result. Consider a graph $G = (V, E)$ with $n \geq 4$ vertices, where n is an even number. Consider that there are $k = \binom{n}{n/2}/2 - (n/2-1)$ objectives. There are $\binom{n}{n/2}/2$ possible ways to separate the vertices of V into two sets of equal size, V_1 and V_2. We consider a set $\mathcal{S}$ containing $\binom{n}{n/2}/2 - (n/2-1)$ of these partitions. To each considered partition (V_1, V_2), we associate an objective for which the weight of each edge (u,v) with $u \in V_1$ and $v \in V_1$ is 0, the weight of each edge (u,v) with $u \in V_2$ and $v \in V_2$ is 0, and the weight of each edge (u,v) with $u \in V_1$ and $v \in V_2$ is 1. The optimal tour for this objective alternates vertices of V_1 and V_2 so its weight is n. Let us consider an algorithm $\mathcal{A}$ which outputs only one tour $T = (x_1, x_2, \ldots, x_n, x_1)$. There are $\frac{n}{2}$ pairs $(x_i, x_{i+n/2})$ (for $i \in \{1, \ldots, \frac{n}{2}\}$). Given a pair $(x_i, x_{i+n/2})$, we denote by V_i the set of vertices $\{x_i, \ldots, x_{i+n/2-1}\}$. There must be an index $i \in \{1, \ldots, \frac{n}{2}\}$ such that $(V_i, V \setminus V_i) \in \mathcal{S}$. Tour T has a weight of 2 with respect to the objective associated with partition $(V_i, V \setminus V_i)$. Thus there is no $\varepsilon > 0$ such that Algorithm $\mathcal{A}$ outputs a $(\frac{2}{n}+\varepsilon)$-approximate tour. □

This proposition is also true if all except one objective c_k fulfill the triangular inequality: the optimal tour for c_k is optimal for c_k and $\frac{2}{n}$-approximate for the other objectives, which fulfill the triangle inequality.

7 Future work

We considered the biobjective Max TSP. It would be interesting to study the cases where there is a fixed number $k \geq 3$ of objectives. There are still gaps between positive and negative

results given in this article. For example, when both objective functions are metric, we provide a polynomial time $(\frac{5}{12} - \epsilon)$-approximation and an upper bound of $\frac{3}{4}$. Maybe both results can be improved. An interesting future work would be to investigate randomized algorithms. Another direct extension of our work is to consider the multiobjective asymmetric Max TSP.

References

[1] Eric Angel, Evripidis Bampis, and Aleksei V. Fishkin. A note on scheduling to meet two min-sum objectives. *Operations Research Letters*, 35(1):69–73, 2007.

[2] Eric Angel, Evripidis Bampis, and Laurent Gourvès. Approximating the Pareto curve with local search for the bicriteria TSP(1,2) problem. *Theoretical Computer Science*, 310(1-3):135–146, 2004.

[3] Eric Angel, Evripidis Bampis, and Laurent Gourvès. Approximation algorithms for the bi-criteria weighted max-cut problem. *Discrete Applied Mathematics*, 154(12):1685–1692, 2006.

[4] Eric Angel, Evripidis Bampis, Laurent Gourvès, and Jérôme Monnot. (Non)-approximability for the multi-criteria TSP(1,2). In *Proceeding of the 15th International Symposium on Fundamentals of Computation Theory (FCT 2005), LNCS 3623*, pages 329–340, 2005.

[5] Eric Angel, Evripidis Bampis, and Alexander Kononov. On the approximate tradeoff for bicriteria batching and parallel machine scheduling problems. *Theoretical Computer Science*, 306(1-3):319–338, 2003.

[6] André Berger, Vincenzo Bonifaci, Fabrizio Grandoni, and Guido Schäfer. Budgeted matching and budgeted matroid intersection via the gasoline puzzle. *Mathematical Programming*, 128(1-2):355–372, 2011.

[7] Markus Bläser, Bodo Manthey, and Oliver Putz. Approximating multi-criteria Max-TSP. In *Proceedings of the 16th Annual European Symposium (ESA 2008), LNCS 5193*, pages 185–197, 2008.

[8] Zhi-Zhong Chen, Yuusuke Okamoto, and Lusheng Wang. Improved deterministic approximation algorithms for max TSP. *Information Processing Letters*, 95(2):333–342, 2005.

[9] Mathias. Ehrgott. *Multicriteria optimization.* LNEMS, Springer-Verlag, 2005.

[10] Thomas Erlebach, Hans Kellerer, and Ulrich Pferschy. Approximating multiobjective knapsack problems. *Management Science*, 48(12):1603–1612, 2002.

[11] Sung-Pil Hong, Sung-Jin Chung, and Bum Hwan Park. A fully polynomial bicriteria approximation scheme for the constrained spanning tree problem. *Operations Research Letters*, 32(3):233–239, 2004.

[12] Lukasz Kowalik and Marcin Mucha. Deterministic 7/8-approximation for the metric maximum TSP. *Theoretical Computer Science*, 410(47-49):5000–5009, 2009.

[13] Bodo Manthey. Multi-criteria TSP: Min and max combined. In *Proceedings of the 7th International Workshop on Approximation and Online Algorithms (WAOA 2009), LNCS 5893*, pages 205–216, 2009.

[14] Bodo Manthey. On approximating multi-criteria TSP. In *Proceedings of the 26th International Symposium on Theoretical Aspects of Computer Science (STACS 2009)*, pages 637–648, 2009.

[15] Bodo Manthey. Deterministic algorithms for multi-criteria TSP. In *Proceedings of the 8th Annual Conference on Theory and Applications of Models of Computation (TAMC 2011), LNCS 6648*, pages 264–275, 2011.

[16] Bodo Manthey and L. Shankar Ram. Approximation algorithms for multi-criteria Traveling Salesman Problems. *Algorithmica*, 53(1):69–88, 2009.

[17] Katarzyna E. Paluch, Marcin Mucha, and Aleksander Madry. A 7/9 - approximation algorithm for the maximum traveling salesman problem. In *Proceedings of the 12th International Workshop on Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX-RANDOM 2009), LNCS 5687*, pages 298–311, 2009.

[18] Christos H. Papadimitriou and Mihalis Yannakakis. On the approximability of trade-offs and optimal access of web sources. In *Proceedings of the 41st Annual Symposium on Foundations of Computer Science (FOCS 2000)*, pages 86–92, 2000.

[19] April Rasala, Clifford Stein, Eric Torng, and Patchrawat Uthaisombut. Existence theorems, lower bounds and algorithms for scheduling to meet two objectives. In *Proceedings of the 13th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2002)*, pages 723–731, 2002.

[20] R. Ravi and Michel X. Goemans. The constrained minimum spanning tree problem. In *Proceeding of the 5th Scandinavian Workshop on Algorithm Theory (SWAT 1996), LNCS 1097*, pages 66–75, 1996.

[21] R. Ravi, Madhav V. Marathe, S. S. Ravi, Daniel J. Rosenkrantz, and Harry B. Hunt III. Many birds with one stone: multi-objective approximation algorithms. In *Proceedings of the 23rd Annual ACM Symposium on Theory of Computing (STOC 1993)*, pages 438–447, 1993.

[22] Clifford Stein and Joel Wein. On the existence of schedules that are near-optimal for both makespan and total weighted completion time. *Operations Research Letters*, 21(3):115–122, 1997.