Differential approximation for optimal satisfiability and related problems

Cristina Bazgan Vangelis Th. Paschos
LAMSADE, Université Paris-Dauphine
Place du Maréchal de Lattre de Tassigny, 75775 Paris, France
{bazgan,paschos}@lamsade.dauphine.fr

Abstract

We study the differential approximation of several optimal satisfiability problems. We prove that MIN SAT is not differential $1/m^{1-\varepsilon}$-approximable for any $\varepsilon > 0$, where m is the number of clauses. Exhibiting that any differential approximation algorithm for MAX MINIMAL VERTEX COVER can be transformed into a differential approximation algorithm for MIN kSAT achieving the same differential performance ratio, we are lead to study the differential approximability of MAX MINIMAL VERTEX COVER and MIN INDEPENDENT DOMINATING SET. Both of them are equivalent for the differential approximation. For these problems we prove a strong inapproximability result, namely, any approximation algorithm has worst-case differential approximation ratio equal to 0.

1 Preliminaries

In this paper we deal with the approximation of classical optimal satisfiability problems as well as with restrictive versions of them. A complete discussion about the numerous applications where these problems are encountered is presented in [BP98]. We also consider some graph-problems as MAX and MIN INDEPENDENT DOMINATING SET and MAX and MIN MINIMAL VERTEX COVER which, as we prove are strongly related to the satisfiability problems we deal with. More precisely, we consider the following **NP**-hard problems.

MAX (MIN) SAT
: **Input:** a set of clauses $C_1, \dots, C_m$ on n variables $x_1, \dots, x_n$.
Output: a truth assignment to the variables that maximizes (minimizes) the number of clauses satisfied.

MAX (MIN) DNF
: **Input:** a set of conjunctions $C_1, \dots, C_m$ on n variables $x_1, \dots, x_n$.
Output: a truth assignment to the variables that maximizes (minimizes) the number of conjunctions satisfied.

MAX NAE 3SAT
: **Input:** a set of conjunctions $C_1, \dots, C_m$ of three literals on n variables $x_1, \dots, x_n$.
Output: a truth assignment to the variables that maximizes the number of conjunctions satisfied in such a way that any one of them has at least one true literal and at least one false literal.

MIN (MAX) MINIMAL VERTEX COVER
: **Input:** a graph $G = (V, E)$.
Output: a minimal vertex cover (a set $S \subseteq V$ such that, $\forall (u, v) \in E$, $u \in S$ or $v \in S$) of minimum (maximum) size.

Min (Max) Independent Dominating Set

Input: a graph $G = (V, E)$.

Output: a maximal independent set (a set $S \subseteq V$ such that, $\forall u, v \in S$, $(u, v) \notin E$ and $\forall u \notin S, \exists v \in S$, $(u, v) \in E$) of minimum (maximum) size.

We study the approximability of all these problems using the so-called *differential approximation ratio* which, informally, for an instance I measures the relative position of the value of an approximated solution in the interval [worst-value feasible solution of I, optimal-value solution of I]. We first recall a few definitions about differential and standard approximabilities. Given an instance x of an optimization problem and a feasible solution y of x, we denote by $m(x, y)$ the value of the solution y, by $\text{opt}(x)$ the value of an optimal solution of x, and by $\omega(x)$ the value of a worst solution of x. The *standard performance, or approximation, ratio* of y is defined as $r(x, y) = \max\{m(x, y)/\text{opt}(x), \text{opt}(x)/m(x, y)\}$, while the *differential performance, or approximation, ratio* of y is defined as $\rho(x, y) = |m(x, y) - \omega(x)|/|\text{opt}(x) - \omega(x)|$.

For a function f, $f(n) > 1$, an algorithm is a *standard $f(n)$-approximation algorithm* for a problem Π if, for any instance x of Π, it returns a solution y such that $r(x, y) \leqslant f(|x|)$, where $|x|$ is the size of x. We say that an optimization problem is *standard constantly approximable* if, for some constant $c > 1$, there exists a polynomial time standard c-approximation algorithm for it. An optimization problem has a *standard polynomial time approximation schema* if it has a polynomial time standard $(1 + \varepsilon)$-approximation, for every constant $\varepsilon > 0$. Similarly, for a function f, $f(n) < 1$, an algorithm is a *differential $f(n)$-approximation algorithm* for a problem Π if, for any instance x of Π, it returns a solution y such that $\rho(x, y) \geqslant f(|x|)$. We say that an optimization problem is *differential constantly approximable* if, for some constant $\delta < 1$, there exists a polynomial time differential δ-approximation algorithm for it. An optimization problem has a *differential polynomial time approximation scheme* if it has a polynomial time differential $(1 + \varepsilon)$-approximation, for every constant $\varepsilon > 0$. We say that two optimisation problems are *differential equivalent* if a differential δ-approximation algorithm for one of them implies a differential δ-approximation algorithm for the other one.

We first study the differential approximability of the optimal satisfiability problems defined above. This study brings to the fore an interesting relationship between Min kSat and Min Minimal Vertex Cover which can be informally described as follows: *any differential approximation algorithm for* Min Minimal Vertex Cover *can be transformed into a differential approximation algorithm for* Min kSat *achieving the same differential performance ratio* (problems verifying this relationship will be called *approximate equivalent*). On the other hand, as we will see just below, Max Minimal Vertex Cover is equivalent, for the differential approximation, to the well-known Min Independent Dominating Set. We are so led to study differential approximation results for Max Minimal Vertex Cover and Min Independent Dominating Set.

All the problems we deal with in this paper have the characteristic that computation of both their optimal and worst solutions is **NP**-hard (for example, considering an instance φ of Max kSat, its worst solution is an assignment satisfying the minimum number of the clauses of φ, i.e., an optimal solution for Min kSat on φ). Remark also that, given a graph $G = (V, E)$, the complement, with respect to V of a minimal vertex cover (resp., maximal independent set) is a maximal independent set (resp., minimal vertex cover) of G. In other words, the objective values of Min (Max) Minimal Vertex Cover and of Min (Max) Independent Dominating Set are linked by affine transformations. Note that the differential approximation ratio is stable for the affine transformation, in the sense that pairs of problems, the objective values of which are linked by affine transformations, are differential approximate equivalent.

We study differential approximation preserving reductions for several optimal satisfiability problems. Combining them with a general result linking approximability of maximization problems in differential and standard approximations, we obtain interesting differential inapproximability results for optimal satisfiability. We also prove that MIN kSAT$(B,\bar{B})$ and MAX kSAT$(B,\bar{B})$ reduce to MIN MINIMAL VERTEX COVER-B' and MIN INDEPENDENT DOMINATING SET-B', respectively. These reductions lead us to study the differential approximation of MIN INDEPENDENT DOMINATING SET. For this problem we prove a strong inapproximability result, informally, *unless* $\mathbf{P}=\mathbf{NP}$*, any approximation algorithm has worst-case approximation ratio equal to 0.* This, in some sense, gives to MIN INDEPENDENT DOMINATING SET the status of one of the hardest problems for the differential approximation. To our knowledge, no such result was previously known. Finally, we prove that one of the optimal satisfiability problems we deal with, the MAX NAE 3SAT is differential approximable within ratio bounded below by 0.649.

2 Approximation preserving reductions for optimal satisfiability

We first prove the differential equivalence for MAX SAT and MIN DNF and for MIN SAT and MAX DNF.

Proposition 1. MAX SAT *and* MIN DNF*, as well as* MIN SAT *and* MAX DNF *are differential equivalent.*

Proof. We construct a reduction from MAX SAT to MIN DNF that preserves the differential approximation ratio. Let I be an instance of MAX SAT on n variables and m clauses. The instance I' of MIN DNF contains m clauses and the same set of n variables. With each clause $\ell_1 \vee \ldots \vee \ell_t$ of I we associate in I' the conjunction $\bar{\ell}_1 \wedge \ldots \wedge \bar{\ell}_t$, where $\bar{\ell}_i = \bar{x}_j$ if $\ell_i = x_j$ and $\bar{\ell}_i = x_j$ if $\ell_i = \bar{x}_j$. It is easy to see that $\mathrm{opt}(I') = m - \mathrm{opt}(I)$ and $\omega(I') = m - \omega(I)$. Also, if $m(I', y)$ is the value of the solution y in I', then the same solution y has in I the value $m(I, y) = m - m(I', y)$. Thus, $\rho(I, y) = \rho(I', y)$. The reduction from MIN DNF to MAX SAT is the same.

By an exactly similar reduction, one can prove that MIN SAT and MAX DNF are also approximate equivalent. ∎

By the proof of proposition 1 one easily can deduce that for each constant $k \geqslant 2$, MAX kSAT and MIN kDNF as well as MIN kSAT and MAX kDNF are differential equivalent.

Consider an instance I of a maximization problem Π, and assume that an approximation algorithm A computes a feasible solution S in I. Then, $(m_A(I,S) - \omega(I))/(\mathrm{opt}(I) - \omega(I)) \geqslant \delta$ implies $m_A(I,S)/\mathrm{opt}(I) \geqslant \delta + (1-\delta)\omega(I)$ and this together with $\omega(I) \geqslant 0$ lead $m_A(I,S)/\beta(I) \geqslant \delta$. So the following fact holds and will be used in what follows.

Fact 1. Approximation of a maximization problem Π within differential approximation ratio δ, implies approximation of Π within standard approximation ratio $1/\delta$. ∎

Combining the results of proposition 1 and fact 1 with the result of [PY91]: *for* $k \geqslant 2$ *and* $B \geqslant 3$*,* MAX kSAT$(B,\bar{B})$ *and* MAX kDNF$(B,\bar{B})$ *have no standard polynomial time approximation schemata* ([PY91]), one deduces that for $\mathsf{k} \geqslant 2$ and $\mathsf{B} \geqslant 3$, MAX kSAT$(\mathsf{B},\bar{\mathsf{B}})$, MAX kDNF$(\mathsf{B},\bar{\mathsf{B}})$, MIN kSAT$(\mathsf{B},\bar{\mathsf{B}})$ and MIN kDNF$(\mathsf{B},\bar{\mathsf{B}})$ have no differential polynomial time approximation schemata, unless $\mathbf{P}=\mathbf{NP}$.

3 MIN SAT and MIN VERTEX COVER

MIN VERTEX COVER is as the MIN MINIMAL VERTEX COVER defined in section 1 modulo the fact that the feasible solutions for the former are not mandatorily minimal. In what follows, by reduction from MIN VERTEX COVER, we establish an inapproximability result for MIN SAT.

Proposition 2. *Unless* $\mathbf{co}-\mathbf{RP}=\mathbf{NP}$*,* MIN SAT *is not differential* $1/m^{1-\varepsilon}$*-approximable for any* $\varepsilon > 0$*, where* m *is the number of clauses of the instance.*

Proof. Let $G = (V, E)$ be a graph on n vertices and denote by $V = \{1, \ldots, n\}$ its vertex set. In order to construct an instance I of Min Sat, at each edge $(i, j) \in E, i < j$ we associate a variable x_{ij}. For each vertex i we define a clause C_i, where $C_i = (\vee_{j:(i,j)\in E \wedge i<j} x_{ij}) \vee (\vee_{j:(i,j)\in E \wedge i>j} \bar{x}_{ji})$.

From a vertex cover C of G we define an assignment as follows. For each $i \notin C$ and each $(i, j) \in E$, $x_{ji} = 1$ if $i > j$ and $x_{ij} = 0$ if $i < j$. Since C is a vertex cover, this definition is not contradictory. If $i \notin C$, then C_i is not satisfied and so $\mathrm{opt}(I) \leqslant \mathrm{opt}(G)$.

Given an assignment v of I, let $C = \{i : C_i \text{ is satisfied}\}$. Note that set C is a vertex cover since for $(i, j) \in E$, at least one of C_i and C_j is satisfied and so at least one of the vertices i, j appears in C. So, at each assignment v of I, we associate in G a vertex cover C with $m(G, C) = m(I, v)$. This also proves that $\mathrm{opt}(I) = \mathrm{opt}(G)$.

Finally, using $\omega(I) \leqslant \omega(G)$, it is easy to show that $\rho(G) \geqslant \rho(I)$.

We have seen that Min Vertex Cover is differential equivalent to Max Independent Set (which is as Max Independent Dominating Set modulo the fact that the independent set to compute has not to be minimal). On the other hand since the worst solution for Max Independent Set is the empty set (in other words, $\omega(I) = 0$, $\forall I$), standard and differential approximation ratios coincide. Furthermore, Max Independent Set is not differential $1/n^{1-\varepsilon}$-approximable for any $\varepsilon > 0$, unless $\mathbf{co-RP} = \mathbf{NP}$ ([Has96]). Consequently, Min Vertex Cover is not differential $1/n^{1-\varepsilon}$-approximable for any $\varepsilon > 0$, unless $\mathbf{co-RP} = \mathbf{NP}$ and the result claimed follows. ∎

From the above proof the following corollary is also deduced: Min Sat$(\mathsf{B}, \bar{\mathsf{B}})$ for $\mathsf{B} \geqslant 1$ is not differential $1/\mathsf{m}^{1-\varepsilon}$-approximable for any $\varepsilon > 0$, unless $\mathbf{co-RP} = \mathbf{NP}$.

4 A positive differential approximation result for Max NAE 3Sat

We show in this section that a restrictive version of Max NAE 3Sat, the one on satisfiable instances is differential constantly approximable by the standard 1.096-approximation algorithm of [Zwick98].

Proposition 3. Max NAE 3Sat *on satisfiable instances is differential 0.649-approximable. On the other hand,* Max NAE 3Sat *is not differential 0.917-approximable.*

Proof. Consider a satisfiable instance φ of Max NAE 3Sat defined on m clauses; obviously, $\mathrm{opt}(\varphi) = m$. Run the standard 1.096-approximation algorithm of [Zwick98] on φ to obtain a solution C satisfying $m(\varphi, C) \geqslant m/1.096$. On the other hand any random assignment by values in $\{0, 1\}$ of the variables of φ, where any of the two values is assigned with probability 1/2, will feasibly satisfy $3m/4$ clauses (in other words, the assignments $(1, 1, 1)$ and $(0, 0, 0)$ are to be excluded from the eight possible assignments for each 3-clause); consequently, $\omega(\varphi) \leqslant 3m/4$.

Using the values for $\mathrm{opt}(\varphi)$, $m(\varphi, C)$ and $\omega(\varphi)$, we get $(m(\varphi, C) - \omega(\varphi))/(\mathrm{opt}(\varphi) - \omega(\varphi)) \geqslant ((m/1.096) - (3m/4))/(m - (3m/4)) = 0.712/1.096 > 0.649$.

In order to show the inapproximability result of the second part of the proposition, if one uses fact 1 together with the result of [Zwick98] that Max NAE 3Sat *is not standard approximable within 1.090, unless* ***P=NP***, the 0.917 differential inapproximability bound is immediately deduced and completes the proof. ∎

5 Optimal satisfiability and Min Independent Dominating Set

We now show that Min kSat$(B, \bar{B})$ is differential reducible to Min Minimal Vertex Cover-B'. Note that an analogous result, dealing with standard approximation, is presented in [CST96] between Min Sat and Min Vertex Cover. But this result does not work for the differential approximation.

Proposition 4. MIN kSAT$(B, \bar{B})$ *is differential reducible to* MIN MINIMAL VERTEX COVER-B' *and* MAX kSAT$(B, \bar{B})$ *is differential reducible to* MIN INDEPENDENT DOMINATING SET-B'.

Proof. Let I be an instance of MIN kSAT$(B, \bar{B})$ with n variables and m clauses. In the instance G of MIN MINIMAL VERTEX COVER, with each clause C_i of I we associate a vertex i. We draw an edge between i and j if there is a variable x such that C_i contains x and C_j contains $\bar{x}$. The vertex-degrees of the so constructed graph are bounded above by $B' = kB$.

From an assignment v of I we define a vertex cover C as the set of vertices that correspond to clauses satisfied by v. So, $\text{opt}(G) \leq \text{opt}(I)$.

From a vertex cover C of G we define a partial assignment v as follows: if $i \notin C$ and $x_j \in C_i$ then $x_j = 0$, and if $i \notin C$ and $\bar{x}_j \in C_i$ then $x_j = 1$. Hence, if $i \notin C$ then C_i is not satisfied by v. By the way v has been defined, the number of the non satisfied clauses in I is greater than, or equal to, the number of vertices that are not in C, i.e., $m(I, v) \leq m(G, C)$. This, together with $opt(G) \leq opt(I)$ proved just above, implies $\text{opt}(G) = \text{opt}(I)$.

If C is a minimal vertex cover (for each $i \in C$ there exists $j \notin C$ such that $(i, j) \in E$), then $m(I, v) = m(G, C)$ since the clause C_i is satisfied by v when $i \in C$. Consequently, in particular, $\omega(I) = \omega(G)$ and this concludes the proof of the first differential reducibility claimed.

By a proof similar to the one of proposition 2, one can show that MAX kSAT$(B, \bar{B})$ reduces to MAX MINIMAL VERTEX COVER-B'. Since the former is differential equivalent to MIN INDEPENDENT DOMINATING SET-B', the proof of the second differential reducibility claimed and of the proposition. is concluded ∎

The result above of naturally leads us to study the differential approximation of MIN INDEPENDENT DOMINATING SET. By a rather technical and lengthy way, we can establish a strongly negative differential approximation result showing that *any polynomial approximation algorithm for* MIN INDEPENDENT DOMINATING SET *has (worst-case) differential approximation ratio equal to 0.* This can be done by constructing a reduction from SAT to MIN INDEPENDENT DOMINATING SET such that the graph obtained for the latter has only two distinct feasible solutions (the optimal and the worst one). We show that if an approximation algorithm guarantees any differential approximation ratio different from 0 for MIN INDEPENDENT DOMINATING SET, then it correctly answers *yes* if the instance of SAT is satisfiable, *no* otherwise. Since SAT is **NP**-complete, one concludes that such an approximation algorithm cannot exist for MIN INDEPENDENT DOMINATING SET.

6 Final remarks

We have given in this paper differential inapproximability results for optimal satisfiability problems, as well as for MIN INDEPENDENT DOMINATING SET. For this problem we have shown that any polynomial time approximation algorithm has worst-case differential approximation ratio 0. This result brings MIN INDEPENDENT DOMINATING SET to the status of one of the hardest problems for the differential approximation.

Differential approximation for optimal satisfiability misses until now in positive results besides the one of section 4 on the satisfiable instances of MAX NAE 3SAT. Achievement of non-trivial positive results is a major open problem for us. It seems that, in the opposite of the standard approximation, obtaining constant differential approximation ratios for optimal satisfiability is a rather hard task.

References

[BP98] R. Battiti and M. Protasi, *Algorithms and heuristics for MAX-SAT*, in D. Z. Du and P. M. Pardalos (eds.) *Handbook of Combinatorial Optimization*, Kluwer Academic

Publishers, 1998.

[CST96] P. Crescenzi, R. Silvestri and L. Trevisan, *To weight or not to weight: where is the question?*, Proceedings of the 4th Israeli Symposium on Theory of Computing and Systems, 1996, 68–77.

[Has96] J. Hastad, *Clique is hard to approximate within* $n^{1-\varepsilon}$, Proceedings of the 37th Ann. IEEE Symposium on Foundations of Computer Science, 1996, 627–636.

[PY91] C. Papadimitriou and M. Yannakakis, *Optimization, approximation and complexity classes*, Journal of Computer and System Science 43 (1991), 425–440.

[Zwick98] U. Zwick, *Approximation algorithms for constraint satisfaction problems involving at most three variables per constraint*, Proceedings of the 9th Ann. ACM-SIAM Symposium on Discrete Algorithms, 1998, 201–210.