Differential approximation for some routing problems

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Abstract. We study vehicle routing problems with constraints on the distance traveled by each vehicle or on the number of vehicles. The objective is to minimize the total distance traveled by vehicles. We design constant differential approximation algorithms for some of these problems. In particular we obtain differential bounds: $\frac{1}{2}$ for METRIC 3VRP, $\frac{3}{5}$ for METRIC 4VRP, $\frac{2}{3}$ for METRIC kVRP with $k \geq 5$, $\frac{1}{2}$ for the non-metric case for any $k \geq 3$, and $\frac{1}{3}$ for CONSTRAINED VRP. We prove also that MIN-SUM EkTSP is $\frac{2}{3}$ differential approximable and has no differential approximation scheme, unless $\mathbf{P} = \mathbf{NP}$.
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1 Introduction

Vehicle routing problems that involve the periodic collection and delivery of goods and services as mail delivery or trash collection are of great practical importance. Simple variants of these real problems can be modeled naturally with graphs. Unfortunately even simple variants of vehicle routing problems are **NP**-hard. In this paper we consider approximation algorithms, and measure their efficiencies in two ways. One is the *standard* measure giving the ratio $\frac{apx}{opt}$, where *opt* and *apx* are the values of an optimal and approximate solution, respectively. The other measure is the *differential* measure, that compares the worst ratio of, on the one hand, the difference between the cost of the solution generated by the algorithm and the worst cost, and on the other hand, the difference between the optimal cost and the worst cost. Formally, the differential measure gives the ratio $\alpha = \frac{wor-apx}{wor-opt}$, where *wor* is the value of the optimal solution for the complementary problem. In [11], the measure $1 - \alpha$ is considered and it is called there z-approximation. Justification for this measure can be found for example in [1, 5, 18, 11, 14].

The main subject of this paper is differential approximation of the routing problems. In these problems n *customers* have to be served by *vehicles* of limited capacity from a common *depot*. A solution consists of a set of routes, where each starts at the depot and returns there after visiting a subset of customers, such that each customer is visited exactly once. We refer to a problem as a VEHICLE

ROUTING PROBLEM (VRP) if there is a constraint on the (possibly weighted) number of customers visited by a vehicle. This constraint reflects the assumption that the vehicle has a finite capacity and that it *collects* from the customers (or *distributes* among them) a commodity. The goal is to find a solution such that the total length of the routes is as small as possible. In other cases, the vehicle is just supposed to *visit* the customers, for example, in order to serve them. In such cases we refer to the problem as a TSP problem.

The problems that are considered here generalize the (undirected) TRAVELING SALESMAN PROBLEM (TSP). Differential approximation algorithms for the TSP are given by Hassin and Khuller [11] and Monnot [14]. We will sometimes use these algorithms to generate approximations for the problems of this paper. However, we note an important difference. In the TSP, adding a constant k to all of the edge length does not affect the set of optimal solutions or the value of the differential ratio. The reason is that every solution contains exactly n edges and therefore every solution value increases by exactly the same value, namely $n \cdot k$. In particular, this means that for the purpose of designing algorithms with bounded differential ratio, it doesn't matter whether d is a metric or not (it can be made a metric by adding a suitable constant to the edge lengths). In contrast, in some of problems dealt with here, the number of edges used by a solution is not the same for every solution and therefore it may turn out, as we will see, that in some cases the metric version is easier to approximate.

It is easy to see that 2VRP is polynomial time solvable. For $k \geq 3$, METRIC kVRP was proved **NP**-hard by Haimovich and Rinnooy Kan [8]. In [9], Haimovich, Rinnooy Kan and Stougie gave a $\frac{5}{2} - \frac{3}{2k}$ standard approximation for Metric kVRP. We study for the first time the differential approximability of kVRP. More exactly we give a $\frac{1}{2}$ differential approximation for the non-metric case for any $k \geq 3$. We improve it to $\frac{3}{5}$ for METRIC 4VRP and to $\frac{2}{3}$ for METRIC kVRP with $k \geq 5$. An approximation lower bound of $\frac{2219}{2220}$ is given here for METRIC nVRP with length 1 and 2 using a lower bound of TSP(1,2) [6].

MIN-SUM EkTSP is a generalization of TSP where we search to cover the customers by *exactly k* vehicles such that the total length is minimum. Bellmore and Hong [3] showed that when we search for a solution with *at most k* cycles then MIN-SUM kTSP is equivalent to TSP on an extended graph. As for MIN TSP, MIN-SUM EkTSP is differential equivalent to METRIC MIN-SUM kTSP or MAX-SUM EkTSP. We show in this paper that METRIC MIN-SUM EkTSP is $\frac{2}{3}$ differential approximable and it has no differential approximation scheme unless **P = NP**.

The paper is organized as follows: In section 2 we give the necessary definitions. In section 3 we give a constant differential approximation algorithm for GENERAL kVRP, and a better constant differential approximation for the metric case. In section 4 the main result is a constant differential approximation for CONSTRAINED VRP. In section 5 we show that MIN-SUM EkTSP is constant differential approximable and has no differential approximation scheme.

2 Terminology

We first recall a few definitions about differential approximability. Given an instance x of an optimization problem and a feasible solution y of x, we denote by $val(x, y)$ the value of the solution y, by $opt(x)$ the value of an optimal solution of x, and by $wor(x)$ the value of a worst solution of x. The *differential approximation ratio* of y is defined as $\delta(x, y) = \frac{|val(x,y)-wor(x)|}{|opt(x)-wor(x)|}$. This ratio measures how the value of an approximate solution $val(x, y)$ is located in the interval between $opt(x)$ and $wor(x)$. More exactly it is equivalent for a minimization problem to prove $\delta(x, y) \geq \varepsilon$ and $val(x, y) \leq \varepsilon opt(x) + (1 - \varepsilon) wor(x)$.

For a function f, $f(n) < 1$, an algorithm is a *differential $f(n)$-approximation algorithm* for a problem Q if, for any instance x of Q, it returns a solution y such that $\delta(x, y) \geq f(|x|)$. We say that an optimization problem is *constant differential approximable* if, for some constant $\delta < 1$, there exists a polynomial time differential δ-approximation algorithm for it. An optimization problem has a *differential polynomial time approximation scheme* if it has a polynomial time differential $(1 - \varepsilon)$-approximation, for every constant $\varepsilon > 0$. We say that two optimization problems are *differential equivalent* (reps., *standard equivalent*) if a differential δ-approximation (reps., standard δ-approximation) algorithm for one of them implies a differential δ-approximation (reps., standard δ-approximation) algorithm for the other one.

We consider in this paper several routing problems. The problems are defined on a complete undirected graph denoted $G = (V, E)$. The vertex set V consists of a *depot vertex* 0, and *customer vertices* $\{1, \ldots n\}$. There is also a function $d : E \rightarrow R$, where $d_{i,j} \geq 0$ denotes the *length* of edge $(i, j) \in E$. In the rest of the paper we call a such graph a *complete valued graph*. We refer to the version of the problem in which d is assumed to satisfy the triangle inequality as the *metric case*. The output to the problems consists of simple cycles, $C_1, \ldots, C_p$, such that $V(C_i) \cap V(C_j) = \{0\}$, $\forall i \neq j$, and $\cup_{i=1}^{p} V(C_i) = V$. We call such a set of cycles a *p-tour*. We now describe the problems. For each one we specify the input, the problem's constraints, and the output.

kVRP
Input: A complete valued graph.
Constraint: $|C_j| \leq k + 1$, $j = 1, \ldots, p$.
Output: A p-tour minimizing the total length of the cycles.

CONSTRAINED VRP
Input: A complete valued graph and a metric $\ell : E \rightarrow R_+$, and $\lambda > 0$.
Constraint: $\sum_{(i,j) \in C_q} \ell_{i,j} \leq \lambda$, $q = 1, \ldots, p$.
Output: A p-tour minimizing the total length of the cycles.

kWVRP
Input: A complete valued graph and a function $w : \{1, \ldots, n\} \to R$ where w_i denotes the weight of i.
Constraint: $\sum_{i \in C_j} w_i \leq k$, $j = 1, \ldots, p$.
Output: A p-tour minimizing the total length of the cycles.

MIN-SUM EkTSP
Input: A complete valued graph.
Constraint: $p = k$.
Output: A p-tour minimizing the total length of the cycles.

For a problem Q, we denote by Q$(1, 2)$ the version of Q where lengths are 1 and 2. We will use in this paper the following problems:
MIN TSP PATH$(1,2)$ is the variant of MIN TSP$(1,2)$ problem where in place of a tour we ask for a Hamiltonian path of minimum length. We can prove that MIN TSP PATH$(1,2)$ where the graph induced by edges of weight 1 is Hamiltonian and cubic has no differential approximation scheme, unless $\mathbf{P} = \mathbf{NP}$. The result follows since MIN TSP PATH$(1,2)$ on these instances has no standard approximation scheme, unless $\mathbf{P} = \mathbf{NP}$, Bazgan [2].
PARTITIONING INTO PATHS OF LENGTH k (kPP): Given a graph $G = (V, E)$ with $|V| = (k + 1)q$, is there a partition of V into q disjoint sets $V_1, \ldots, V_q$ of $k + 1$ vertices each, so that each subgraph induced by V_i has an Hamiltonian path? 2PP have been proved **NP**-complete in [7] whereas, more generally, the **NP**-completeness of kPP is proved in Kirkpatrick and Hell [12] as a special case of G-PARTITION PROBLEM.

A *binary 2-matching* (also called *2-factor* or *cycle cover*) is a subgraph in which each vertex of V has a degree of exactly 2. A *minimum binary 2-matching* is one with minimum total edge weight. Hartvigsen [10] has shown how to compute a minimum binary 2-matching in $O(n^3)$ time. More generally, a *binary f-matching*, where f is a vector of size $n + 1$, is a subgraph in which each vertex i of V has a degree of exactly f_i. A *minimum f-matching* is one with minimum total edge weight and is computable in polynomial time, Cook et al. [4].

3 kVRP

nVRP is standard equivalent with TSP. So, when the distance is not metric and using the result of Sahni and Gonzalez [17] we deduce that nVRP is not $2^{p(n)}$ standard approximable for any polynomial p, unless $\mathbf{P} = \mathbf{NP}$. In fact for any $k \geq 5$ constant the problem is as hard to approximate as nVRP.

3.1 General kVRP

When d is a metric, the reduction of TSP to nVRP is straightforward, and it easily follows that computing *opt* is **NP**-hard. However, there is no such reduction between the corresponding maximization problems MAX TSP and MAX nVRP leading to the conclusion that computing *wor* is also **NP**-hard.

Proposition 1. *Computing a worst solution for kVRP is* **NP**-*hard for any $k \geq 3$ even if the distance function takes only two values.*

Proof. We use a reduction from PARTITIONING INTO PATHS OF LENGTH k (kPP). Let $G = (V, E)$ with $V = \{1, \ldots, (k+1)q\}$ be an instance of kPP. We construct an instance of kVRP by the following way: we add the depot vertex 0 and set $d_e = 3$ if $e \in E$ and $d_e = 1$ otherwise. It is easy to verify that the answer to kPP is positive if and only if $wor \geq q(3k + 2)$.

In the following we give a $\frac{1}{2}$ differential approximation for non-metric kVRP. We apply ideas similar to those in Hassin and Khuller [11].

We first compute a lower bound LB. Then we generate a feasible solution for G with value $apx = LB + \delta_1$. Next, we generate another feasible solution of value $bad = LB + \delta_2$ where $\delta_2 \geq \delta_1$. This proves that the approximate solution with value apx is an α-differential approximation where

$$\alpha = \frac{wor - apx}{wor - opt} \geq \frac{bad - apx}{bad - opt} \geq \frac{\delta_2 - \delta_1}{bad - LB} = \frac{\delta_2 - \delta_1}{\delta_2}, \tag{1}$$

since for a minimization problem $wor \geq bad \geq apx \geq opt \geq LB$. To generate LB we replace 0 by a complete graph with a set V_0 of $2n$ vertices and zero length edges. The distance between a vertex of V_0 and a vertex i of $V \setminus V_0$ is the same as the distance between 0 and i. Denote the resulting graph by G'. Compute in G' a minimum weight binary 2-matching M.

Lemma 1. *Let LB denote the weight of M. Then $opt_{VRP} \geq LB$.*

Theorem 1. *kVRP is $\frac{1}{2}$ differential approximable.*

Proof. We transform each cycle of M into a cycle in G, replacing the vertices of V_0 by a single occurrence of the depot vertex 0. We work on each cycle of M separately. For each cycle we describe solutions sol_1, sol_2 and sol_3 such that $\delta(sol_1) + \delta(sol_2) = \delta(sol_3)$. We define apx to be the VRP solution obtained by concatenating the shortest of sol_1 and sol_2 for every cycle, and we define bad similarly but using sol_3. We obtain therefore that $\delta(bad) \geq 2\delta(apx)$, and the theorem is proved by (1). In this proof, sol_3 will be always described by the cycles $(0, 1, 0), \ldots, (0, m, 0)$.

First, consider a cycle that does not contain the depot, and w.l.o.g. denote its vertices by $(1, \ldots, m)$. The construction depends on the parity of m.

Suppose that m is even ($m \neq 2$ since the 2-matching is binary). Let sol_1 consist of the cycles $(0, 1, 2, 0), \ldots, (0, m-1, m, 0)$. Let sol_2 consist of the cycles $(0, m, 1, 0), \ldots, (0, m-2, m-1, 0)$ (See Figure 1 for $m = 6$).

Suppose now that m is odd. We modify sol_1 by choosing $(0, m-2, m-1, m)$ as the last cycle (see Figure 2). We modify sol_2 by using $(0, m-1, 0)$ as the last cycle.

Consider a cycle $(0, 1, \ldots, m, 0)$ in M. If $m \leq k$ we don't change it. Otherwise:

If $m \equiv 0 \bmod 3$ then let sol_1 consist of the cycles $(0, 1, 0), (0, 2, 3, 0), \ldots, (0, m-2, 0), (0, m-1, m, 0)$. Let sol_2 consist of the cycles $(0, m, 1, 2, 0), \ldots, (0, m-3, m-2, m-1, 0)$.
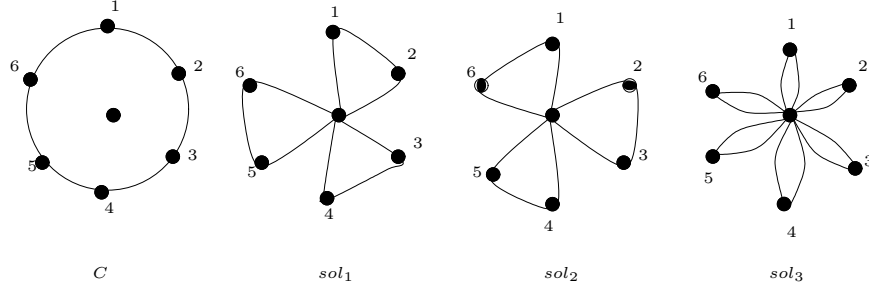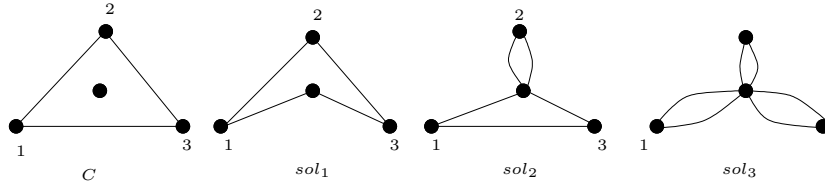
Fig. 1. $m = 6$

Fig. 2. $m = 3$

If $m \equiv 1 \bmod 3$ then let sol_1 consist of the cycles $(0, 1, 0),(0, 2, 3, 0),\ldots,(0, m-3, 0),(0, m-2, m-1, m, 0)$. Let sol_2 consist of the cycles $(0, m, 1, 2, 0),\ldots,(0, m-4, m-3, m-2, 0)$, and $(0, m-1, 0)$.

If $m \equiv 2 \bmod 3$ then let sol_1 consist of the cycles $(0, 1, 0),(0, 2, 3, 0),\ldots,(0, m-3, m-2, 0),(0, m-1, m, 0)$. Let sol_2 consist of the cycles $(0, m, 1, 2, 0),\ldots,(0, m-5, m-4, m-3, 0)$, and $(0, m-2, m-1, 0)$.

Let $\delta(sol_i)$ denote the added cost of sol_i with respect to the length of C. Since M was computed to have a minimum cost, $\delta(sol_i) \geq 0$ and we have $\delta(sol_3) = \delta(sol_1) + \delta(sol_2)$ which complete the proof.

3.2 METRIC kVRP

When d is a metric, computing a worst solution becomes easy since any feasible cycle can be broken into two cycle without decrease the weight and then:

Lemma 2. $wor_{VRP} = 2 \sum_{i=1}^{n} d_{0,i}$

In Theorem 1 we have shown that kVRP is $\frac{1}{2}$ differential approximable. We now show that in the metric case, the same bound can be achieved by a simpler algorithm: we compute a minimum weight perfect matching M on the subgraph induced by $\{1, \ldots, n\}$, if n is even, or by $\{0, 1, \ldots, n\}$ if n is odd. We link each endpoint different of 0 of M to the depot. It is easy to see that $opt_{VRP} \geq 2d(M)$ by walking around an optimum solution for kVRP and by shortcut it in order to obtain a Hamiltonian cycle. Using Lemma 2 and the construction of the

approximate solution, we obtain: $apx = d(M) + \sum_{i=1}^{n} d_{0,i} \leq \frac{1}{2}opt_{VRP} + \frac{1}{2}wor_{VRP}$ proving that the result is a $\frac{1}{2}$ differential approximation.

In Haimovich et al. [9], a $\frac{5}{2} - \frac{3}{2k}$ standard approximation for METRIC kVRP is obtained by reduction to METRIC TSP and using Christofides' algorithm. We proceed similarly for the differential case.

Theorem 2. METRIC kVRP *is* $\delta \cdot \frac{k-1}{k}$ *differential approximable, where* δ *is the differential approximation ratio for* METRIC TSP.

Proof. Our algorithm modifies the *Optimal Tour Partitioning* heuristic of Haimovich, Rinnooy Kan and Stougie [9]: first construct a tour T on V using the differential approximation algorithm for TSP of value $val(T)$. W.l.o.g., assume that this tour is described by the sequence $(0, 1, \ldots, n, 0)$. We produce k solutions sol_i for $i = 1, \ldots, k$ and we select the best solution. The first cycle of sol_i is formed by the sequence $(0, 1, \ldots, i, 0)$ and then each other cycle (except eventually the last) of sol_i has exactly k consecutive vertices (for instance, the second cycle is $(0, i+1, \ldots, i+k, 0)$) and finally, the last cycle is formed by the unvisited vertices (connecting n to the depot 0). Denote by apx_i for $i = 1, \ldots, k$ the values of the k solutions and by apx the value of the best one.

Consider $sol_1, \ldots, sol_k$; each edge of $T \setminus \{(0, 1), (0, n)\}$ appear exactly $(k-1)$ times and each edge $(0, j)$ for $j \neq 1, n$ appears exactly twice. Finally, the edges $(0, 1)$ and $(0, n)$ appears exactly $(k+1)$ times. Using Lemma 2, we deduce: $apx \leq \frac{1}{k} \sum_{i=1}^{k} apx_i \leq \frac{(k-1)}{k} val(T) + \frac{2}{k} wor_{VRP}$. By hypothesis, T satisfies: $val(T) \leq (1-\delta)wor_{TSP} + \delta opt_{TSP}$ and since it is possible to construct from an optimum solution of VRP a solution of TSP with smaller a value (using the triangle inequality), it follows that $opt_{TSP} \leq opt_{VRP}$. Also, by connecting the depot twice with each customer, we can construct from a solution of TSP a solution of VRP with greater value, and therefore $wor_{TSP} \leq wor_{VRP}$.

Using the previous inequalities we obtain that

$$apx \leq \delta \frac{k-1}{k} opt_{VRP} + \left(1 - \delta \frac{k-1}{k}\right) wor_{VRP}.$$

Since the best known differential approximation algorithm for TSP is $\frac{2}{3}$ [11, 14] then the algorithm of Theorem 2 is an $\frac{2}{3} \cdot \frac{k-1}{k}$ differential approximation algorithm for metric kVRP. So, for $k \geq 5$, we obtain a ratio strictly better that the bound produced by the Theorem 1 or by the previous algorithm build on a matching. Now, we will improve all the previous bounds for $k > 3$ since we give a $\frac{3}{5}$ (resp., $\frac{2}{3}$) differential approximation for METRIC 4VRP (resp., METRIC kVRP for $k \geq 5$).

Theorem 3. METRIC kVRP *is* $min\{\frac{2}{3}, \frac{k-1}{k+1}\}$ *differential approximable.*

Proof. Our algorithm works as follow: we compute a minimum weight binary 2-matching $M = (C_1, \ldots, C_q)$ on the subgraph induced by $\{1, \ldots, n\}$. Then, for each cycle of it, we produce several solutions and we take the best one among them. These different solutions depend on the size m_i of the cycle C_i of M.

Without loss of generality, assume that this cycle is described by the sequence $(1, \ldots, m_i, 1)$ with $m_i \geq 3$. Let $wor_i = 2 \sum_{j=1}^{m_i} d_{0,j}$.

If $m_i \leq k$, then we produce m_i solutions $sol_1, \ldots, sol_{m_i}$ where sol_j is obtained just by deleting the edge $(j, j+1)$ (mod m_i) and by connecting j and $j+1$ to the depot. Since $wor_i \geq d(C_i)$ by the triangle inequality and $m_i \geq 3$ we have: $apx_i = \min_j d(sol_j) \leq \frac{m_i - 1}{m_i} d(C_i) + \frac{1}{m_i} wor_i \leq \frac{2}{3} d(C_i) + \frac{1}{3} wor_i$.

Now, assume that $m_i = kp + r$ with $p \geq 1$ and $0 \leq r \leq k - 1$. We still produce m_i solutions $sol_1, \ldots, sol_{m_i}$ but in this case in a different way. In order to build the solution sol_j, we delete from C_i the edge $(j-1, j)$ and the edges $(j - 1 + r + k\ell, j + r + k\ell)$ for $\ell = 1, \ldots, p$ (the indices are taken mod m_i) and we add for each endpoint of these paths, the link with the depot. Note that when $r = 1$, the first path is the isolated vertex j and then in this case, we add once again the edge $(0, j)$.

If $r = 0$, then we deduce: $apx_i \leq \frac{p(k-1)}{kp} d(C_i) + \frac{p}{kp} wor_i \leq \frac{2}{3} d(C_i) + \frac{1}{3} wor_i$.

If $r \geq 1$, then we obtain: $apx_i \leq \frac{p(k-1)+r-1}{kp+r} d(C_i) + \frac{p+1}{kp+r} wor_i \leq \frac{k-1}{k+1} d(C_i) + \frac{2}{k+1} wor_i$.

Finally, since $d(M) \leq opt_{VRP}$, by taking the minimum between $\frac{2}{3}$ and $\frac{k-1}{k+1}$ and by summing over i the previous inequalities we obtain the expected result.

Since $n\text{VRP}$ and TSP are standard equivalent by using the result of Papadimitriou and Yannakakis [16] we deduce immediately that $n\text{VRP}(1,2)$ has no standard approximation scheme. Also TSP$(1,2)$ has no differential approximation scheme, Monnot et al. [15] but we cannot deduce immediately that $n\text{VRP}(1,2)$ has no differential approximation scheme since wor_{nVRP} and wor_{TSP} can be very far. However, we prove in the following a lower bound for the approximation of $n\text{VRP}(1,2)$.

Theorem 4. METRIC $n\text{VRP}(1, 2)$ *is not $\frac{2219}{2220} - \varepsilon$ differential approximable, for any constant ε, unless $\mathbf{P} = \mathbf{NP}$.*

Proof. Since $wor_{nVRP} \leq 4n \leq 4opt_{nVRP}$, a δ differential approximation for $n\text{VRP}(1, 2)$ gives a $\delta + 4(1 - \delta)$ standard approximation for $n\text{VRP}(1, 2)$. Using the negative result given in Engebretsen and Karpinski[6]: TSP$(1,2)$ is not $\frac{741}{740} - \varepsilon$ standard approximable, then we obtain the expected result.

4 CONSTRAINED VRP

We assume now that each edge is associated with a weight ℓ (where $\ell_{i,j} \geq 0$ denotes the cost/time of traversing the edge (i, j)) satisfying the triangle inequality, and the solution must satisfy that the total weight on each cycle does not exceed λ. Note that if we do not assume that ℓ is a metric then even deciding whether the problem has any feasible solution is **NP**-complete.

Theorem 5. *Deciding the feasibility of* CONSTRAINED VRP *is NP-complete.*

Proof. In order to prove the NP-hardness, we reduce HAMILTONIAN $s - t$ PATH problem to CONSTRAINED VRP. From a graph $G = (V, E)$ with $V = \{1, \ldots, n\}$, we construct a graph G' instance of CONSTRAINED VRP by adding a depot vertex 0. We define the function l as follows: $\ell_{0,s} = \ell_{0,t} = 1$, $\ell_{0,i} = \lambda$, for $i \neq s, t$, $\ell_{i,j} = 1$ if $(i, j) \in E$ and $\ell_{i,j} = \lambda$ if $i, j \in \{1, \ldots, n\}$ and $(i, j) \notin E$. Trivially there is a feasible solution for G' only if $\lambda \geq n + 1$. It is easy to see that CONSTRAINED VRP has a feasible solution iff G contains a Hamiltonian path between s and t.

Therefore, we assume that ℓ satisfies the triangle inequality, and to ensure feasibility we also assume that $2\ell_{0,i} \leq \lambda$ for $i = 1, \ldots, n$. Using similar ideas as those used for kVRP we can prove:

Theorem 6. CONSTRAINED VRP *is $\frac{1}{3}$ differential approximable.*

Proof. We start with a binary 2-matching as described in Lemma 1 except that the initial graph is not a complete undirected graph K_{n+1} but a partial graph G' of it built by deleting the edges (i, j) for $i \neq 0$ and $j \neq 0$ such that $\ell_{0,i} + \ell_{i,j} + \ell_{j,0} > \lambda$. It is easy to observe that M is still a lower bound of an optimal solution of CONSTRAINED VRP. As previously, we work on each cycle of M separately.

First, consider a cycle of M that does not contain the depot, and w.l.o.g. denote its vertices by $(1, \ldots, m)$. The construction depends on the parity of m. If m is even, we produce two same solutions sol_1 and sol_2 of Theorem 1, i.e., $(0, 1, 2, 0), (0, 3, 4, 0), \ldots, (0, m-1, m, 0)$ and $(0, m, 1, 0), (0, 2, 3, 0), (0, 4, 5, 0), \ldots$, $(0, m - 2, m - 1, 0)$. Now, if m is odd, we produce m solutions $sol_1, \ldots, sol_m$ where the solution sol_i for $i = 1, \ldots, m$ consist of the cycles $(0, i, 0), (0, i + 1, i + 2, 0), (0, i + 3, i + 4, 0), \ldots, (0, i - 2, i - 1, 0)$ (the indices are taken mod m).

Second, consider a cycle $(0, 1, \ldots, m, 0)$ in M. If $\sum_{i=1}^{m} w_i \leq \lambda$ we don't change it. Otherwise, we have $m \geq 3$ and we produce two solutions sol_1 and sol_2 depending on the parity of m. Let us consider the two cases:

Suppose that m is even. Let sol_1 consist of the cycles $(0, 1, 2, 0), (0, 3, 4, 0), \ldots$, $(0, m - 1, m, 0)$. Let sol_2 consist of the cycles $(0, 1, 0), (0, 2, 3, 0), \ldots, (0, m - 2, m - 1, 0), (0, m, 0)$. Now, suppose that m is odd. Let sol_1 consist of the cycles $(0, 1, 0), (0, 2, 3, 0), \ldots, (0, m - 1, m, 0)$. Let sol_2 consist of the cycles $(0, 1, 2, 0)$, $(0, 3, 4, 0), \ldots, (0, m - 2, m - 1, 0), (0, m, 0)$.

In all cases the solution $bad = (0, 1, 0), \ldots, (0, n, 0)$ satisfies $\min_i \delta(sol_i) \leq \frac{2}{3}\delta(bad)$ (this relation holds with equality when each cycle has exactly three vertices and does not contain the depot, since in this case $\delta(sol_1) + \delta(sol_2) + \delta(sol_3) = 2\delta(bad)$). Therefore, by (1), the best of the solutions sol_i is a $\frac{1}{3}$ differential approximation.

In Haimovich et al. [9], the authors consider two versions of kVRP with additional constraint on the length of each cycles. In the first problem that we will call here WEIGHTED kVRP (kWVRP), each customer has a weight and we want to find a solution such that the total customer weight on each cycle does not exceed k. In the second, called in [9] MIN METRIC DISTANCE, we want to find a solution such that the total distance on each cycle does not exceed a given bound λ. For each of these two problems, we give a differential reduction preserving approximation scheme from CONSTRAINED VRP.

Lemma 3. *A δ differential approximation solution for* Constrained VRP *(respectively, metric case) is also a δ differential approximation for kWVRP (respectively, metric case).*

Proof. Let $G = (V, E)$ with d and w be an instance of kWVRP. We construct an instance of Constrained VRP as follows. First we set $\lambda = k$. The graph and the function d are the same whereas the function ℓ is defined by: $\ell_{i,j} = \frac{w_i + w_j}{2}$ where we assume that $w_0 = 0$. This function satisfies the triangle inequality. Moreover, let C be a cycle linking the depot to a subset of customers. We have $\sum_{q \in C} w_q \leq k$ iff $\sum_{(i,j) \in C} \ell_{i,j} \leq \lambda$.

Corollary 1. *kWVRP is $\frac{1}{3}$ differential approximable.*

Min Metric Distance is a particular case of Metric Constrained VRP where the function ℓ is exactly the function d. Thus, from Theorem 6 we deduce the corollary:

Corollary 2. Min Metric Distance *is $\frac{1}{3}$ differential approximable.*

5 Min-Sum EkTSP

It is easy to see that Min-Sum EkTSP is differential equivalent to Metric Min-Sum EkTSP since the number of edges in every solution is the same (like in the TSP case). Hence, we add a constant to all the edge lengths and achieve the triangle inequality without affecting the best and worst solutions.

Theorem 7. Metric Min-Sum EkTSP *is $\frac{2}{3}$ differential approximable, $\forall k \geq 1$.*

Proof. Add to every edge incident with the depot a parallel copy. Compute a minimum binary f-matching M on G where $f(0) = 2k$ and $f(v) = 2$ for $v \in V \setminus \{0\}$. Compute by using an algorithm of Hassin and Khuller [11] or Monnot [14] a solution C' for TSP on the subgraph G' of G induced by $V' = V \setminus (\cup_{i=1}^{k-1} V(C_i)) \cup \{0\}$, where $C_1, \ldots, C_k$ are the cycles of M containing the depot 0. The approximate solution *sol* is composed of C' and the cycles $C_1, \ldots, C_{k-1}$. See Figure 3. Since M is an minimum binary f-matching M on G then $M' = M \setminus (\cup_{i=1}^{k-1} C_i)$ is an optimum binary 2-matching on G'. We denote by $r = \sum_{i=1}^{k-1} d(C_i)$; the TSP algorithm gives a solution satisfying $val \leq \frac{2}{3} d(M') + \frac{1}{3} wor_{TSP}(G')$. Since $wor_{kTSP}(G) \geq wor_{TSP}(G') + r$ and $opt_{kTSP}(G) \geq d(M)$, we deduce that the value of *sol* satisfies: $apx = val + r \leq \frac{2}{3}[d(M') + r] + \frac{1}{3}[wor_{TSP}(G') + r] \leq \frac{2}{3} opt_{kTSP}(G) + \frac{1}{3} wor_{kTSP}(G)$

Theorem 8. *Unless $\mathbf{P} = \mathbf{NP}$,* Min-Sum EkTSP(1,2) *has no standard and differential approximation scheme.*

Proof. We reduce Min TSP Path (1,2) on Hamiltonian cubic graphs to Min-Sum E2TSP. From a graph $G = (V, E)$ on n vertices, we construct a graph G' instance of Min-Sum E2TSP. G' consists of two copies of G and a vertex 0 (the
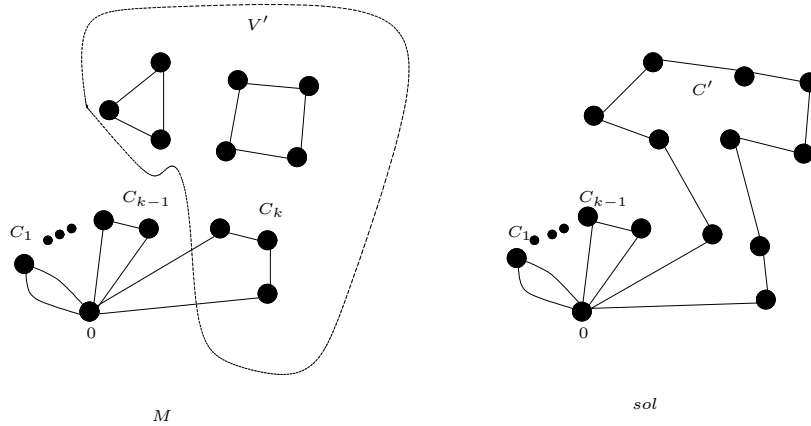
Fig. 3. M and sol

depot). Within a copy, the edges have the same distance as in G; $d_{0,i} = 1$, for each vertex i in one of the two copies; $d_{i,j} = 2$ if i and j are vertices in different copies. We have $opt(G') = 2opt(G) + 4$ and $wor(G') = 2wor(G) + 4$. Given a solution S of G' with two cycles, we can transform it in another one S' that contains exactly two cycles $(0, P_1, 0)$, $(0, P_2, 0)$, each of these two paths are contained in a copy of G and with a better value. The idea for doing this is to remove the edges between the two copies and to replace them by the missing edges in the two copies. We consider as solution for G the path with the smallest value among the two. So, $val = \min\{val(P_1), val(P_2)\} \leq \frac{val(P_1)+val(P_2)}{2} = \frac{val(S')-4}{2} \leq \frac{val(S)-4}{2}$. Since $opt(G) = \frac{opt(G')}{2} - 2$ and $wor(G) = \frac{wor(G')}{2} - 2$ then a δ differential approximation of MIN-SUM E2TSP gives a δ differential approximation for MIN TSP PATH (1,2) on Hamiltonian and cubic graphs and the conclusion follows.

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