On the Complexity Landscape of the Domination Chain

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Abstract. In this paper, we survey and supplement the complexity landscape of the domination chain parameters as a whole, including classifications according to approximability and parameterised complexity. Moreover, we provide clear pointers to yet open questions. As this posed the majority of hitherto unsettled problems, we focus on Upper Irredundance and Lower Irredundance that correspond to finding the largest irredundant set and resp. the smallest maximal irredundant set. The problems are proved NP-hard even for planar cubic graphs. While Lower Irredundance is proved not $c\log(n)$-approximable in polynomial time unless $NP \subseteq \mathrm{DTIME}(n^{\log\log n})$, no such result is known for Upper Irredundance. Their complementary versions are constant-factor approximable in polynomial time. All these four versions are APX-hard even on cubic graphs.

1 Introduction

The well-known domination chain

$$\mathrm{ir}(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq \mathrm{IR}(G)$$

links parameters related to the fundamental notions of independence, domination and irredundance in graphs. It was introduced in [22,12], is thoroughly discussed in the textbook [34] and studied further in many ways, [11,21,38,42] showing only a small selection. These studies cover both combinatorial and computational aspects. We focus on the latter aspects in this paper. In this chain, $\gamma(G)$ and $\Gamma(G)$ are the minimum and maximum cardinalities over all minimal dominating sets in G, $\alpha(G)$ is the maximum cardinality of an independent set, $i(G)$ is the minimum cardinality over all maximal independent sets in G. The less known irredundance parameters are explained below.

With $n(G)$ being the order (number of vertices) of G, we can write co$-\zeta(G) = n(G) - \zeta(G)$. Then, we state the following complementary domination chain:

$$\mathrm{co}-\mathrm{IR}(G) \leq \mathrm{co}-\Gamma(G) \leq \mathrm{co}-\alpha(G) \leq \mathrm{co}-i(G) \leq \mathrm{co}-\gamma(G) \leq \mathrm{co}-\mathrm{ir}(G)\,.$$

Sometimes, the complement problems have received their own names, like Nonblocker, Maximum Enclaveless Set, or Maximum Spanning Star Forest, which all refer to the complement problem of Minimum Domination, or, most likely better known, Minimum Vertex Cover which refers to the complement problem of Maximum Independent Set. We will also use $\tau(G)$ instead of co $-\,\alpha(G)$ to refer to this graph parameter.

Throughout this paper, we will use rather standard terminology from graph theory. For any subset $S \subseteq V$ and $v \in S$ we define the private neighbourhood of v with respect to S as $pn(v,S) := N[v] - N[S - \{v\}]$. Any $w \in pn(v,S)$ is called a *private neighbour of v (with respect to S)*. S is called *irredundant* if every vertex in S has at least one private neighbour, i.e., if $|pn(v,S)| > 0$ for every $v \in S$. A maximal irredundant set is also known as an *upper irredundant set*. IR(G) denotes the cardinality of the largest irredundant set in G, while ir(G) is the cardinality of the smallest maximal irredundant set in G that is the smallest upper irredundant set in G. The domination chain is largely due to the following two combinatorial properties: (1) Every maximal independent set is a minimal dominating set. (2) A dominating set $S \subseteq V$ is minimal if and only if $|pn(v,S)| > 0$ for every $v \in S$. Observe that v can be a private neighbour of itself, i.e., a dominating set is minimal if and only if it is also an irredundant set. Actually, every minimal dominating set is also a maximal irredundant set.

For any $\varepsilon > 0$, a graph $G = (V,E)$ is called *everywhere-ε-dense* if every vertex in G has at least $\varepsilon|V|$ neighbours and *average-ε-dense* if $|E| \geq \varepsilon n^2$, for $0 < \varepsilon < 1/2$.

We first present some combinatorial bounds for IR(G). The same kind of bounds have been derived for $\Gamma(G)$ in [6]. Due to the space constraints, here and in the following some proofs (denoted by (*)) are deferred to the appendix.

Lemma 1. *(*) For any connected graph G with $n > 0$ vertices we have:*

$$\alpha(G) \;\leq\; IR(G) \;\leq\; \max\left\{\alpha(G), \frac{n}{2} + \frac{\alpha(G)}{2} - 1\right\} \tag{1}$$

Lemma 2. *(*) For any connected graph G with $n > 0$ vertices, minimum degree δ and maximum degree Δ, we have:*

$$\alpha(G) \;\leq\; IR(G) \;\leq\; \max\left\{\alpha(G),\; \frac{n}{2} + \frac{\alpha(G)(\Delta-\delta)}{2\Delta} - \frac{\Delta-\delta}{\Delta}\right\} \tag{2}$$

This lemma generalises [35, Proposition 12], which states the property for Δ-regular graphs, where, in particular, $\delta = \Delta$. Eq. 1 immediately yields:

Lemma 3. *Let G be a connected graph. Then,*

$$\frac{\tau(G)}{2} + 1 \leq co - IR(G) \leq \tau(G) \tag{3}$$

2 The complexity of the domination chain

We are studying algorithmic and complexity aspects of the domination chain parameters in this paper. For the basic definitions on classical complexity, approximation and parameterised algorithms we refer to standard texts like [5,26]. For providing hardness proofs in the area of approximation algorithms, L-reductions have become a kind of standard. An optimisation problem APX-hard under L-reduction has no polynomial-time approximation scheme if P $\neq$ NP.

We have summarised what is known (and what is done in this paper) in Tables 1 and 2. Clearly, there is no need to repeat classical complexity results in Table 2. However, observe that the status of parameterised complexity and approximation of these problems and their complementary versions indeed differ. The hitherto unsolved questions regarding Upper Domination have been tackled and largely resolved in [6], which can be seen as a kind of companion paper to this one. Notice that in Table 1, the optimisation problems that correspond to the first three listed graph parameters are minimisation problems (in particular Lower Irredundance wich corresponds to find ir(G)), while the last three are maximisation problems (in particular Upper Irredundance wich corresponds to find IR(G)); this split is indicated by the double lines; this is reversed in Table 2. Also, when considering these problems as parameterised problems, we only consider the standard parameterisation, which is a lower bound on the entity to be maximised or an upper bound on the entity to be minimised. In order to distinguish the problem parameters of the two tables, we use k in Table 1 and ℓ in Table 2. The purpose of this paper is to survey the state of art and to solve most of what was still open until now.

	ir	γ	i	α	Γ	IR
exact $\mathcal{O}^*()$	1.99914^n [11]	1.4864^n [36]	1.3351^n [14]	1.2002^n [43]	1.7159^n [6]	1.9369^n [11]
$\in$ FPT?	W[2]-C [11]	W[2]-C [25]	W[2]-C [25]	W[1]-C [25]	W[1]-H [6]	W[1]-C [27]
non-apx rat.	$c\log(n)$ Th.5	$c\log(n)$ [29]	$n^{1-\varepsilon}$ [33]	$n^{1-\varepsilon}$ [44]	$n^{1-\varepsilon}$ [6]	?
degree restrictions						
apx-ratio	$\frac{3}{2}\Delta$ [23]	$\log(\Delta)+1$ [19]	$\Delta+1$ Obs.4	$\frac{\Delta+3}{5}$ [7]	$\frac{6\Delta^2+2\Delta-3}{10\Delta}$ [6] & Obs.2	
kernel	$\frac{3}{2}\Delta k$ Obs.5	$(\Delta+1)k$ Obs. 4		Δk Obs. 3		
dense-apx	?	APX-H [32]	not $n^{1-\varepsilon}$ Th.9	not $n^{1-\varepsilon}$ Pr.1	not $n^{1-\varepsilon}$ Co.5	APX-H Th.8
cubic graphs						
+planar	NP-C Th.1	NP-C [31]	NP-C [38]	NP-C [31]	NP-C [6]	NP-C Th.2
$\in$ PTAS?	APX-C Co.2	APX-C [2]	APX-H Co.4	APX-C [2]	APX-C [6]	APX-C Co.3

Table 1. Status of various problems related to the domination chain

	co − ir	co − γ	co − i	τ	co − Γ	co − IR
apx-rat.	2 Obs.1	$\frac{240}{193}$ [4]	$\sqrt{n}$ [13]	2 (folklore)	4 [6]	4 Th.6
non-apx rat.	?	$\frac{260}{259}$ [40]	$n^{\frac{1}{2}-\varepsilon}$ [13]	2 (UGC) [37]	?	?
kernel	$2\ell - 1$ [11]	$\frac{5}{3}\ell + 3$ [24]	ℓ^2 [30](Sec.4.3)	2ℓ [26]	$\ell^2 + \ell$ [6]	3ℓ [11]
FPT-$\mathcal{O}^*()$	3.841^ℓ [11]	2.0226^ℓ [24]	1.5874^ℓ [13]	1.2738^ℓ [18]	4.3077^ℓ [6]	2.8752^ℓ [11]
degree restrictions						
$3 \leq \Delta \leq d$	APX-C Co.2	APX-C [8]	$1.5d$-apx [13]	APX-C [41]	APX-C [6]	
dense	?	?	APX-C Th.9	APX-C [20]	APX-C Co.5	APX-C Th.8

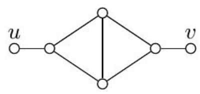
Table 2. Status of various problems related to the complementary domination chain

3 On the classical complexity of irredundant set problems

In this section, we prove that Lower Irredundance and Upper Irredundance (also their complementary versions) are NP-hard on planar cubic graphs.

Theorem 1. *(*) Lower Irredundance is NP-hard on planar cubic graphs.*

Proof. We use the same construction as in [38], where Minimum Domination on planar cubic graphs is reduced to Minimum Independent Domination, that is: Given a planar cubic graph $G = (V, E)$, construct G' from G by replacing every $(u, v) \in E$ by the following planar cubic subgraph with four new vertices:



The argumentation [38] shows that $i(G') = \gamma(G) + |E|$ which automatically gives us $\text{ir}(G') \leq \gamma(G) + |E|$. One can also proof that $\text{ir}(G') \geq \gamma(G) + |E|$ which means that Minimum Domination on G has a solution of cardinality at most k if and only if Lower Irredundance on G' has a solution of cardinality at most $k + |E|$. Details of this proof can be found in the appendix. □

Interesting side note to this proof is that ir, γ and i coincide on G'. Since especially ir and i are known to differ arbitrarily even on cubic graphs [45], this is obviously due to the special structure of G'. It contains induced $K_{1,3}$ (every original vertex with its neighbourhood), so the result for $\text{ir} = \gamma = i$ from [28] does not apply. This makes this construction an interesting candidate to study the characterisation of the graph class for which $\text{ir} = i$. With a different construction, we can show the same type of result for Upper Irredundance.

Theorem 2. *(*) Upper Irredundance is NP-hard on planar cubic graphs.*

4 A special flavour of minimax / maximin problems

Half of the parameters in the domination chain can be defined as either, in case of minimax problems, looking for the smallest of all (inclusion-wise) maximal vertex sets with a certain property ($i(G)$ is the size of the smallest maximal independent set; similarly, $\mathrm{ir}(G)$ is defined), or, in case of maximin problems, looking for the largest of all minimal vertex sets with a certain property ($\Gamma(G)$ is an example). Also, the complementary problems share this flavour; for instance, $\mathrm{co} - i(G)$ can be seen as looking for the largest of all minimal vertex covers.

Typical exact algorithms for maximisation problems fix certain subsets to be part of the solution. In the decision variant, when a parameter value that lower-bounds the size of the solution is part of the input, we might have a sufficient number of vertices in our partial solution and now want to (rather immediately) announce that a sufficiently large solution exists. This is not a problem for determining $\alpha(G)$ or $\mathrm{IR}(G)$, but this may become problematic in the case of maximin problems. In the following we consider the extension-problem for the other two maximin problems related to the domination-chain: $\mathrm{co} - i(G)$ and $\mathrm{co} - \mathrm{ir}(G)$. The first one can formally be stated as follows:

Minimal Vertex Cover Extension
Input: A graph $G = (V, E)$, a set $S \subseteq V$.
Question: Does G possess a minimal vertex cover S' with $S' \supseteq S$?

Observe that this extension problem can also be seen as a kind of subset problem for independent sets by rephrasing the question to: Is there a maximal independent set S' for G with $S' \subseteq V - S$? In more general terms, one can view the extension-version of some maximin problem as exclusion-version of the complementary minimax problem.

Theorem 3. *(*) Minimal Vertex Cover Extension is NP-hard even restricted to planar cubic graphs.*

Proof. Consider the following simple reduction from satisfiability: For a formula $c_1 \wedge \cdots \wedge c_m$ over variables $x_1, \ldots, x_n$, let $G = (V, E)$ be the graph with vertices $v_i, \bar{v}_i$ for every $i = 1, \ldots, n$ and $c_1, \ldots, c_m$ and edges connecting every clause with its literals and connecting v_i with $\bar{v}_i$ for every i. For this graph, the set $S = \{c_1, \ldots, c_m\}$ can be extended to a minimal vertex cover if and only if the formula $c_1 \wedge \cdots \wedge c_m$ is satisfiable. A more sophisticated construction yields a planar cubic graph G as input for Minimal Vertex Cover Extension. Details of this proof can be found in the appendix. □

The maximin problem $\mathrm{co} - \mathrm{ir}(G)$ can also be considered with respect to extension. Since complements of irredundant sets are rather uncomfortable, we describe this problem in terms of the complementary problem $\mathrm{ir}(G)$:

Minimal Co-Irredundant Extension
Input: A graph $G = (V, E)$, a set $S \subseteq V$.
Question: Does G possess a maximal irredundant set S' with $S' \subseteq V - S$?

Theorem 4. *(*) Minimal Co-Irredundant Extension is NP-hard.*

5 Approximation results

In this section, after studying the approximation on general graphs, we consider bounded degree graphs and cubic graphs.

Theorem 5. *(*) For any $c > 0$, there is no $c\log(n)$-approximation for* Lower Irredundance *unless $NP \subseteq \mathrm{DTIME}(n^{\log\log n})$.*

For the little studied complement of Lower Irredundance we observe:

Observation 1 *For any graph G without isolated vertices one can compute a minimal dominating set of cardinality at most $\frac{n}{2}$ in polynomial time for an arbitrary spanning forest of G. The complement of this dominating set is consequently a 2-approximation for* Co-Lower Irredundance.

Using Lemma 3, one can use known exact or approximation algorithms for Minimum Vertex Cover and also results from parameterized approximation such as [15] to deduce:

Theorem 6. *(*)* Co-Upper Irredundance *can be approximated with factor 4 in polynomial, factor 3 in $O^*(1.2738^{\tau(G)})$ and factor 2 in $O^*(1.2738^{\tau(G)})$ or $O^*(1.2002^n)$ time.*

There is a kind of methodology to link optimisation problems related to the domination chain to those related to the complementary domination chain, which can be stated as follows.

Theorem 7. *Assume that the optimisation problem associated to some graph parameter ζ of the domination chain is APX-hard on cubic graphs. Then, the optimisation problem associated to the complement problem of ζ is also APX-hard on cubic graphs.*

Proof. We claim that the reduction that acts as the identity on graph (instances) and complements solution sets is an L-reduction. Given a cubic graph $G = (V, E)$ of order n with $m = \frac{3}{2}n$ edges as an instance of the optimisation problem belonging to ζ (and also to the complement problem). Let us distinguish the two optima by writing $\mathrm{opt}_\zeta(G)$ and $\mathrm{opt}_{\mathrm{co}-\zeta}(G)$, respectively. Then, $\mathrm{opt}_{\mathrm{co}-\zeta}(G) = n - \mathrm{opt}_\zeta(G)$. Similarly, if S' is a solution to G in the complement problem, then $n - |S'|$ is the size of the solution $S := V \setminus S'$ of the original problem. Hence,

$$\left|\mathrm{opt}_\zeta(G) - |S|\right| = \left|(n - \mathrm{opt}_{\mathrm{co}-\zeta}(G)) - (n - |S'|)\right| = \left|\mathrm{opt}_{\mathrm{co}-\zeta} - |S'|\right| .$$

Moreover, as $\mathrm{ir}(G) \geq \frac{2n}{9}$ according to [23], which yields $\mathrm{opt}_\zeta(G) \geq \frac{2n}{9}$ by the domination chain,

$$\mathrm{opt}_{\mathrm{co}-\zeta}(G) \leq n \leq \frac{9}{2}\,\mathrm{opt}_\zeta(G),$$

which proves the claim. □

Theorem 3.3 in [2] shows that Minimum Domination, restricted to cubic graphs, is APX-hard. We can use Theorem 7 to immediately deduce:

Corollary 1. *The complement problem corresponding to* MINIMUM DOMINATION *is APX-hard when restricted to cubic graph instances.*

This sharpens earlier results [8] that only considered the subcubic case.

Corollary 2. LOWER IRREDUNDANCE *restricted to cubic graphs is APX-hard. Similarly,* CO-LOWER IRREDUNDANCE *is APX-hard on cubic graphs.*

Proof. The reduction from Theorem 1 can be seen as an L-reduction from the APX-hard MINIMUM DOMINATION problem on cubic graphs [2] to LOWER IRREDUNDANCE on cubic graphs. Observe that $\gamma(G) \geq \frac{n}{4}$ and $|E| = \frac{3}{2}n$ for any cubic graph G, which gives $\mathrm{ir}(G') = \gamma(G) + |E| \leq 7\gamma(G)$. Furthermore, any maximal irredundant set of cardinality val' for G' can be used to compute a dominating set for G of cardinality $val = val' - |E|$, which yields $val - \gamma(G) = val' - \mathrm{ir}(G')$. Together with Theorem 7 the result for CO-LOWER IRREDUNDANCE follows. □

The computations in the previous proof can be carried out completely analogously for UPPER IRREDUNDANCE and CO-UPPER IRREDUNDANCE.

Corollary 3. *(*)* UPPER IRREDUNDANCE *is APX-hard on cubic graphs. Similarly,* CO-UPPER IRREDUNDANCE *is APX-hard on cubic graphs.*

Manlove's NP-hardness proof for MINIMUM INDEPENDENT DOMINATION on cubic planar graphs [38] turns out to be an L-reduction, so that with Theorem 7 we can conclude:

Corollary 4. MINIMUM INDEPENDENT DOMINATION *and* MAXIMUM MINIMAL VERTEX COVER *are both APX-hard on cubic graphs.*

This improves on earlier results for MAXIMUM MINIMAL VERTEX COVER, for instance, the APX-hardness shown in [39] for graphs of maximum degree bounded by five.

6 Further algorithmic observations

Most of the previously collected results have been hardness results; here we complement some of them by simple algorithmic results.

Observation 2 *The approximation-results for* UPPER DOMINATION *restricted to graphs of bounded degree from [6] are based on eq. 2 and the fact that every maximal independent set is an upper dominating set which is also true for* UPPER IRREDUNDANCE. *The approximation by a suitable independent set yields the same approximation-ratio here which especially means that* UPPER IRREDUNDANCE *can be approximated within factor at most* $\frac{6\Delta^2+2\Delta-3}{10\Delta}$ *for any graph* G *of bounded degree* Δ.

Observation 3 *With Brooks' Theorem one can always find an independent set of cardinality at least $\frac{n}{\Delta}$ for any graph G of bounded degree Δ. From a parameterised point of view, this immediately gives a Δk-kernel for* Maximum Independent Set, Upper Domination *and* Upper Irredundance *for the natural parameter k of these problems, since any bounded-degree graph with more than Δk vertices is a trivial "yes"-instance.*

Observation 4 *Bounded degree Δ implies $\gamma \geq \frac{n}{\Delta+1}$, which means that any greedy solution yields a $(\Delta+1)$-approximation for* Minimal Maximum Independent Set *($i(G)$ in domination chain) and* Minimum Domination. *For standard parameterisation this also yields a $(\Delta+1)k$ kernel for these problems since graphs with more than $(\Delta+1)k$ vertices are trivial "no"-instances.*

Lower Irredundance is the only problem for which these consequences of bounded degree are less obvious. A more thorough investigation of lower irredundant sets in [23] yields the bound $\mathrm{ir}(G) \geq \frac{2n}{3\Delta}$.

Observation 5 *The bound from [23] implies that any greedy maximal irredundant set for a graph of bounded degree Δ is a 1.5Δ-approximation for* Lower Irredundance. *Parameterised by $k = ir(G)$, any graph with more than $1.5\Delta k$ vertices is a trivial "no"-instance which yields a $1.5\Delta k$ kernel.*

Notice that, although the kernel results indicated in the previous two observations look weak at first glance, they allow for lower bound results based on the assumption that $P \neq NP$ according to [17].

7 Consequences for everywhere dense graphs

In [3], Arora et al. presented a unified framework for proving polynomial time approximation schemes for (average) dense graphs, mainly for Max Cut type problems, and for Min Bisection for everywhere dense graphs. Concerning the problems from the domination chain Minimum Vertex Cover and Minimum Domination were studied; in [20], Minimum Vertex Cover is proved APX-hard on everywhere dense graphs and in [32], it is proved that Minimum Domination is NP-hard on (average) dense graphs. We will show inapproximation results for more domination-chain problems on everywhere dense graphs. Interestingly, we can make use of our reductions for sparse (cubic) graphs:

Theorem 8. *For any $\varepsilon > 0$,* Upper Irredundance *and* Co-Upper Irredundance *are APX-hard for everywhere-ε-dense graphs.*

Proof. We construct an L-reduction from (Co-)Upper Irredundance on cubic graphs to (Co-)Upper Irredundance on everywhere-ε-dense graphs. Given a connected cubic graph $G = (V, E)$ on n vertices, we construct a dense graph G' by joining a clique C of $\lceil \frac{\varepsilon n - 3}{1-\varepsilon} \rceil$ new vertices to G. G' has minimum degree $\varepsilon n'$, where $n' = n + \lceil \frac{\varepsilon n-3}{1-\varepsilon} \rceil = \lceil \frac{\varepsilon n - 3 + n - \varepsilon n}{1-\varepsilon} \rceil = \lceil \frac{n-3}{1-\varepsilon} \rceil$ is the number of vertices of G'. Any vertex $v \in V$ has $3 + \lceil \frac{\varepsilon n-3}{1-\varepsilon} \rceil = \lceil \frac{\varepsilon n - 3 + 3 - 3\varepsilon}{1-\varepsilon} \rceil = \lceil \frac{\varepsilon(n-3)}{1-\varepsilon} \rceil$ many neighbours in G'. Any vertex in the added clique has an even higher degree if $n \geq 4$. As any maximal irredundant set of G' that contains a vertex of C is a singleton set, $\mathrm{opt}(G') = \mathrm{opt}(G)$ and, w.l.o.g., any maximum irredundant set in G' is a subset of V which makes it a maximal irredundant set of G.

For Co-Upper Irredundance, we have $\mathrm{opt}(G') = \mathrm{opt}(G) + \lceil \frac{\varepsilon n-3}{1-\varepsilon} \rceil$ and, given any solution S' in G', we can transform it into a new one containing all new vertices and some vertices from V. The set $S' \cap V$ is a solution for G. In a cubic graph, the optimum value of the complement of an upper irredundant set is at least $n/4$ using inequality (3) and the fact that $\tau(G) \geq n/2$ (as G is connected and non-trivial) and thus $\mathrm{opt}(G) \geq n/4$. Thus $\mathrm{opt}(G') \leq \mathrm{opt}(G) + \frac{\varepsilon n-3}{1-\varepsilon} \leq \mathrm{opt}(G) + \frac{4\varepsilon\, \mathrm{opt}(G)-3}{1-\varepsilon} \leq \frac{1+3\varepsilon}{1-\varepsilon} \mathrm{opt}(G)$. □

Observe that the arguments and the computations of the previous proof are also valid for Co-Upper Domination. Since it is also APX-hard on cubic graphs [6] we can conclude the same result. Almost the same reduction is an E-reduction when we start with a general instance for Upper Domination (just adding more vertices in order to be sure that G' is everywhere-ε-dense). Since Upper Domination is not $n^{1-\delta}$-approximable for any $\delta > 0$, if P $\neq$ NP on general graphs [6] we can conclude the same result for everywhere-dense graphs.

Corollary 5. *For any $\varepsilon > 0$,* Co-Upper Domination *is APX-hard and* Upper Domination *is not $n^{1-\delta}$-approximable for any $\delta > 0$, if P $\neq$ NP, for everywhere-ε-dense graphs.*

The inapproximability result from [44] with the above reduction yields:

Proposition 1. *For any $\varepsilon > 0$,* Maximum Independent Set *is not $n^{1-\delta}$-approximable for any $\delta > 0$, if P $\neq$ NP, for everywhere-ε-dense graphs.* [5]

Theorem 9. *For any $\varepsilon > 0$,* Maximum Minimal Vertex Cover *is APX-hard and* Minimum Maximal Independent Set *is not $n^{1-\delta}$-approximable for any $\delta > 0$, if P $\neq$ NP, for everywhere-ε-dense graphs.*

Proof. We give an E-reduction from Minimum Maximal Independent Set on general graphs to Minimum Maximal Independent Set on everywhere-ε-dense graphs. Consider for a graph G the family $\{G^j : j \in \mathbb{N}\}$, recursively defined by $G^0 := G$ and $G^{j+1} := G^j + G^j$ ("+" denotes graph join). If the order of G is n, the order of G^j is $2^j n$ for every $j \in \mathbb{N}$. Also every $v \in G^j$ has degree at least $n(2^j - 1)$ which means that G^j is $(1-1/2^j)$-dense. Let V be the vertices of G and $V \cup V'$ be the vertices of $G+G$. For any independent set S of $G+G$ either $S \subseteq V$ or $S \subseteq V'$, which means that independent sets in $G+G$ always yield equivalent independent sets in G and hence $i(G) = i(G+G)$. Inductively, this argument implies $i(G) = i(G^j)$ for all $j \in \mathbb{N}$. For j such that $j \geq \log_2(1/(1-\varepsilon))$, the graph G^j hence yields the aforementioned E-reduction since any independent set in G^j yields an independent set in G of the same size.

[5] We were informed about this fact by Marek Karpiński.

Starting with a cubic graph G, G^j yields an L-reduction from Maximum Minimal Vertex Cover on cubic graphs, which is APX-hard by Corollary 4, to Maximum Minimal Vertex Cover on everywhere-ε-dense graphs, since for cubic graphs $\text{co}-i(G) \geq \frac{n}{2}$ and hence $\text{co}-i(G^j) < 2^j n \leq 2^{j+1}\text{co}-i(G)$. □

8 Summary, open problems and prospects

We have presented a sketch of the complexity landscape of the domination chain. As can be seen from our tables, the status of most combinatorial problems has now been solved. However, there are still several question marks in these tables, and also the positive (algorithmic) results implicitly always ask for possible improvements.

For the investigation of complexity aspects of graph parameters, chains of inequalities like the domination chain help to unify proofs, but also to find spots that have not been investigated yet. Also, the idea of looking at the complementary chain should work out in each case. An example of a similar chain of parameters is the Roman domination chain [16]. Most of what we know is concerning Roman domination and its complementary version, which is also called the differential of a graph; see [1,8,9,10].

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9 Appendix: Omitted (standard) definitions

9.1 Basic notions of graph theory

Throughout this paper, we only deal with undirected simple graphs $G = (V, E)$. In the following, we explain the main notions of graph theory that we use, but refer to any textbook in that area for yet unexplained standard notions. The number of vertices $|V|$ is also known as the order of G. As usual, $N(v)$ denotes the open neighbourhood of v in a graph G, and $N[v]$ is the closed neighbourhood of v in G, i.e., $N[v] = N(v) \cup \{v\}$. These notions can be easily extended to vertex sets X, e.g., $N(X) = \bigcup_{x \in X} N(x)$. The cardinality of $N(v)$ is also known as the degree of v, denoted as $deg(v)$. The maximum degree in a graph is usually written as Δ. A graph of maximum degree three is called subcubic, and if actually all degrees equal three, it is called a cubic graph.

Given a graph $G = (V, E)$, a subset S of V is a *dominating set* if every vertex $v \in V \setminus S$ has at least one neighbour in S, i.e., if $N[S] = V$. A dominating set is minimal if no proper subset is a dominating set. Likewise, a vertex set I is *independent* if $N(I) \cap I = \emptyset$. An independent set is maximal if no proper superset is independent. In the following we use classical notations: $\gamma(G)$ and $\Gamma(G)$ are the minimum and maximum cardinalities over all minimal dominating sets in G, $\alpha(G)$ is the maximum cardinality of an independent set, $i(G)$ is the minimum cardinality of a maximal independent set, and $\tau(G)$ is the size of a minimum vertex cover, which equals $|V| - \alpha(G)$ by Gallai's identity. A minimal dominating set D of G with $|D| = \Gamma(G)$ is also known as an *upper dominating set* of G.

9.2 *E*-reductions

A problem A is called *E-reducible* to a problem B, if there exist polynomial time computable functions f, g and a constant β such that

- f maps an instance I of A to an instance I' of B such that $\mathrm{opt}(I)$ and $\mathrm{opt}(I')$ are related by a polynomial factor, i.e. there exists a polynomial p such that $\mathrm{opt}(I') \leq p(|I|)\,\mathrm{opt}(I)$,
- g maps any solution S' of I' to one solution S of I such that $\varepsilon(I, S) \leq \beta\varepsilon(I', S')$.

An important property of an E-reduction is that it can be applied uniformly to all levels of approximability; that is, if A is E-reducible to B and B belongs to $\mathcal{C}$ then A belongs to $\mathcal{C}$ as well, where $\mathcal{C}$ is a class of optimisation problems with any kind of approximation guarantee.[6]

[6] See also S. Khanna, R. Motwani, M. Sudan, and U. Vazirani. On syntactic versus computational views of approximability. *SIAM Journal on Computing*, 28:164–191, 1998.

9.3 On the structure of maximal irredundant sets

Any maximal irredundant set D for a graph $G = (V, E)$ can be associated with a partition of the set of vertices V into five sets F, I, P, O, R given by: $I := \{v \in D \colon v \in pn(v, D)\}$, $F := D - I$, $P \in \{B \subseteq N(F) \cap (V - D) \colon |pn(v, D) \cap B| = 1$ for all $v \in F\}$ with $|F| = |P|$, $O = N[D] - (D \cup P)$, $R = V - N[D]$. This representation is not necessarily unique since there might be different choices for the sets P and O but for every partition of this kind, the following properties hold:

1. Every vertex $v \in F$ has at least one neighbour in F, called a **f**riend.
2. The set I is an **i**ndependent set in G.
3. The subgraph induced by the vertices $F \cup P$ has an edge cut set separating F and P that is, at the same time, a perfect matching; hence, P can serve as the set of **p**rivate neighbours for F.
4. The neighbourhood of a vertex in I is always a subset of O, which are otherwise the **o**utsiders.
5. Each vertex in R has at least one neighbour in P.

Lemma 4. *For any connected graph G with $n > 0$ vertices and a maximum irredundant set D with an associated partition (F, I, P, O, R) as defined above, if $|D| = IR(G) > \alpha(G)$ then $|I| \leq \alpha(G) - 2$.*

Proof. Let G be a connected graph with $n > 0$ vertices and let D be a maximum irredundant set with an associated partition (F, I, P, O, R). We first show that if $\text{IR} > \alpha(G)$ then $|F| \geq 2$ (in fact, one can show that then $|F| \geq 3$ but that is not necessary for our proof). Indeed, if $|F| = 0$, then D is also an independent set, and thus $\text{IR}(G) = \alpha(G)$, and according to our definition of partition (F, I, P, O, R), we have $|F| \neq 1$ (see Property 1 of this partition).

Now, if $|F| \geq 2$ then the subgraph of G induced by $F \cup P$ contains an independent set of size 2 consisting of a vertex in F, say v, and a vertex in P, say u, such that v and u are not adjacent. Since in the original graph G, there are no edges between the vertices in I and the vertices in $F \cup P$ (Property 4), $I \cup \{u, v\}$ forms an independent set of size $|I| + 2$. This sets a lower bound on the independence number and we have $\alpha(G) \geq |I| + 2$.

From the above, it follows that if $\text{IR}(G) > \alpha(G)$ then $|I| \leq \alpha(G) - 2$. □

10 Appendix: Omitted proofs

10.1 Proof of Lemma 1

We consider a graph G with $n > 0$ vertices and let D be a maximum irredundant set with associated partition (F, I, P, O, R). We examine separately the following two cases:

1. $\text{IR}(G) = \alpha(G)$. Then we trivially have $\text{IR}(G) \leq \alpha(G)$.

2. $\mathrm{IR}(G) > \alpha(G)$.
   From the fact that $|F| = |P|$ (from Property 3) we have $|F| = \frac{n-|I|-|O|-|R|}{2} \leq \left\lfloor \frac{n-|I|}{2} \right\rfloor$ and thus

$$\mathrm{IR}(G) = |F| + |I| \leq \left\lfloor \frac{n+|I|}{2} \right\rfloor$$

   From the above and Lemma 4 we have

$$\mathrm{IR}(G) \leq \left\lfloor \frac{n+|I|}{2} \right\rfloor \leq \left\lfloor \frac{n+\alpha(G)-2}{2} \right\rfloor \leq \frac{n}{2} + \frac{\alpha(G)}{2} - 1$$

This concludes the proof of the claim.

10.2 Proof of Lemma 2

Let G be a connected graph with $n > 0$ vertices, maximum degree Δ and let D be a maximum irredundant set with associated partition (F, I, P, O, R). Our argument is similar to the one in Lemma 1, as the case $\mathrm{IR}(G) = \alpha(G)$ is again trivial, assume $\mathrm{IR}(G) > \alpha(G)$. Again, we obtain:

$$\mathrm{IR}(G) = |F| + |I| = \frac{n + |I| - |O| - |R|}{2}$$

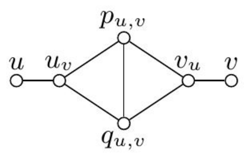
We next derive an improved lower bound on $|O|$. Let e be the number of edges adjacent with vertices from I. As G is of minimum degree δ, we have $e \geq \delta|I|$. As the vertices in I are only adjacent with the vertices in O, there are at least e edges that have exactly one end vertex in O. Since G has maximum degree Δ, we have that $|O| \geq \lceil \frac{e}{\Delta} \rceil \geq \left\lceil \frac{\delta|I|}{\Delta} \right\rceil$.

From the above and Lemma 4 we have

$$\begin{aligned}\mathrm{IR}(G) &\leq \left\lfloor \frac{n+|I|-\lceil \frac{\delta|I|}{\Delta} \rceil - |R|}{2} \right\rfloor \leq \frac{n + |I| - \frac{\delta|I|}{\Delta}}{2} = \frac{n + \frac{(\Delta-\delta)|I|}{\Delta}}{2} \\ &\leq \frac{n + \frac{(\Delta-\delta)}{\Delta}(\alpha(G)-2)}{2} = \frac{n}{2} + \frac{\Delta-\delta}{2\Delta}\alpha(G) - \frac{\Delta-\delta}{\Delta}\end{aligned}$$

10.3 Proof of Theorem 1

Recall that we build G' from G by adding four new vertices for every edge. Let $u_v, v_u, p_{u,v}, q_{u,v}$ be the names of the new vertices added for $(u, v) \in E$ assigned like in the picture below:



Suppose S is a maximal irredundant set for G'. Consider the sets $D := S \cap V$ and $V_{u,v} := S \cap \{u_v, p_{u,v}, q_{u,v}, v_u\}$ for all $(u,v) \in E$ and perform the following two altering-steps (no changes on S):

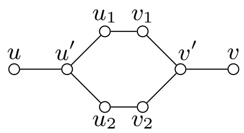
1. For every edge (u,v) in E such that $S \cap \{u_v, v_u, p_{u,v}, q_{u,v}\} = \emptyset$, delete u from D and add u_v to $V_{u,v}$. Maximality of S implies $u, v \in S$ and $pn(u,S) = \{u_v\}$ and $pn(v,S) = \{v_u\}$. For all edges $(u,w) \in E$ with $w \neq v$ consequently $u_w \notin pn(u,S)$ which means $S \cap \{u_w, p_{u,w}, q_{u,w}\} \neq \emptyset$. This especially ensures that for all such edges (u,v) one of the vertices u, v remains in S.
2. For all $w \in V$ such that $N_G[w] \cap D = \emptyset$, we know that w was not deleted by step one since otherwise its neighbour from the edge which triggered step one would be in D. This means $w \notin S$, so by maximality there is a vertex $v \in S \cup \{w\}$ such that $pn(v, S \cup \{w\}) = \emptyset$. Since S is irredundant, such a vertex v is either in $\{w_z, p_{w,z}, q_{w,z} : z \in N_G(w)\}$ or $v = w$. While there exists a vertex $w \in V$ such that $N_G[w] \cap D = \emptyset$ apply alterations according to the following cases for v:
   (a) If $v = w_z$ for some $z \in N_G(w)$, irredundance of S yields $pn(w_z, S) \subseteq \{w, w_z\}$ so especially $q_{w,z} \notin pn(w_z, S)$ so $|V_{w,z}| = 2$. Delete w_z from $V_{w,z}$ and add w to D.
   (b) If $v = q_{w,z}$ for some $z \in N_G(w)$ (symmetrically for $p_{w,z}$), irredundance of S yields $pn(q_{w,z}, S) = \{w_z\}$ which means $z_w \in S$. Delete z_w from $V_{w,z}$ and add z to D.
   (c) If $v = w$, especially $w \notin pn(w, S \cap \{w\})$, so there is a vertex $x \in N_G(w)$ such that $w_x \in S$. If $|V_{w,x}| = 2$, delete w_x from $V_{w,x}$ and add w to D. Otherwise, $\{x_w, p_{x,w}, q_{x,w}\} \cap S = \emptyset$, so x was not deleted in step one, so $x \notin S$. Maximality requires $pn(v', S \cup \{x_w\}) = \emptyset$ for some $v' \in S \cup \{x_w\}$. Since $x_w \notin N[S]$, this means $v' = x_y$ for some $y \in N_G(x)$ and $pn(x_y, S) = \{x\}$. This implies $S \cap \{q_{x,y}, p_{x,y}\} \neq \emptyset$, so especially $|V_{x,y}| = 2$. Delete x_y from $V_{x,y}$ and add x to D.

Observe that this process of deleting a vertex from some $V_{x,y}$ is not done twice for the same edge (x,y). For every such $V_{x,y}$ we always either delete x_y and add x to D or delete y_x and add y to D. The first two cases (a) and (b) are only considered if neither x nor y are in D, so clearly no exchange has happened in a previous step. In the third case, S only contains x_y or y_x, so $V_{x,y}$ will be used at most once to add x or y, respectively. After these steps clearly D is a dominating set for G. Since $|V_{x,y}| \geq 1$ for all $(x,y) \in E$ and $|S| = |D| + \sum_{(x,y)\in E} |V_{x,y}|$, this dominating set has a cardinality of at most $|S| - |E|$. With the choice of S as a maximal irredundant set for G' being arbitrary, we can conclude that $\mathrm{ir}(G') \geq \gamma(G) + |E|$.

10.4 Proof of Theorem 2

The proof of this result is split into two parts: in (A), we prove NP-hardness on subcubic planar graphs, and then in (B) we show how to modify the construction to get NP-hardness also in the case of cubic graphs.

(A) We reduce from the NP-hard Maximum Independent Set on cubic planar graphs. Let $G = (V, E)$ be a cubic planar input graph for Maximum Independent Set. Construct a subcubic planar graph G' from G by replacing every edge $(u, v) \in E$ by the following subgraph with the set of six new vertices $V_{u,v} := \{u', u_1, u_2, v_1, v_2, v'\}$.



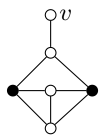
Any maximum independent set S of G can be extended to a maximum independent set in G' of cardinality $|S| + 3|E|$, since for every edge $(u, v) \in E$ either $u \notin S$ or $v \notin S$, so either $\{u', v_1, v_2\}$ or $\{v', u_1, u_2\}$ can be added to S without violating independence. The resulting maximal independent set is also a maximal irredundant set in G'.

Let, on the other hand, S be a maximal irredundant set for G'. For the induced C_6 of an edge-gadget, one can easily verify by checking all possibilities that S can contain at most three out of the six vertices in $V_{u,v}$ without violating irredundance. If $\{u, v\} \subset S$ for an edge $(u, v) \in E$, S can contain at most two vertices from $V_{u,v}$. Consider the sets $S' = S \cap V$ and $R = \emptyset$. While there is an edge $(u, v) \in E$ with $\{u, v\} \subset S'$, delete u from S' and add it to R. When this deleting-process ends, S' is an independent set in G. With $E_1 := \{(u, v) \colon \{u, v\} \subset S\}$ and $E_2 = E - E_1$, the cardinality of S' can be estimated by:

$$|S'| = |S| - \sum_{(u,v)\in E} |V_{u,v} \cap S| - |R| \geq |S| - 3|E_2| - 2|E_1| - |E_1| = |S| - 3|E|.$$

In conclusion, for any $k \in \mathbb{N}$, G has an independent set of cardinality at least k if and only if G' has an irredundant set of cardinality at least $k + 3|E|$. With a planar (sub)cubic input graph G, the constructed input graph G' is planar and subcubic.

(B) In the previous construction, the resulting graph G' is already subcubic and planar and does not contain degree-one vertices. For every vertex of degree two in G', add the following subgraph:



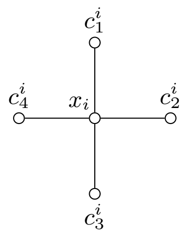
Any choice of two vertices within such a new subgraph dominates all of its new vertices, so any irredundant set for the new graph contains at most two vertices from any of the new subgraphs. If the original vertex v chooses its new neighbour to be a private neighbour in some irredundant set S, then S can only contain one of the new vertices this subgraph. Deleting v from S and choosing, for example, the two black vertices in the above picture to belong to S instead does neither violate irredundance nor change the cardinality of S. Let T be the set of degree-two vertices in G' and let G'' denote the graph build from G' by adding the above subgraph to every vertex $v \in T$. The graph G'' is planar, cubic and has an irredundant set of cardinality $k + 2|T|$ if and only if G' has an irredundant set of cardinality k.

10.5 Proof of Theorem 3

If there exists an assignment ϕ for $x_1, \ldots, x_n$ which satisfies $c_1 \wedge \cdots \wedge c_m$, the set $S' := S \cup \{v_i \colon \phi(x_i) = 0\} \cup \{\bar{v}_i \colon \phi(x_i) = 1\}$ is a minimal vertex cover for G. Since for each edge $(v_i, \bar{v}_i)$ either $v_i \in S'$ or $\bar{v}_i \in S'$ and all vertices c_j are in S', every edge is covered. Every vertex $v_i \in S'$ (or $\bar{v}_i \in S'$) uniquely covers the edge $(v_i, \bar{v}_i)$, so $S' - \{v_i\}$ (or $S' - \{\bar{v}_i\}$) is not a vertex cover. Since ϕ is satisfying for $c_1 \wedge \cdots \wedge c_m$, every clause c_j has at least one literal which is not in S'. This means that the edge corresponding to this satisfying literal is uniquely covered by $c_j \in S'$ or, in other words, that $S' - \{c_j\}$ is not a vertex cover, so S' is minimal.

Let, on the other hand, S' be a minimal vertex cover for G which extends S. Every edge $(v_i, \bar{v}_i)$ has to be covered by S' which means that $S' \cap \{v_i, \bar{v}_i\} \neq \emptyset$. The setting $\phi(x_i) = 0$ if $\bar{v}_i \notin S'$ and $\phi(x_i) = 1$ if $v_i \notin S'$ is hence not contradictory. Minimality requires that there is an edge (c_j, x) for which $x \notin S'$ for each $j = 1, \ldots, m$. The only possible vertices x for these *private* edges are literals of the clause c_j which means that there is either a $v_i \notin S'$ or a $\bar{v}_i \notin S'$ which is literal in c_j. In terms of satisfiability, this implies that the assignment ϕ can be extended to a satisfying assignment for $c_1 \wedge \cdots \wedge c_m$.

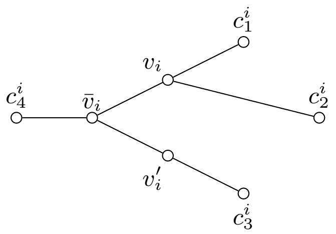
To prove hardness for the restriction to planar cubic graphs, consider reducing not from general satisfiability but from the NP-hard 4-Bounded Planar 3-Connected SAT (4P3C3SAT)[7]. For a 4P3C3SAT-formula $c_1 \wedge \cdots \wedge c_m$, the associated graph $G = (V, E)$ with vertex-set $\{c_1, \ldots, c_m\} \cup \{x_1, \ldots, x_n\}$ and edges connecting each clause to the three variables which occur in it is planar and the vertices x_i have degree at most four. Fix some planar embedding of G and let $c^i_1, c^i_2, c^i_3, c^i_4$ be the (possibly not existing) clauses in which x_i appears, arranged in the chosen planar embedding in clockwise order:



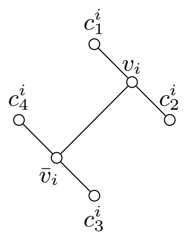
[7] see J. Kratochvíl. A special planar satisfiability problem and a consequence of its NP-completeness. *Discrete Applied Mathematics*, 52:233–252, 1994.

Create the graph G' from G by replacing each x_i according to the following cases:

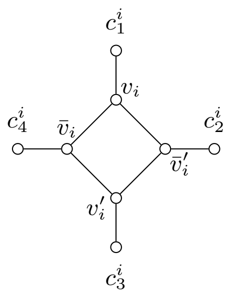
(a) If the variable x_i appears positively in clauses c_1^i, c_2^i, c_3^i and negated in c_4^i, replace x_i by:



(b) If the variable x_i appears positively in clauses c_1^i, c_2^i and negated in c_3^i, c_4^i, replace x_i by:



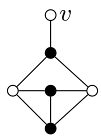
(c) If the variable x_i appears positively in clauses c_1^i, c_3^i and negated in c_2^i, c_4^i, replace x_i by:



All other cases are rotations of the above three cases and/or invert the roles of v_i and $\bar{v}_i$ and v_i' and $\bar{v}_i'$.

By the same argumentation as above for general satisfiability, the set $S = \{c_1, \ldots, c_m\}$ can be extended to a minimal vertex cover for G' if and only if the formula $c_1 \wedge \cdots \wedge c_m$ is satisfiable.

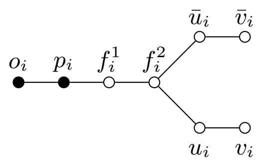
G' is planar and subcubic. For a planar cubic instance add the following subgraph to every vertex v of degree two and put the black vertices to S:



All new edges introduced by these additional subgraphs are already covered by the vertices added to S and these new vertices in S obviously do not cover any original edge. These additional subgraphs consequently do not affect the possibility to turn S into a minimal vertex cover for G'.

10.6 Proof of Theorem 4

Just like for Minimal Vertex Cover Extension, we reduce from satisfiability. Given a formula $c_1 \wedge \cdots \wedge c_m$ over variables $x_1, \ldots, x_n$, construct a graph $G = (V, E)$ with vertices $\{k_1, \ldots, k_m\} \cup \{o_i, p_i, f_i^1, f_i^2, u_i, \bar{u}_i, v_i, \bar{v}_i \colon i = 1, \ldots, n\}$ and edges connecting k_j to v_i or $\bar{v}_i$ if x_i or $\bar{x}_i$ is literal in c_j respectively. Further, the vertices $\{o_i, p_i, f_i^1, f_i^2, u_i, \bar{u}_i, v_i, \bar{v}_i \colon i = 1, \ldots, n\}$ induce i non-connected subgraphs of the following structure:



With this construction, $c_1 \wedge \cdots \wedge c_m$ is satisfiable if and only if there is an irredundant set for G which does not contain any vertex from $S := \{k_1, \ldots, k_m\} \cup \{o_i, p_i \colon 1 \leq i \leq m\}$.

Suppose $c_1 \wedge \cdots \wedge c_m$ is satisfiable by some assignment ϕ. We claim that $S' := \{u_i \colon \phi(x_i) = 1\} \cup \{\bar{u}_i \colon \phi(x_i) = 0\} \cup \{f_i^1, f_i^2 \colon i = 1, \ldots, n\}$ is a maximal irredundant set in G. The private neighbourhoods with respect to S' are: $pn(f_i^1, S') = \{p_i\}, pn(f_i^2, S') = \{u_i, \bar{u}_i\} - S', pn(u_i, S') = \{v_i\}, pn(\bar{u}_i, S') = \{\bar{v}_i\}$, all of which are non-empty which means that S' is irredundant. About maximality, adding a vertex v from $V - S'$ to S' means one of the following cases:

1. $v = o_i$ or $v = p_i$ for some i yields $pn(f_i^1, S' \cup \{v\}) = \emptyset$.
2. $v = \bar{u}_i$ or $v = \bar{v}_i$ for some i with $\phi(x_i) = 1$ yields $pn(f_i^2, S' \cup \{v\}) = \emptyset$.
3. $v = u_i$ or $v = v_i$ for some i with $\phi(x_i) = 0$ yields $pn(f_i^2, S' \cup \{v\}) = \emptyset$.
4. $v = v_i$ for some i with $\phi(x_i) = 1$ yields $pn(u_i, S' \cup \{v\}) = \emptyset$.
5. $v = \bar{v}_i$ for some i with $\phi(x_i) = 0$ yields $pn(\bar{u}_i, S' \cup \{v\}) = \emptyset$.
6. $v = k_j$ for some j implies $pn(u_i, S' \cup \{v\}) = \emptyset$ (or $pn(\bar{u}_i, S' \cup \{v\}) = \emptyset$) for each i with x_i (or $\bar{x}_j$) literal in c_j. Since ϕ is a satisfying assignment for $c_1 \wedge \cdots \wedge c_m$, ϕ satisfies at least one literal of each c_j and the corresponding u_i (or $\bar{u}_i$) is consequently in S'.

Overall, any choice of $v \in V - S'$ yields some $w \in S'$ such that $pn(w, S' \cup \{v\}) = \emptyset$ which proves maximality of S'.

Let S' be a maximal irredundant set for G with $S' \cap S = \emptyset$. Maximality of S' requires at least one vertex w for which $pn(w, S' \cup \{o_i\}) = \emptyset$ for each $i = 1, \ldots, n$. For any set which does not contain o_i or p_i, the only possible choice for such a vertex w is f_i^1. $pn(f_i^1, S' \cup \{o_i\}) = \emptyset$ especially requires that f_i^1 has at least one neighbour in $S' \cup \{o_i\}$ which means $f_i^2 \in S'$ for all $i = 1, \ldots, n$. Irredundance of S' requires at least one private neighbour for f_i^2, which means that either $S' \cap \{u_i, v_i\} = \emptyset$ or $S' \cap \{\bar{u}_i, \bar{v}_i\} = \emptyset$. This allows to define the (partial) assignment:

$$\phi(x_i) = \begin{cases} 1 & \text{if } S' \cap \{u_i, v_i\} \neq \emptyset \\ 0 & \text{if } S' \cap \{\bar{u}_i, \bar{v}_i\} \neq \emptyset \end{cases}$$

Suppose there is a clause c_j which is not satisfied by this assignment. This means that for all neighbours v_i (or $\bar{v}_i$) of k_j, $S' \cap \{v_i, u_i\} = \emptyset$ (or $S' \cap \{\bar{v}_i, \bar{u}_i\} = \emptyset$). This however means that $pn(k_j, S' \cup \{k_j\}) \supset \{k_j\}$ and further, k_j can only affect private neighbourhodds from vertices in $\{u_i, v_i\}$ (or $\{\bar{u}_i, \bar{v}_i\}$) for indices i such that x_i (or $\bar{x}_i$) is literal in c_j, non of which are in S'. In other words, if c_j is not satisfied by ϕ, $S' \cup \{k_j\}$ is irredundant, a contradiction to the maximality of S'.

10.7 Proof of Theorem 5

We show an E-reduction from Minimum Domination. Let $G = (V, E)$ be a connected graph as an instance for Minimum Domination. Consider the associated bipartite graph $G' = (V \cup V', E')$ where V' is a copy of V and $E' = \{(u, v'), (u', v) \colon (u, v) \in E\} \cup \{(v, v') \colon v \in V\}$. Let G'' be the graph built from G' by turning V into a clique as instance for Lower Irredundance.

Any minimum dominating set D of G is also dominating for G'' and, because of minimality, every $v \in D$ has at least one private neighbour. In G'' every $v \in D$ consequently has a private neighbour in V' since $pn_{G''}(v, D) = \{w' \colon w \in pn_G(v, D)\}$. So D is a maximal irredundant set in G'', hence $\gamma(G) \geq \mathrm{ir}(G'')$.

For any maximal irredundant set D'' for G'', the set $D = \{v \colon \{v, v'\} \cap D'' \neq \emptyset\}$ is a dominating set for G. Assume there is a vertex $w \in V$ such that $w \notin N_G[D]$. Because w' is not adjacent to any vertex in D'', we have $w' \in pn_{G''}(w', D'' \cup \{w'\})$. Suppose there is some $v \in D''$ such that $pn_{G''}(v, D'') \subset V$. This means that $D'' \cap V = \{v\}$ and $\{x' \colon x \in N[v]\} \subset D''$. Then however $pn_{G''}(x', D'') = \emptyset$ for any $\{x' \colon x \in N[v]\}$ (there is at least one neighbour since G is connected) which is not possible. So $pn_{G''}(v, D'' \cup \{w'\}) \supseteq pn_{G''}(v, D'') - V \neq \emptyset$ for all $v \in D''$ which would make $D'' \cup \{w'\}$ irredundant, a contradiction to the maximality of D''. Since $|D| \leq |D''|$ we conclude $\gamma(G) \leq \mathrm{ir}(G'')$ and then $\gamma(G) = \mathrm{ir}(G'')$. In conclusion, the non-approximability of Minimum Domination of Feige[8] transfers to Lower Irredundance.

[8] see: U. Feige. A threshold of $\ln n$ for approximating set cover. *Journal of the ACM*, 45:634–652, 1998.

10.8 Proof of Theorem 6

Given a graph G on n vertices, we first find a vertex cover V' in G using any 2-approximation algorithm, and define $S' = V \setminus V'$. Set S' is an independent set and let S be a maximal independent set containing S'. The set $V \setminus S$ is a vertex cover of size $|V \setminus S| \leq |V'| \leq 2\tau(G) \leq 4(n - \mathrm{IR}(G))$, see Lemma 3. Moreover, $V \setminus S$ is the complement of a maximal independent set which also makes it the complement of a maximal irredundant set, so overall a feasible solution for Co-Upper Irredundance with $|V \setminus S| \leq 4(n - \mathrm{IR}(G))$. The claimed running time for the factor-2 approximation stems from the best parameterised and exact algorithms for Minimum Vertex Cover.[9]

10.9 Proof of Corollary 3

The reductions from Theorem 2 can be seen as L-reductions and since the composition of L-reductions is an L-reduction and from the APX-hard Maximum Independent Set problem on cubic graphs we obtain the APX-hardness of Upper Irredundance on cubic graphs. Observe that $\alpha(G) \geq \frac{n}{4}$ and $|E| = \frac{3}{2}n$ for any cubic graph G, which gives $\mathrm{IR}(G') = \alpha(G) + 3|E| \leq 19\alpha(G)$. Furthermore, any maximal irredundant set of cardinality val' for G' can be used to compute a dominating set for G of cardinality $val = val' - 3|E|$, which yields $val - \alpha(G) = val' - \mathrm{IR}(G')$. Moreover, $\mathrm{IR}(G'') = \mathrm{IR}(G') + 8|E|$ and $\mathrm{IR}(G') \geq \frac{19n}{4}$, gives $\mathrm{IR}(G'') \leq \frac{67\mathrm{IR}(G')}{19}$. Furthermore, any maximal irredundant set of cardinality val'' for G'' can be used to compute a maximal irredundant set for G' of cardinality $val' = val'' - 8|E|$, which yields $val' - \mathrm{IR}(G') = val'' - \mathrm{IR}(G'')$. Together with Theorem 7, the result for Co-Upper Irredundance follows.

[9] The current published records are held by the following two papers: J. Chen, I. A. Kanj, and G. Xia. Improved upper bounds for vertex cover. *Theoretical Computer Science*, 411(40–42):3736–3756, 2010. M. Xiao and H. Nagamochi. Exact algorithms for maximum independent set. In L. Cai, S.-W. Cheng, and T. W. Lam, editors, *Algorithms and Computation - 24th International Symposium, ISAAC*, volume 8283 of *LNCS*, pages 328–338. Springer, 2013.