Poly-APX- and PTAS-completeness in standard and differential approximation
(Extended abstract)

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Abstract. We first prove the existence of natural **Poly-APX**-complete problems, for both standard and differential approximation paradigms, under already defined and studied suitable approximation preserving reductions. Next, we devise new approximation preserving reductions, called FT and DFT, respectively, and prove that, under these reductions, natural problems are **PTAS**-complete, always for both standard and differential approximation paradigms. To our knowledge, no natural problem was known to be **PTAS**-complete and no problem was known to be **Poly-APX**-complete until now. We also deal with the existence of intermediate problems under FT- and DFT-reductions and we show that such problems exist provided that there exist **NPO**-intermediate problems under Turing-reduction. Finally, we show that MIN COLORING is **APX**-complete for the differential approximation.

1 Introduction

Many **NP**-complete problems are decision versions of natural optimization problems. Since, unless **P** = **NP**, such problems cannot be solved in polynomial time, a major question is to find polynomial algorithms producing solutions "close to the optimum" (in some pre-specified sense). Here, we deal with polynomial approximation of **NPO** problems (see [1] for a formal definition), i.e., of optimization problems the decision versions of which are in **NP**. As usual, we deal with problems the solution-values (or objective values) of which are integer.

For a problem Π in **NPO**, we distinguish three different versions of it: in the constructive version denoted also by Π, the goal is to determine the best solution y^* of an instance x; in the evaluation version Π_e, we are only interested in determining the value of y^*; finally, the decision version Π_d is as dealt in [2].

A polynomial approximation algorithm A for an optimization problem Π is a polynomial time algorithm that produces, for any instance x of Π, a feasible solution $y = \mathtt{A}(x)$. The quality of y is estimated by computing the so-called approximation ratio. Two approximation ratios are commonly used in order to evaluate the approximation capacity of an algorithm: the standard ratio and the differential ratio. Given an instance x of an optimization problem Π, let $\mathrm{opt}(x)$

be the value of an optimal solution, and $\omega(x)$ be the value of a worst feasible solution. This value is the optimal value of the same optimization problem (with respect to the set of instances and the set of feasible solutions for any instance) defined with the opposite objective (minimize instead of maximize, and vice-versa) with respect to Π. For a feasible solution y of x, denote by $m(x, y)$ its value. The *standard approximation ratio* of y is defined as $r(x, y) = m(x, y)/\text{opt}(x)$. The *differential approximation ratio* of y is defined as $\delta(x, y) = |m(x, y) - \omega(x)|/|\text{opt}(x) - \omega(x)|$. Following the above, standard approximation ratios for minimization problems are greater than, or equal to, 1, while for maximization problems these ratios are smaller than, or equal to 1. On the other hand, differential approximation ratio is always at most 1 for any problem.

By means of approximation ratios, **NPO** problems are classified with respect to their approximability properties. Particularly interesting approximation classes are, for the standard approximation paradigm, the classes **Poly-APX** (the class of the problems approximated within a ratio that is a polynomial, or the inverse of a polynomial when dealing with maximization problems, on the size of the instance), **APX** (the class of constant-approximable problems), **PTAS** (the class of problems admitting polynomial time approximation schemata) and **FPTAS** (the class of problems admitting fully polynomial time approximation schemata). We are referred to [1] for formal definitions. Analogous classes can be defined under the differential approximation paradigm: **Poly-DAPX**, **DAPX**, **DPTAS** and **DFPTAS**, are the differential counterparts of **Poly-APX**, **APX**, **PTAS** and **FPTAS**, respectively. Note that **FPTAS** $\subsetneq$ **PTAS** $\subsetneq$ **APX** $\subsetneq$ **Poly-APX**, and **DFPTAS** $\subsetneq$ **DPTAS** $\subsetneq$ **DAPX** $\subsetneq$ **Poly-DAPX**; these inclusions are strict unless **P** = **NP**.

During last two decades, several approximation preserving reductions have been introduced and, using them, hardness results in several approximability classes have been studied. We quote here four approximation preserving reductions that are central to our paper: PTAS, DPTAS, F and E (see also [3] for short definitions of them).

The P-reduction defined in [4] and extended in [5, 6] (been renamed PTAS-reduction) allows existence of **APX**-complete problems as MAX INDEPENDENT SET-B, or MIN METRIC TSP, etc (see [1, 2] for formal definitions about **NPO** problems mentioned in the paper).

In differential approximation, analogous results have been obtained in [7] under DPTAS-reduction. Natural problems such as MAX INDEPENDENT SET-B, or MIN VERTEX COVER-B are shown to be **DAPX**-complete.

Under F-reduction ([4]), only one (not very natural) problem (derived from MAX VARIABLE-WEIGHTED SAT) is known to be **PTAS**-complete. **DPTAS**-completeness has been done until now, but in any case F-reduction does not allow it.

Finally, the E-reduction ([8]) allows existence of **Poly-APX-PB**-complete problems but the existence of **Poly-APX**-complete problems has been left open.

An **NPO** problem Π is *polynomially bounded* if and only if there exists a polynomial q such that, for any instance x and for any feasible solution $y \in \mathrm{Sol}(x)$, $m(x, y) \leqslant q(|x|)$. It is *diameter polynomially bounded* if and only if there exists a polynomial q such that, for any instance x, $|\mathrm{opt}(x) - \omega(x)| \leqslant q(|x|)$. The notion of diameter boundness is very useful and intuitive when dealing with the differential approximation paradigm. The class of polynomially bounded **NPO** problems will be denoted by **NPO-PB**, while the class of diameter polynomially bounded **NPO** problems will be denoted by **NPO-DPB**. Analogously, for any (standard or differential) approximation class **C**, we will denote by **C-PB** (resp., **C-DPB**) the subclass of polynomially bounded (resp., diameter polynomially bounded) problems of **C**.

The main results of this paper deal with the existence of complete problems for **Poly-APX**, **Poly-DAPX**, **FPTAS** and **DFPTAS**. **Poly-APX**-completeness is shown via PTAS-reduction ([6]), while **Poly-DAPX**-completeness is shown via DPTAS-reduction ([7, 9]). We define two new reductions, called FT and DFT, respectively, and show that, using them, natural problems as MAX PLANAR INDEPENDENT SET, MIN PLANAR VERTEX COVER, or BIN PACKING are complete for **PTAS** (the two first ones), or for **DPTAS** (all the three). Next, we study the existence of intermediate[1] problems for these reductions. We prove that such problems exist provided that there exist intermediate problems in **NPO** under the seminal Turing-reduction (see [1] for its definition). Finally, we prove that MIN COLORING is **DAPX**-complete under DPTAS-reduction. This is the first problem that is **DAPX**-complete but not **APX**-complete.

Results are given here without detailed proofs which can be found in [3].

2 Poly-APX-completeness

As mentioned in [8], the nature of the E-reduction does not allow transformation of a non-polynomially bounded problem into a polynomially bounded one. In order to extend completeness in the whole **Poly-APX** we have to use a larger (less restrictive) reduction than E. In what follows, we show that PTAS-reduction can do it. Before continuing, we need the following notions defined in [8].

A problem $\Pi \in$ **NPO** is said *additive* if and only if there exist an operator $\oplus$ and a function f, both computable in polynomial time, such that:

- $\oplus$ associates with any pair $(x_1, x_2) \in \mathcal{I}_\Pi \times \mathcal{I}_\Pi$ an instance $x_1 \oplus x_2 \in \mathcal{I}_\Pi$ with $\mathrm{opt}(x_1 \oplus x_2) = \mathrm{opt}(x_1) + \mathrm{opt}(x_2)$;
- with any solution $y \in \mathrm{sol}_\Pi(x_1 \oplus x_2)$, f associates two solutions $y_1 \in \mathrm{sol}_\Pi(x_1)$ and $y_2 \in \mathrm{sol}_\Pi(x_2)$ such that $m(x_1 \oplus x_2, y) = m(x_1, y_1) + m(x_2, y_2)$.

Let **Poly** be the set of functions from $\mathbb{N}$ to $\mathbb{N}$ bounded by a polynomial. A function $F : \mathbb{N} \to \mathbb{N}$ is *hard* for **Poly** if and only if for any $f \in$ **Poly**, there exist three constants k, c and n_0 such that, for any $n \geqslant n_0$, $f(n) \leqslant kF(n^c)$.

[1] For two complexity classes $\mathbf{C_1}$ and $\mathbf{C_2}$, $\mathbf{C_1} \subseteq \mathbf{C_2}$, and a reduction R preserving membership in $\mathbf{C_1}$, a problem is called $\mathbf{C_2}$-intermediate, if it is neither $\mathbf{C_2}$-complete under R, nor it belongs to $\mathbf{C_1}$.

A maximization problem $\Pi \in \mathbf{NPO}$ is *canonically hard for* ***Poly-APX*** if and only if there exist a transformation T from 3SAT to Π, two constants n_0 and c and a function F, hard for **Poly**, such that, given an instance x of 3SAT on $n \geqslant n_0$ variables and a number $N \geqslant n^c$, instance $x' = T(x, N)$ belongs to $\mathcal{I}_\Pi$ and verifies the following properties:

1. if x is satisfiable, then $\mathrm{opt}(x') = N$, otherwise $\mathrm{opt}(x') = N/F(N)$;
2. given a solution $y \in \mathrm{sol}_\Pi(x')$ such that $m(x', y) > N/F(N)$, one can polynomially determine a truth assignment satisfying x.

Note that, since 3SAT is **NP**-complete, a problem Π is canonically hard for **Poly-APX**, if any decision problem $\Pi' \in \mathbf{NP}$ reduces to Π along Items 1 and 2 just above.

Theorem 1. *If $\Pi \in$ **NPO** is additive and canonically hard for* ***Poly-APX****, then any problem in* ***Poly-APX*** *PTAS-reduces to Π.*

Proof (Sketch). Let Π' be a maximization problem of **Poly-APX** and let $\mathtt{A}$ be an approximation algorithm for Π achieving approximation ratio $1/c(\cdot)$, where $c \in$ **Poly** (the case of minimization will be dealt later). Let Π be an additive problem, canonically hard for **Poly-APX**, let F be a function hard for **Poly** and let k and c' be such that (for $n \geqslant n_0$, for a certain value n_0) $nc(n) \leqslant k(F(n^{c'}) - 1)$. Let, finally, $x \in \mathcal{I}_{\Pi'}$, $\varepsilon \in]0, 1[$ and $n = |x|$. Set $m = m(x, \mathtt{A}(x))$; then $m \geqslant \mathrm{opt}_{\Pi'}(x)/c(n)$. We uniformly partition the interval $[0, mc(n)]$ of possible values for $\mathrm{opt}_{\Pi'}(x)$ into $q(n) = 2c(n)/\varepsilon$ sub-intervals (remark that q is a polynomial). Consider, for $i \in \{1, \ldots, q(n)\}$, the set of instances $\mathcal{I}_i = \{x : \mathrm{opt}_{\Pi'}(x) \geqslant imc(n)/q(n)\}$.

Set $N = n^{c'}$. We construct, for any i, an instance χ_i of Π such that: if $x \in \mathcal{I}_i$, then $\mathrm{opt}_\Pi(\chi_i) = N$, otherwise, $\mathrm{opt}_\Pi(\chi_i) = N/F(N)$. We define $f(x, \varepsilon) = \chi = \oplus_{1 \leqslant i \leqslant q(n)} \chi_i$. Observe that $c(n)/q(n) = \varepsilon/2$.

Let y be a solution of χ and let j be the largest i for which $m(\chi_i, y_i) > N/F(N)$, where y_i is the track of y on χ_i. Then, one can compute a solution ψ' of x such that $m(x, \psi') \geqslant jm\varepsilon/2$. We define

$$\psi = g(x, y, \varepsilon) = \mathrm{argmax}\left\{m\left(x, \psi'\right), m(x, \mathtt{A}(x))\right\}$$

Note that $m(x, \psi) \geqslant \max\{m, jm\varepsilon/2\}$.

We show in [3] that $r(x, \psi) \geqslant r(\chi, y)(1 - (3\varepsilon/4))$, i.e., reduction just sketched is a PTAS-reduction with $c(\varepsilon) = \varepsilon/(4 - 3\varepsilon)$. ∎

For the case where the problem Π' (in the proof of Theorem 1) is a minimization problem, one can reduce it to a maximization problem (for instance using the E-reduction of [8], p. 12) and then one can use the reduction of Theorem 1. Since the composition of an E- and a PTAS-reduction is a PTAS-reduction, the result of Theorem 1 applies also for minimization problems.

Combination of Theorem 1, of remark just above and of the fact that MAX INDEPENDENT SET *is additive and canonically hard for* ***Poly-APX*** ([8]), produces the following concluding theorem.

Theorem 2. MAX INDEPENDENT SET *is* ***Poly-APX****-complete under PTAS-reduction.*

3 Poly-APX-completeness under the differential paradigm

The fact that function f (instance-transformation) of DPTAS-reduction ([7]) is multi-valued allows us to relax the constraint that a **Poly-DAPX**-complete problem has to be additive; we simply impose that it is canonically hard for **Poly-APX**.

Theorem 3. *If a (maximization) problem $\Pi \in$ **NPO** is canonically hard for **Poly-APX**, then any problem in **Poly-DAPX** DPTAS-reduces to Π.*

Proof (Sketch). Let Π be canonically hard for **Poly-APX**, for some function F hard for **Poly**, let $\Pi' \in$ **Poly-DAPX** be a maximization problem and let $\mathtt{A}$ be an approximation algorithm for Π' achieving differential approximation ratio $1/c(\cdot)$, where $c \in$ **Poly**. Finally, let x be an instance of Π' of size n.

We will use the central idea of [7] (see also [9] for more details). We define a set $\Pi'_{i,l}$ of problems derived from Π'. For any pair (i,l), $\Pi'_{i,l}$ has the same set of instances and the same solution-set as Π'; for any instance x and any solution y of x, set $m_{i,l}(x,y) = \max\{0, \lfloor m(x,y)/2^i \rfloor - l\}$. Considering x as instance of any of the problems $\Pi'_{i,l}$, we will build an instance $\chi_{i,l}$ of Π, obtaining so a multi-valued function f. Our central objective is, informally, to determine a set of pairs (i,l) such that we will be able to build a "good" solution for Π' using "good" solutions of $\chi_{i,l}$.

Let $\varepsilon \in]0,1[$; set $M_\varepsilon = 1 + \lfloor 2/\varepsilon \rfloor$ and let c' and k be such that (for $n \geqslant n_0$ for some n_0) $nc(n) \leqslant kF(n^{c'})$ (both c' and k may depend on ε). Assume finally, without loss of generality, that $n \geqslant k$ and set $N = n^{c'}$. Then, $1/F(N) \leqslant 1/c(n)$. Set $m = m(x, \mathtt{A}(x))$. In [7], a set $\mathcal{F}$ of pairs (i,l) is built such that: $|\mathcal{F}|$ is polynomial with n and, furthermore, there exists a pair (i_0, l_0) in $\mathcal{F}$ such that:

$$\delta_{i_0,l_0}(x,y) \geqslant 1 - \varepsilon \Longrightarrow \delta(x,y) \geqslant 1 - 3\varepsilon \quad (1)$$

$$\mathrm{opt}_{i_0,l_0}(x,y) \leqslant M_\varepsilon \quad (2)$$

Let q be an integer. Consider, for any pair $(i,l) \in \mathcal{F}$, the set of instances $\mathcal{I}^q_{i,l} = \{x \in \mathcal{I}_{\Pi'_{i,l}} : \mathrm{opt}_{i,l}(x) \geqslant q\}$. More precisely, consider these instance-sets for $q \in \{0, \ldots, M_\varepsilon\}$. For any pair $(i,l) \in \mathcal{F}$ and for any $q \in \{0, \ldots, M_\varepsilon\}$, one can build an instance $\chi^q_{i,l}$ of Π such that: $\mathrm{opt}_\Pi(\chi^q_{i,l}) = N$ if $\mathrm{opt}_{i,l}(x) \geqslant q$, $\mathrm{opt}_\Pi(\chi^q_{i,l}) = N/F(N)$, otherwise. We set $f : f(x,\varepsilon) = (\chi^q_{i,l}, (i,l) \in \mathcal{F}, q \in \{0, \ldots, M_\varepsilon\})$.

Let $y = (y^q_{i,l}, (i,l) \in \mathcal{F}, q \in \{0, \ldots, M_\varepsilon\})$ be a solution of $f(x,\varepsilon)$. Set $L_y = \{(i,l,q) : m(\chi^q_{i,l}, y^q_{i,l}) > N/F(N)\}$. For each $(i,l,q) \in L_y$, one can determine a solution $\psi^q_{i,l}$ of x (seen as instance of $\Pi'_{i,l}$) with value at least q. Set $\psi = g(x,y,\varepsilon) = \mathrm{argmax}\{m(x, \mathtt{A}(x)), m(x, \psi^q_{i,l}), (i,l,q) \in L_y\}$.

Consider now a pair (i_0, l_0) verifying (1) and (2) and set $q_0 = \mathrm{opt}_{i_0,l_0}(x)$. We can show ([3]) that if $\delta(\chi^{q_0}_{i_0,l_0}, y^{q_0}_{i_0,l_0}) \geqslant 1 - 3\varepsilon$, then $\delta(x,\psi) \geqslant 1 - 3\varepsilon$. Considering $\varepsilon' = 3\varepsilon$ and $c(\varepsilon') = \varepsilon'$, the reduction just sketched is a DPTAS-reduction. ∎

Using the fact that MAX INDEPENDENT SET is canonically hard for **Poly-APX**, Theorem 3 directly exhibits the existence of a **Poly-DAPX**-complete problem.

Theorem 4. MAX INDEPENDENT SET *is* ***Poly-DAPX****-complete under* DPTAS*-reduction.*

Note that we could obtain the **Poly-DAPX**-completeness of canonically hard problems for **Poly-APX** even if we forbade DPTAS-reduction to be multi-valued. However, in this case, we should assume (as in Section 2) that Π is additive (and the proof of Theorem 3 would be much longer).

4 PTAS-completeness

In order to study **PTAS**-completeness, we introduce a new reduction, called FT-reduction, preserving membership in **FPTAS**.

Let Π and Π' be two **NPO** maximization problems. Let $\square_{\alpha}^{\Pi'}$ be an oracle for Π' producing, for any $\alpha \in]0,1]$ and for any instance x' of Π', a feasible solution $\square_{\alpha}^{\Pi'}(x')$ of x' that is an $(1-\alpha)$-approximation for the standard ratio.

Definition 1. *Π FT-reduces to Π' (denoted by $\Pi \leq_{\mathsf{FT}} \Pi'$) if and only if, for any $\varepsilon > 0$, there exists an algorithm $\mathtt{A}_{\varepsilon}(x, \square_{\alpha}^{\Pi'})$ such that:*

- *for any instance x of Π, $\mathtt{A}_{\varepsilon}$ returns a feasible solution which is a $(1-\varepsilon)$-standard approximation;*
- *if $\square_{\alpha}^{\Pi'}(x')$ runs in time polynomial in both $|x'|$ and $1/\alpha$, then $\mathtt{A}_{\varepsilon}$ is polynomial in both $|x|$ and $1/\varepsilon$.*

For the case where at least one among Π and Π' is a minimization problem it suffices to replace $1-\varepsilon$ or/and $1-\alpha$ by $1+\varepsilon$ or/and $1+\alpha$, respectively.

Clearly, FT-reduction transforms a fully polynomial time approximation schema for Π' into a fully polynomial time approximation schema for Π, i.e., it preserves membership in **FPTAS**.

The F-reduction is a special case of FT-reduction since the latter explicitly allows multiple calls to oracle $\square$. Also, FT-reduction seems allowing more freedom in the way Π is transformed into Π'; for instance, in F-reduction, function g transforms an optimal solution for Π' into an optimal solution for Π, i.e., F-reduction preserves optimality; this is not the case for FT-reduction. This freedom will allow us to reduce non polynomially bounded **NPO** problems to **NPO-PB** ones. In fact, it seems that FT-reduction is larger than F. This remains to be confirmed. Such proof is not trivial and is not tackled here.

In what follows, given a class $\mathbf{C} \subseteq \mathbf{NPO}$ and a reduction R, we denote by $\overline{\mathbf{C}}^{\mathsf{R}}$ the closure of $\mathbf{C}$ under R, i.e., the set of problems in **NPO** that R-reduce to some problem in **C**.

The basic result of this section (Theorem 5) follows immediately from Lemmata 1 and 2. Lemma 1 introduces a property of Turing-reduction for **NP**-hard problems. In Lemma 2, we transform (under certain conditions) a Turing-reduction into a FT-reduction. Proofs of the two lemmata are given for maximization problems. The case of minimization is completely analogous.

Lemma 1. *If an* ***NPO*** *problem* Π' *is* ***NP****-hard, then any* $\Pi \in$ ***NPO*** *Turing-reduces to* Π'.

Proof. Let Π be an **NPO** problem and q be a polynomial such that $|y| \leqslant q(|x|)$, for any instance x of Π and for any feasible solution y of x. Assume that encoding $n(y)$ of y is binary. Then $0 \leqslant n(y) \leqslant 2^{q(|x|)} - 1$. We consider the following problem $\hat{\Pi}$ (see also [5]) which is the same as Π up to its objective function that is defined by $m_{\hat{\Pi}}(x,y) = 2^{q(|x|)+1} m_{\Pi}(x,y) + n(y)$.

Clearly, if $m_{\hat{\Pi}}(x,y_1) \geqslant m_{\hat{\Pi}}(x,y_2)$, then $m_{\Pi}(x,y_1) \geqslant m_{\Pi}(x,y_2)$. So, if y is an optimal solution for x (seen as instance of $\hat{\Pi}$), then it is also an optimal solution for x (seen, this time as instance of Π).

Remark now that for $\hat{\Pi}$, the evaluation problem $\hat{\Pi}_e$ and the constructive problem $\hat{\Pi}$ are equivalent. Indeed, given the value of an optimal solution y, one can determine $n(y)$ (hence y) by computing the remainder of the division of this value by $2^{q(|x|)+1}$.

Since Π' is **NP**-hard, we can solve the evaluation problem $\hat{\Pi}_e$ if we can solve the (constructive) problem Π'. Indeed, we can solve $\hat{\Pi}_e$ using an oracle solving, by dichotomy, the decision version $\hat{\Pi}_d$ of $\hat{\Pi}$; $\hat{\Pi}_d$ reduces to the decision version Π'_d of Π' by a Karp-reduction (see [1, 2] for a formal definition of this reduction); finally, one can solve Π'_d using an oracle for the constructive problem Π'. So, with a polynomial number of queries to an oracle for Π', one can solve both $\hat{\Pi}_e$ and $\hat{\Pi}$, and the proof of the lemma is complete. ∎

We now show how, starting from a Turing-reduction (that only preserves optimality) between two **NPO** problems Π and Π' where Π' is polynomially bounded, one can devise an FT-reduction transforming a fully polynomial time approximation schema for Π' into a fully polynomial time approximation schema for Π.

Lemma 2. *Let* $\Pi' \in$ ***NPO-PB****. Then, any* ***NPO*** *problem Turing-reducible to* Π' *is also* FT*-reducible to* Π'.

Proof. Let Π be an **NPO** problem and suppose that there exists a Turing-reduction between Π and Π'. Let $\square_{\alpha}^{\Pi'}$ be an oracle computing, for any instance x' of Π' and for any $\alpha > 0$, a feasible solution y' of x' such that $r(x',y') \geqslant 1 - \alpha$. Moreover, let p be a polynomial such that for any instance x' of Π' and for any feasible solution y' of x', $m(x',y') \leqslant p(|x'|)$.

Let x be an instance of Π. The Turing-reduction claimed gives an algorithm solving Π using an oracle for Π'. Consider now this algorithm where we use, for any query to the oracle with the instance x' of Π', the approximate oracle $\square_{\alpha}^{\Pi'}(x')$, with $\alpha = 1/(p(|x'|) + 1)$. This algorithm produces an optimal solution, since a solution y' being an $(1-(1/(p(|x'|)+1)))$-approximation for x' is an optimal one (recall that we deal with problems having integer-valued objective functions). Indeed,

$$\frac{m_{\Pi'}(x',y')}{\operatorname{opt}_{\Pi'}(x')} \geqslant 1 - \frac{1}{p(|x'|)+1} \Longrightarrow m_{\Pi'}(x',y') > \operatorname{opt}_{\Pi'}(x') - 1$$
$$\Longrightarrow m_{\Pi'}(x',y') = \operatorname{opt}(x')$$

It is easy to see that this algorithm is polynomial when $\Box_{\alpha}^{\Pi'}(x')$ is polynomial in $|x'|$ and in $1/\alpha$. Furthermore, since any optimal algorithm for Π can be a posteriori seen as a fully polynomial time approximation schema, we immediately conclude $\Pi \leq_{\mathsf{FT}} \Pi'$ and the proof of the lemma is complete. ∎

Combination of Lemmata 1 and 2, immediately derives the basic result of the section expressed by the following theorem.

Theorem 5. *Let Π' be an **NP**-hard a problem of **NPO**. If $\Pi' \in$ **NPO-PB**, then any **NPO** problem FT-reduces to Π'.*

From Theorem 5, one can immediately deduce the two corollaries that follow.

Corollary 1. $\overline{\textbf{\textit{PTAS}}}^{\mathsf{FT}} = \textbf{\textit{NPO}}$.

Corollary 2. *Any polynomially bounded problem in **PTAS** is **PTAS**-complete under FT-reduction.*

For instance, MAX PLANAR INDEPENDENT SET and MIN PLANAR VERTEX COVER are in both **PTAS** ([10]) and **NPO-PB**. What has been discussed in this section concludes then the following result.

Theorem 6. MAX PLANAR INDEPENDENT SET *and* MIN PLANAR VERTEX COVER *are **PTAS**-complete under FT-reduction.*

Remark that the results of Theorem 6 cannot be trivially obtained using the F-reduction of [4].

5 DPTAS-completeness

In order to study **DPTAS**-completeness we will again use a new reduction called DFT-reduction. Since it is very similar to the FT-reduction of Section 4 (up to consideration differential ratios instead of standard ones), its definition is omitted.

Let us note that one of the basic features of differential approximation ratio is that it is stable under affine transformations of the objective functions of the problems dealt. In this sense, problems for which the objective functions of the ones are affine transformations of the objective functions of the others are approximate equivalent for the differential approximation paradigm (this is absolutely not the case for standard paradigm). The most notorious case of such problems is the pair MAX INDEPENDENT SET and MIN VERTEX COVER. Affine transformation is nothing else than a very simple kind of differential-approximation preserving reduction, denoted by AF, in what follows. Two problems Π and Π' are affine equivalent if $\Pi \leq_{\mathsf{AF}} \Pi'$ and $\Pi' \leq_{\mathsf{AF}} \Pi$. Obviously affine transformation is both a DPTAS- and a DFT-reduction (as this latter one is derived from Definition 1).

Results of this section are derived analogously to the case of the **PTAS**-completeness of Section 4: we show that any **NP**-hard problem, that belongs to both **NPO-DPB** and **DPTAS**, is **DPTAS**-complete. The basic result of this paragraph (Theorem 7) is an immediate consequence of Lemma 1 and of the following Lemma 3, differential counterpart of Lemma 2 (see [3] for the proof).

Lemma 3. *If $\Pi' \in$ **NPO-DPB**, then any **NPO** problem Turing-reducible to Π' is also DFT-reducible to Π'.*

Theorem 7. *Let $\Pi' \in$ **NPO-DPB** be an **NP**-hard problem. Then, any problem in **NPO** is DFT-reducible to Π'.*

Corollary 3. $\overline{\mathbf{DPTAS}}^{\mathsf{DFT}} = \mathbf{NPO}$.

Corollary 4. *Any **NPO-DPB** problem in **DPTAS** is **DPTAS**-complete under DFT-reductions.*

The following concluding theorem deals with the existence of **DPTAS**-complete problems.

Theorem 8. MAX PLANAR INDEPENDENT SET, MIN PLANAR VERTEX COVER *and* BIN PACKING *are **DPTAS**-complete under DFT-reductions.*

Proof. For **DPTAS**-completeness of MAX PLANAR INDEPENDENT SET, just observe that, for any instance G, $\omega(G) = 0$. So, standard and differential approximation ratios coincide for this problem; moreover, it is in both **NPO-PB** and **NPO-DPB**. Then, the inclusion of MAX PLANAR INDEPENDENT SET in **PTAS** suffices to conclude its membership in **DPTAS** and, by Corollary 4, its **DPTAS**-completeness.

MAX PLANAR INDEPENDENT SET and MIN PLANAR VERTEX COVER are affine equivalent; hence, the former AF-reduces to the latter. Since AF-reduction is a particular kind of DFT-reduction, the **DPTAS**-completeness of MIN PLANAR VERTEX COVER is immediately concluded.

Finally, since BIN PACKING $\in$ **DPTAS** ([11]) and also BIN PACKING belongs to **NPO-DPB**, its **DPTAS**-completeness immediately follows. ∎

6 About intermediate problems under FT- and DFT-reductions

FT-reduction is weaker than the F-reduction of [4]. Furthermore, as mentioned before, this last reduction allows existence of **PTAS**-intermediate problems. The question of existence of such problems can be posed for FT-reduction too. In this section, we handle it via the following theorem.

Theorem 9. *If there exists an **NPO**-intermediate problem for the Turing-reduction, then there exists a problem **PTAS**-intermediate for FT-reduction.*

Proof (Sketch). Let $\Pi \in$ **NPO** be intermediate for the Turing-reduction. Suppose that Π is a maximization problem (the minimization case is completely similar). Let p be a polynomial such that, for any instance x and any feasible solution y of x, $m(x, y) \leqslant 2^{q(|x|)}$. Consider the following maximization problem $\widetilde{\Pi}$ where:

- instances are the pairs (x, k) with x an instance of Π and k an integer in $\{0, \dots 2^{q(|x|)}\}$;
- for an instance (x, k) of $\widetilde{\Pi}$, its feasible solutions are the feasible solutions of the instance x of Π;
- the objective function of $\widetilde{\Pi}$ is:

$$m_{\widetilde{\Pi}}((x,k),y) = \begin{cases} |(x,k)| & \text{if } m(x,y) \geqslant k \\ |(x,k)| - 1 & \text{otherwise} \end{cases}$$

It suffices now to show the three following properties:

1. $\widetilde{\Pi} \in$ **PTAS**;
2. if $\widetilde{\Pi}$ were in **FPTAS**, then Π would be polynomial;
3. if $\widetilde{\Pi}$ were **PTAS**-complete, then Π would be ***NPO**-complete under Turing-reductions*[2].

Obviously, if Properties 1, 2 and 3 hold ([3]), then the theorem is concluded since their combination deduces that if Π is ***NPO**-intermediate under Turing-reductions*, then $\widetilde{\Pi}$ is **PTAS**-intermediate, under FT. ∎

We now state an analogous result about the existence of **DPTAS**-intermediate problems under DFT-reduction.

Theorem 10. *If there exists an **NPO**-intermediate problem under Turing-reduction, then there exists a problem **DPTAS**-intermediate, under DFT-reduction.*

Proof. The proof is analogous to one of Theorem 9, up to modification of definition of $\widetilde{\Pi}$ (otherwise, $\widetilde{\Pi} \notin$ **DPTAS**, because the value of the worst solution of an instance (x, k) is $|(x, k)| - 1$). We only have to add, for any instance (x, k) of $\widetilde{\Pi}$, a new feasible solution y_x^0 with value $m_{\widetilde{\Pi}}((x, k), y_x^0) = 0$. Then, the result claimed is got in exactly the same way as in the proof of Theorem 9. ∎

7 A new DAPX-complete problem not APX-complete

All **DAPX**-complete problems given in [7] are also **APX**-complete under the **E**-reduction ([8]). An interesting question is if there exist **DAPX**-complete problems that are not also **APX**-complete for some standard-approximation preserving reduction. In this section, we positively answer this question by the following theorem.

Theorem 11. MIN COLORING *is **DAPX**-complete under DPTAS-reductions.*

Proof. Consider problem MAX UNUSED COLORS. For this problem, standard and differential approximation ratios coincide and coincide also with differential ratio of MIN COLORING. So, MAX UNUSED COLORS $\leq_{\mathsf{AF}}$ MIN COLORING.

[2] We emphasize this expression in order to avoid confusion with usual **NPO**-completeness considered under the strict-reduction ([12]).

As proved in [13], MAX UNUSED COLORS is **MAX-SNP**-hard under L-reduction, a particular kind of E-reduction. Also, $\overline{\textbf{MAX-SNP}}^{\mathsf{E}} = \textbf{APX-PB}$ ([8]). MAX INDEPENDENT SET-B belongs to **APX-PB**, so, MAX INDEPENDENT SET-B E-reduces to MAX UNUSED COLORS. E-reduction is a particular kind of PTAS-reduction, so, MAX INDEPENDENT SET-B $\leq_{\mathsf{PTAS}}$ MAX UNUSED COLORS.

Standard and differential approximation ratios for MAX INDEPENDENT SET-B, on the one hand, standard and differential approximation ratios for MAX UNUSED COLORS, and differential ratio of MIN COLORING, on the other hand, coincide. So, MAX INDEPENDENT SET-B $\leq_{\mathsf{DPTAS}}$ MIN COLORING.

DPTAS- and AF-reductions just exhibited, together with the fact that their composition is obviously a DPTAS-reduction, establish immediately the **DAPX**-completeness of MIN COLORING. ∎

As we have already mentioned, MIN COLORING is, until now, the only problem known to be **DAPX**-complete but not **APX**-complete. In fact, in standard approximation paradigm, it belongs to the class **Poly-APX** and is inapproximable, in a graph of order n, within $n^{1-\varepsilon}$, $\forall\varepsilon > 0$, unless **NP** coincides with the class of problems that could be optimally solved by slightly super-polynomial algorithms ([14]).

8 Conclusion

We have defined suitable reductions and obtained natural complete problems for important approximability classes, namely, **Poly-APX**, **Poly-DAPX**, **PTAS** and **DPTAS**. Such problems did not exist until now. This work extends also the ones in [7, 9] further specifying and completing a structure for differential approximability. The only among the most notorious approximation classes for which we have not studied completeness is **Log-DAPX** (the one of the problems approximable within differential ratios of $O(1/\log|x|)$). This is because, until now, no natural **NPO** problem is known to be differentially approximable within inverse logarithmic ratio. Work about definition of **Log-DAPX**-hardness is in progress.

Another point that deserves further study, is the structure of approximability classes beyond **DAPX** that are defined not with respect to the size of the instance but to the size of other parameters as natural as $|x|$. For example, dealing with graph-problems, no research is conducted until now on something like **Δ-APX**-, or **Δ-DAPX**-completeness where Δ is the maximum degree of the input graph. Such works miss to both standard and differential approximation paradigms. For instance, a question we are currently trying to handle is if MAX INDEPENDENT SET is, under some reduction, **Δ-APX**-complete, or **Δ-DAPX**-complete. Such notion of completeness, should lead to achievement of inapproximability results (in terms of graph-degree) for several graph-problems.

Finally, the existence of natural **PTAS**-, or **DPTAS**-intermediate problems (as BIN PACKING for **APX** under AP-reduction) for F-, FT- and DFT-reductions remains open.

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