On the Complexity of Global Constraint Satisfaction*

(Extended Abstract)

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Abstract. We study the computational complexity of decision and optimization problems that may be expressed as boolean contraint satisfaction problem with the global cardinality constraints. In this paper we establish a characterization theorem for the decision problems and derive some new approximation hardness results for the corresponding global optimization problems.

1 Introduction

Constraints of the global nature arise naturally in some optimization problems. For example, Min Bisection can be viewed as Min Cut with the restriction that the two sets of vertices that determine the cut must be of equal size. It is known that Min Cut is polynomial while Min Bisection is *NP*-hard. Min Bisection, Max Bisection and other optimization problems can be written as boolean constraint satisfaction problems where a feasible solution is a balanced assignment (where the number of variables set to 1 is the same as the number of variables set to 0). It was an increased interest in global optimization problems recently, cf. [HZ01,FL01,JS04].

In this paper we study the complexity of decision and optimization problems of the balanced versions of boolean constraint satisfaction problems depending on the type of constraints. Schaefer [Sch78] established a dichotomy theorem for the boolean constraint satisfaction problems distinguishing six polynomial time solvable cases. For the decision versions we show that if the set of constraints contains only equations of width 2 or it contains only conjunctions of literals, then the balanced version is polynomial time solvable and otherwise it is *NP*-complete.

Creignou [Cre95] and Khanna and Sudan [KS96] established a dichotomy theorem for maximization versions of boolean constraint satisfaction problems that classify the problems into polynomially solvable or *APX*-hard. The balanced versions of these problems where also studied. Sviridenko [Svi01] proved that the balanced version of Max Sat is $1/(1-\frac{1}{e})$-approximable. For the balanced version

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of MAX 2SAT, Blaser and Manthey [BM02] established a 1.514-approximation factor and Hofmeister [Hof03] a 4/3-approximation factor. Lower bound were also studied for these problems. Holmerin [Hol02] showed that the balanced version of MAX E4-0H-LIN2 (see for the definition Section 2) cannot be approximated within 1.0957 in polynomial time, unless $P{=}NP$. Also Holmerin and Khot [HK03] showed that balanced version of MAX E3-0H-LIN2 is hard to approximate within $\frac{4}{3}-\varepsilon$ and in [HK04] they improved their result showing that this problem is hard to approximate within $2-\varepsilon$, for any $\varepsilon > 0$, if $NP \not\subseteq \cap_{\delta>0}\ DTIME(2^{n^{\delta}})$, thus obtaining the best possible inapproximability factor result for this problem. We prove in this paper that all the cases that were considered by Creignou [Cre95] and Khanna and Sudan [KS96] in the dichotomy theorem become APX-hard and also that most of the trivial maximization constraint satisfaction problems have their balanced version APX-hard.

Khanna, Sudan and Trevisan [KST97] established a classification theorem for minimization versions of boolean constraint satisfaction problems. The complexity of approximation of MIN BISECTION was for long time widely open. Feige and Krautghamer [FK00] established an approximation algorithm for this problem within $O(\log^2 n)$ approximation factor. This result has been recently improved to $O(\log^{1.5} n)$ by the recent result of Arora, Rao and Vazirani [ARV04]. Very recently, Khot [Kho04] established that under the assumption that $NP \not\subseteq \cap_{\delta>0}\ BTIME(2^{n^{\delta}})$, for $BTIME$ denoting randomized polynomial time, MIN BISECTION has no polynomial time approximation scheme. Under the assumption that refuting SAT formulas is hard to approximate on average, Feige [Fei02] proved also that MIN BISECTION is hard to approximate below $\frac{4}{3}$.

Holmerin [Hol02] studied the hardness of approximating some generalizations of MIN BISECTION. In particular he showed that the balanced version of MIN E4-1H-LIN2 is not $(2-\varepsilon)$-approximable for any $\varepsilon > 0$, unless $P{=}NP$. We prove several inapproximability result for balanced minimization problems. In particular, using the inapproximability result for DENSEST k SUBGRAPH established by Khot [Kho04], we prove that the balanced version of MIN MONOTONE-E2SAT has no polynomial time approximation scheme, if $NP \not\subseteq \cap_{\delta>0}\ BTIME(2^{n^{\delta}})$.

The paper is organized as follows: in Section 2 we introduce some preliminary notation and definitions, and Section 3 contains our results on decision problems. In Sections 4 and 5 we present our results concerning maximization and minimization optimization problems.

2 Preliminaries

We refer a general reader to [KST97,KSW97,KSTW01,CKS01] for a background on the boolean constraint satisfaction problems.

A constraint is a boolean function $f : \{0,1\}^k \rightarrow \{0,1\}$. A constraint application is a pair $< f, (i_1, \ldots, i_r) >$ where r is the arity of f and the $i_\ell \in [n]$ indicate to which r of the n boolean variables a given constraint is applied. This constraint application will be denoted in the following by $f(x_{i_1}, \ldots, x_{i_r})$.

Let $\mathcal{F} = \{f_1, \ldots, f_t\}$ be a finite collection of boolean functions. An $\mathcal{F}$-set of constraints on n boolean variables $x_1, \ldots, x_n$ is a collection of constraint applications $\{f_j(x_{j_1}, \ldots, x_{j_{r_j}})\}_{j=1}^{m}$ for some integer m, where $f_j \in \mathcal{F}$ and r_j is the arity of f_j. We say that an assignment satisfies an $\mathcal{F}$-set of constraints if it satisfies every constraint in the collection.

The satisfiability problem CSP($\mathcal{F}$) consists of deciding whether there exists an assignment that satisfies a given $\mathcal{F}$-set of constraints. kCSP($\mathcal{F}$) (respectively, EkCSP($\mathcal{F}$)) is the variant of CSP($\mathcal{F}$) where each boolean function f_j is a function of at most (respectively, exactly) k variables, for $j \leq t$. The problems Max (Min) CSP($\mathcal{F}$) consist of finding a boolean assignment that maximizes (minimizes) the number of constraints that are satisfied. Max (Min) kCSP($\mathcal{F}$) (respectively, Max (Min) EkCSP($\mathcal{F}$)) are variants of Max (Min) CSP($\mathcal{F}$) where each constraint depends on at most (respectively, exactly) k literals.

Given a problem A, the Balanced version of A is the problem A with a new set of feasible solutions being assignments where the number of variables set to *true* (denoted by 1) is the same as the number of variables set to $false$ (denoted by 0). Such assignments will be called *balanced* assignments.

We consider also a generalization of this problem. Given a problem A, the α-Balanced version of A, $0 < \alpha < 1$, is the problem A with a new set of feasible solutions being assignments with the number of *true* variables being an α ratio of the total number of variables. Such assignments will be called in the following α-balanced.

In this paper we study the complexity of decision and optimization problems related to Balanced CSP($\mathcal{F}$) depending on the type of constraints defined by a class $\mathcal{F}$.

We will use basic notation of [Sch78]. We refer to [KMSV94] for the precise definition of an E-reduction.

3 Complexity of Decision Problems

The decision complexity of boolean constraint satisfaction problems is well established. In particular, Schaefer [Sch78] established the following remarkable dichotomy theorem:

Theorem 1 (Dichotomy Theorem for CSP($\mathcal{F}$) [Sch78]). *Given an $\mathcal{F}$-set of constraints, the problem CSP($\mathcal{F}$) is polynomial time computable if $\mathcal{F}$ satisfies one of the conditions below, and CSP($\mathcal{F}$) is NP-complete otherwise.*

1. *Every function in $\mathcal{F}$ is 0-valid.*
2. *Every function in $\mathcal{F}$ is 1-valid.*
3. *Every function in $\mathcal{F}$ is weakly positive.*
4. *Every function in $\mathcal{F}$ is weakly negative.*
5. *Every function in $\mathcal{F}$ is affine.*
6. *Every function in $\mathcal{F}$ is bijunctive.*

Motivated by the above result, we aim at formulating analogous result for balanced problems. Firstly we show that for any $\mathcal{F}$-set of constraints, Balanced CSP($\mathcal{F}$) is at least as difficult as CSP($\mathcal{F}$).

Lemma 1. *If* CSP($\mathcal{F}$) *is NP-complete, then* Balanced CSP($\mathcal{F}$) *is also NP-complete.*

We turn now to a polynomial time case. We formulate our result in slightly more general setting of the α-balanced problems.

Theorem 2. *For any* $0 < \alpha < 1$, α-Balanced E2-Lin2 *is solvable in polynomial time.*

Proof. Let us consider first $\alpha = \frac{1}{2}$. Given an instance I of Balanced E2-Lin2 on n variables and m equations, we construct some equivalence classes on the set of literals by considering the equations one after another as follows. Given an equation $x_i \oplus x_j = 0$ ($x_i \oplus x_j{=}1$), we distinguish the following cases.

- If literals $x_i, \bar{x}_i, x_j, \bar{x}_j$ do not appear in a class, then we construct a new class and we put together x_i and x_j (x_i and $\bar{x}_j$ respectively).
- If either x_i or $\bar{x}_i$ appears in a class C_k and $x_j, \bar{x}_j$ do not appear in a class, then
  - if $x_i \in C_k$ then we introduce x_j ($\bar{x}_j$ respectively) in C_k.
  - if $\bar{x}_i \in C_k$ then we introduce $\bar{x}_j$ (x_j respectively) in C_k.
- If literals x_i or $\bar{x}_i$ and x_j or $\bar{x}_j$ appear in the same class C_k then I is not satisfiable if $\{x_i, \bar{x}_j\} \subseteq C_k$ or $\{\bar{x}_i, x_j\} \subseteq C_k$ ($\{x_i, x_j\} \subseteq C_k$ or $\{\bar{x}_i, \bar{x}_j\} \subseteq C_k$ respectively).
- If either x_i or $\bar{x}_i$ appears in a class C_k and either x_j or $\bar{x}_j$ appears in a class C_ℓ then
  - if $x_i \in C_k$ and $x_j \in C_\ell$ then we put together the literals of both classes C_k and C_ℓ (we put together the literals of the class C_k with the negated literals of the class C_ℓ).
  - if $x_i \in C_k$ and $\bar{x}_j \in C_\ell$ then we put together the literals of the class C_k with the negated literals of the class C_ℓ (we put together the literals of both classes C_k and C_ℓ).

Suppose that at the end we obtain t equivalence classes $C_1, \ldots, C_t$. Denote by a_{2i-1} and a_{2i} the number of literals that appear positive and respectively negative in C_i. Balanced E2-Lin2 on I consists of deciding if there exists a partition of these $2t$ integers in two equal size sets P and N such that P and N contain exactly one of a_{2i-1}, a_{2i} for $i = 1, \ldots, t$. This problem in solvable in polynomial time by dynamic programming [GJ76]. If such a partition P, N exists then the following assignment is balanced and satisfies I:

- if $a_{2i-1} \in P$ then we assign to the positive variables of C_i the value 1 and to the negated variables of C_i the value 0.
- if $a_{2i-1} \in N$ then we assign to the positive variables of C_i the value 0 and to the negated variables of C_i the value 1.

If $\alpha \neq \frac{1}{2}$ then as below we construct equivalence classes $C_1, \ldots, C_t$ and compute integers $a_1, \ldots, a_{2t}$. We add two other integers $a_{2t+1} = n|1 - 2\alpha|$, $a_{2t+2} = 0$ and solve the above partition problem on this new instance. □

The above result contrast interestingly with Theorem 4.

We define a new problem to be used later.

kOnes(E3-bH-Lin2), $b \in \{0, 1\}$
Input: A set of equations of the type $x_{i_1} \oplus x_{i_2} \oplus x_{i_3} = b$ on n boolean variables $x_1, \ldots, x_n$.
Question: Is there an assignment with exactly k variables set to 1, satisfying all equations ?

Max Ones($\mathcal{F}$) consists of determining an assignment that satisfies all constraints of $\mathcal{F}$-type and maximizes the number of variables assigned to 1.

Theorem 3 ([KSW97,KSTW01]). *If every function in $\mathcal{F}$ is of the type $x_{i_1} \oplus x_{i_2} \oplus x_{i_3} = 0$ or if every function in $\mathcal{F}$ is of the type $x_{i_1} \oplus x_{i_2} \oplus x_{i_3} = 1$ then the problem* Max Ones($\mathcal{F}$) *is APX-complete.*

A consequence of the previous theorem is that kOnes(E3-0H-Lin2) and kOnes(E3-1H-Lin2) are *NP*-complete.

Proposition 1. Balanced E3-0H-Lin2 *and* Balanced E3-1H-Lin2 *are both NP-complete.*

Proof. We construct a reduction between kOnes(E3-0H-Lin2) and Balanced E3-0H-Lin2. Given an instance I of kOnes(E3-0H-Lin2) on n variables $x_1, \ldots, x_n$ we construct an instance I' on $2n$ variables $x_1, \ldots, x_n$, $y_1, \ldots, y_{n-k}$, $z_1, \ldots, z_k$ as follows. For each equation $x_{i_1} \oplus x_{i_2} \oplus x_{i_3} = 0$ from I, we associate in I' the same equation and let us denote in the following this set of equations by A. We add also to I' the following set of equations, called B, $y_i \oplus y_j \oplus z_\ell = 0$ for every $i, j \in \{1, \ldots, n-k\}, \ell \in \{1, \ldots, k\}$, $i \neq j$. It is easy to see that in order for an assignment to satisfy B the variables y must have the same value and the variables z must have the same value. Thus, an assignment that satisfies B has $z_i = 0$, $i = 1, \ldots, k$. Suppose that $y_i = 0$, $i = 1, \ldots, n-k$ then since the assignment must be balanced we have $x_i = 1$, $i = 1, \ldots, n$ but in this case the equations in A are not satisfied. So, $y_i = 1$, $i = 1, \ldots, n-k$ and due to the balanced condition, the restriction of this assignment to variables x satisfies I and contains exactly k variables 1 and $n-k$ variables 0. In the similar way kOnes(E3-1H-Lin2) is reducible to Balanced E3-1H-Lin2. □

Theorem 4. *For any $k \geq 3$, $b \in \{0, 1\}$,* Balanced Ek-bH-Lin2 *is NP-complete.*

Proof (sketch). For every odd k, we can construct a reduction between Balanced Ek-0H-Lin2 and Balanced E$(k+2)$-1H-Lin2. Using an equivalence between Balanced Ek-0H-Lin2 and Balanced Ek-1H-Lin2 for every odd k and Proposition 1, we derive the *NP*-completeness of Balanced Ek-bH-Lin2,

$b \in \{0,1\}$, for every odd k. For every odd k, and $b \in \{0,1\}$, we can construct a reduction between BALANCED Ek-bH-LIN2 and BALANCED E$(k+1)$-bH-LIN2. Thus we can derive the *NP*-completeness of BALANCED Ek-H-LIN2 for every even $k \geq 4$. $\square$

MONOTONE-2SAT is a trivial problem. In contrast to this, we show that α-BALANCED MONOTONE-E2SAT is, in fact, *NP*-hard.

Theorem 5. α-BALANCED MONOTONE-E2SAT *is NP-complete, for any* $\alpha > 0$.

Proof. We reduce α-CLIQUE (cf. [GJ76]) to α-BALANCED MONOTONE-E2SAT. An instance of α-CLIQUE has an input a graph on n vertices and we have to decide if it contains a clique of size at least αn. The reduction is as follows: given a graph $G = (V, E)$ on n vertices, we construct an instance I on n boolean variables $x_1, \ldots, x_n$, one for each vertex of G. For any $i, j \in V$ such that $(i, j) \notin E$, we add the clause $\bar{x}_i \vee \bar{x}_j$. It is clear that if C is a clique in G of size αn, then the assignment $x_i = 1$ if $i \in C$ and $x_i = 0$ if $i \notin C$ satisfies each clause of I since for each $(i, j) \notin E$, x_i or x_j is false. Conversely, if an α-balanced assignment satisfies I, then the set $C = \{i : x_i = 1\}$ is a clique of size αn. Since α-CLIQUE is *NP*-hard [GJ76], α-BALANCED MONOTONE-E2SAT is *NP*-hard as well. $\square$

Theorem 6. BALANCED MONOTONE-EkSAT *is NP-complete for any* $k \geq 3$.

Proof (sketch). A reduction from EkSAT yields. $\square$

Since BALANCED AND is trivial we can formulate the following

Theorem 7 (Characterization Theorem for BALANCED CSP$(\mathcal{F})$**).** *Given an $\mathcal{F}$-set of constraints, the problem* α-BALANCED CSP$(\mathcal{F})$ *is polynomial time solvable (if every function in $\mathcal{F}$ is affine with width 2 or if every function in $\mathcal{F}$ is a conjunction of literals), otherwise it is NP-complete.*

4 Approximation of Global Maximum Constraint Satisfaction

We state first the following known classification theorem of MAX CSP$(\mathcal{F})$ (cf. [Cre95,KS96]).

Theorem 8 (Characterization Theorem for MAX CSP$(\mathcal{F})$ **[Cre95,KS96]).** MAX CSP$(\mathcal{F})$ *is either polynomial time computable or is APX-complete. Moreover, it is in P if and only if $\mathcal{F}$ is either 0-valid or 1-valid or 2-monotone.*

Some upper bounds have been established for these balanced versions of MAX CSP$(\mathcal{F})$. α-BALANCED MAX SAT was proven to be $1/(1 - \frac{1}{e})$-approximable ([Svi01]). α-BALANCED MAX 2SAT was proven to be 1.514-approximable ([BM02]) and BALANCED MAX 2SAT was proven to be 4/3-approximable ([Hof03]).

We state first the following direct lemma:

Lemma 2. MAX CSP$(\mathcal{F})$ *is E-reducible to* BALANCED MAX CSP$(\mathcal{F})$.

The following lemma shows that the three polynomial cases for Max CSP($\mathcal{F}$) became difficult for the balanced version.

Theorem 9. Balanced Max Monotone-EkSat *is APX-hard, for* $k \geq 2$.

A particular case of the following problem is equivalent to a Balanced Max CSP($\mathcal{F}$) problem for some particular $\mathcal{F}$ as it will be proved later.

We introduce now a new problem.

Densest k Subgraph
Input: A graph $G = (V, E)$ on n vertices where n is even.
Output: A subset $S \subseteq V$ of size k that maximize the number of edges with both extremities in S.

The hardness of the approximation of Densest k Subgraph remained open for long time. Recently, Khot [Kho04] was able to establish such a result using a special PCP technique.

Theorem 10 ([Kho04]). Densest k Subgraph *has no polynomial time approximation scheme if NP* $\not\subseteq \cap_{\delta>0}$ *BTIME*(2^{n^δ}).

More precisely, Khot [Kho04] has proved the previous result for Densest k Subgraph when $k = cn$ for c a constant.

Proposition 2. Densest k Subgraph *is E-reducible to* Densest $\frac{n}{2}$ Subgraph.

Proposition 3. Balanced Max Monotone-E2AND *is E-equivalent to* Densest $\frac{n}{2}$ Subgraph.

Theorem 11. Balanced Max Monotone-EkAND, $k \geq 2$, *has no polynomial time approximation scheme if NP* $\not\subseteq \cap_{\delta>0}$ *BTIME*(2^{n^δ}).

Proof (sketch). We can E-reduce Balanced Max Monotone-EkAND to Balanced Max Monotone-E$(k+1)$AND, for $k \geq 2$, and thus using Propositions 2, 3 and Theorem 10 (Khot's result [Kho04]) the result follows. $\square$

We consider in the following the balanced version of affine constraints.

Max E2-1H-Lin2, that is Max Cut, is known to be APX-hard [PY91] and Balanced Max E2-1H-Lin2 that is Max Bisection is known to be APX-hard [PY91,Has97]. Each instance of Max E2-0H-Lin2 is satisfied by the trivial assignment 0. We show a relation between the complexity of Balanced Max Monotone-E2AND and Balanced Max E2-0H-Lin2 (or Balanced Max Uncut).

Proposition 4. Balanced Max Monotone-E2AND *is E-reducible to* Balanced Max E2-0H-Lin2.

Thus we establish an inapproximability result for Balanced Max Uncut.

Theorem 12. Balanced Max UnCut *has no polynomial time approximation scheme if* $NP \not\subseteq \cap_{\delta>0}$ $BTIME(2^{n^\delta})$.

Proof. The result is a consequence of Propositions 3, 4 and Theorem 10. □

When k is odd, Max Ek-bH-Lin2 is trivial since the assignment b for all variables satisfies all equations. When k is even, Max Ek-0H-Lin2 is also trivial since the assignment 0 for all variables satisfies all equations. For $k \geq 4$ even, Max Ek-1H-Lin2 is not know to be hard to approximate.

Theorem 13. Balanced Max Ek-bH-Lin2 *is APX-hard, for* $k \geq 3$, $b \in \{0, 1\}$.

Proof. We construct an E-reduction between Balanced Max E2-1H-Lin2 and Balanced Max E3-1H-Lin2. Given an instance I of Balanced Max E2-1H-Lin2 on n variables $x_1, \ldots, x_n$ and m equations, we construct an instance I' on $3n$ variables $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, $z_1, \ldots, z_n$ as follows. For each equation $x_{i_1} \oplus x_{i_2} = 1$ from I, we associate in I' the equations $x_{i_1} \oplus x_{i_2} \oplus z_\ell = 1$, for $\ell = 1, \ldots, n$ and let us call in the following this set of equations A. We add also to I' the following equations $z_i \oplus z_j \oplus y_\ell = 1$ for $i \neq j$, $i, j, \ell \in \{1, \ldots, n\}$. This last set of equations is called B. It is easy to see that $opt(I') \geq n \times opt(I) + |B|$ since the same assignment for variables x, $z_i = 0$, $i = 1, \ldots, n$ and $y_i = 1$, $i = 1, \ldots, n$ satisfies in A, $n \times opt(I)$ equations. Since there are $\Theta(n^2)$ equations of the type $z_i \oplus z_j \oplus y_\ell = 1$ and $z_i \oplus z_j \oplus y_t = 1$ for some fixed i and j and $\ell, t \in \{1, \ldots, n\}$ then $y_\ell = y_t$ and so all variables y have the same values and this value is 1. We can prove similarly that all variables z have the same values. Since the assignment is balanced, then variables z have values 0 and thus the variables x form a balanced solution.

For every odd k, we can construct an E-reduction between Balanced Max Ek-0H-Lin2 and Balanced Max E$(k+2)$-1H-Lin2. Using an E-equivalence between Balanced Max Ek-0H-Lin2 and Balanced Max Ek-1H-Lin2 for every odd k, we derive the APX-hardness of Balanced Max Ek-bH-Lin2, $b \in \{0, 1\}$, for every odd k. For every odd k and $b \in \{0, 1\}$, we can construct an E-reduction between Balanced Max Ek-bH-Lin2 and Balanced Max E$(k+1)$-bH-Lin2. Thus we can derive the APX-hardness of Balanced Max Ek-H-Lin2 for every even $k \geq 4$. □

Balanced Max Ek-bH-Lin2 was studied for particular cases of k and $b = 0$. More precisely, Holmerin [Hol02] proved that Balanced Max E4-0H-Lin2 cannot be approximated within 1.0957 in polynomial time, unless P=NP. Also Holmerin and Khot showed in [HK03] that Balanced Max E3-0H-Lin2 is hard to approximate within $\frac{4}{3} - \varepsilon$ and in [HK04] they improved their result showing that Balanced Max E3-0H-Lin2 is hard to approximate within $2 - \varepsilon$ if $NP \not\subseteq \cap_{\delta>0}$ $DTIME(2^{n^\delta})$, thus obtaining the best possible inapproximability bound result for this problem (under this assumption).

Theorem 14 (Characterization Theorem for Balanced Max CSP($\mathcal{F}$)**).** Balanced Max CSP($\mathcal{F}$) *is APX-hard.*

5 Approximation of Global Minimum Constraint Satisfaction

A classification theorem for MIN CSP($\mathcal{F}$) was formulated in [KST97].

We can show directly, like for the decision and maximization constraint satisfaction problems, that the balanced version of a minimization problem is at least as hard as an underlying problem.

Lemma 3. MIN CSP($\mathcal{F}$) *is E-reducible to* BALANCED MIN CSP($\mathcal{F}$).

MIN MONOTONE-EkSAT for $k \geq 2$ are trivial problems. For the balanced situation we formulate

Proposition 5. BALANCED MAX MONOTONE-E2AND *is E-reducible to* BALANCED MIN MONOTONE-E2SAT.

We derive now

Theorem 15. BALANCED MIN MONOTONE-E2SAT *has no polynomial time approximation scheme if NP* $\not\subseteq \cap_{\delta>0}$ *BTIME*(2^{n^δ}).

Proof. The result is a consequence of Proposition 5 and Theorem 11. □

We first show that a hardness approximation result for BALANCED MIN MONOTONE-E2SAT implies a hardness approximation result for MIN BISECTION.

Proposition 6. BALANCED MIN MONOTONE-E2SAT *is E-reducible to* MIN BISECTION.

Proof. Given an instance I of BALANCED MIN MONOTONE-E2SAT on n variables $x_1, \ldots, x_n$ and m clauses, we construct an instance I' of MIN BISECTION on $n+2$ variables $x_1, \ldots, x_n$ and two new variables y and z and $3m$ equations as follows : for each clause $x_1 \vee x_2$ we add 3 equations $x_1 \oplus x_2 = 1, x_1 \oplus z = 1, x_2 \oplus z = 1$. We have $opt(I') \leq 2opt(I)$ since the assignment satisfying $opt(I)$ clauses in I and $z = 0$ and $y = 1$ satisfies $2opt(I)$ equations in I'. Given a balanced assignment v for I' satisfying val' equations, we can consider $z = 0$. If $y = 1$ then, the restriction of v on x variables is balanced and satisfies $\frac{val'}{2}$ clauses. If $y = 0$ then the restriction of v on x variables satisfies $\frac{val'}{2}$ clauses in I but is not balanced. Observe that the balanced assignment obtained by changing the value of an x variable from 1 to 0 satisfies at most $\frac{val'}{2}$ clauses. □

We establish now an E-reduction between BALANCED MAX UNCUT and MIN BISECTION.

Proposition 7. BALANCED MAX E2-0H-LIN2 *is E-reducible to* BALANCED MIN E2-1H-LIN2.

We formulate now

Theorem 16. BALANCED MIN MONOTONE-EkSAT, $k \geq 2$, *has no polynomial time approximation scheme if NP* $\not\subseteq \cap_{\delta>0}$ *BTIME*(2^{n^δ}).

Proof (sketch). We can E-reduce BALANCED MIN MONOTONE-EkSAT to BALANCED MIN MONOTONE-E$(k+1)$SAT, for $k \geq 2$, and thus using Theorem 15, the result follows. □

MIN E2-0H-LIN2 is MIN UNCUT that is known to be APX-hard by [GVY93] and thus BALANCED MIN E2-0H-LIN2 is BALANCED MIN UNCUT is also APX-hard. MIN E2-1H-LIN2 that is MIN CUT is polynomial solvable. BALANCED MIN E2-1H-LIN2 is MIN BISECTION for which the hardness of approximation was proved very recently [Kho04]. For $k \geq 3$, MIN Ek-1H-LIN2 is trivial since the assignment 0 for all variables satisfies no equation. When k is odd, MIN Ek-0H-LIN2 is also trivial since the assignment 1 for all variables satisfies no equation and when $k \geq 4$ is even, it is not known if MIN Ek-0H-LIN2 is hard to approximate.

Theorem 17 ([HK03]). BALANCED MIN E3-bH-LIN2, $b \in \{0,1\}$, *is NP-hard to approximate within any constant factor.*

The proof of the above result uses a PCP technique. We can prove without using directly a PCP method a somewhat weaker result

Theorem 18. BALANCED MIN Ek-bH-LIN2, $b \in \{0,1\}$, *is APX-hard for every* $k \geq 3$.

References

[AKK95] S. Arora, D. Karger and M. Karpinski, *Polynomial time approximation schemes for dense instances of NP-hard problems*, Proceedings of 27th Annual ACM Symposium on the Theory of Computing, 1995, 284–293; also published in Journal of Computer and System Sciences 58, 1999, 193–210.

[ARV04] S. Arora, S. Rao and U. Vazirani, *Expander flows and a $\sqrt{\log n}$-approximation to sparsest cut*, Proceedings of the 36th Annual ACM Symposium on Theory of Computing (STOC 2004), 222 – 231.

[BFK03] C. Bazgan, W. Fernandez de la Vega and M. Karpinski, *Polynomial time approximation schemes for dense instances of the minimum constraint satisfaction*, Random Structures and Algorithms, 23(1), 2003, 73–91.

[BM02] M. Blaser and B. Manthey, *Improved Approximation Algorithms for Max 2Sat with Cardinality Constraint*, The 13th Annual International Symposium on Algorithms and Computation (ISAAC 2002), LNCS 2518, 2002, 187–198.

[Cre95] N. Creignou, *A dichotomy theorem for maximum generalized satisfiability problems*, Journal of Computer and System Sciences, 51(3), 1995, 511–522.

[CKS01] N. Creignou, S. Khanna and M. Sudan, *Complexity Classifications of Boolean Constraint Satisfaction Problems*, SIAM Monographs on Discrete Mathematics and Applications, 2001.

[Fei02] U. Feige, *Relations between average case complexity and approximation complexity*, Proceedings of the 34th Annual ACM Symposium on Theory of Computing (STOC 2002), 534–543.

[FK00] U. Feige and R. Krauthgamer, *A polylogarithmic approximation of the Minimum Bisection*, Proceedings of the 41st Annual Symposium on Foundations of Computer Science, 2000, 105–115.

[FL01] U. Feige and M. Langberg, *Approximation Algorithms for Maximization Problems arising in Graph Partitioning*, Journal of Algorithms 41 (2001), 174–211.

[GJ76] M. R. Garey and D. S. Johnson, *Computers and intractability. A guide to the theory of NP-completeness*, Freeman, C.A. San Francisco (1979).

[GVY93] N. Garg, V. Vazirani and M. Yannakakis, *Approximate max-flow min-(multi)cut theorems and their applications*, Proceedings of the 25th Annual ACM Symposium on Theory of Computing (STOC 1993), 698–707.

[HZ01] E. Halperin, U. Zwick, *A unified framework for obtaining improved approximation algorithms for maximum graph bisection problems*, Proceedings of the 8th Conference on Integer Programming and Combinatorial Optimization (IPCO 2001), 210–225.

[Has97] J. Håstad, *Some optimal inapproximability results*, Proceedings of the 29th Annual ACM Symposium on Theory of Computing (STOC 1997), 1-10.

[Hof03] T. Hofmeister, *An Approximation Algorithm for Max 2Sat with Cardinality Constraint*, Proceedings of the 11th Annual European Symposium on Algorithms, 301–312, 2003.

[Hol02] J. Holmerin, *PCP with Global Constraints – Balanced Homogeneous Linear Equations*, Manuscript, 2002.

[HK03] J. Holmerin and S. Khot, *A strong inapproximability result for a generalization of Minimum Bisection*, Proceedings of the 18th IEEE Conference on Computational Complexity, 371–378, 2003.

[HK04] J. Holmerin and S. Khot, *A new PCP Outer Verifier with Applications to Homogeneous Linear Equations and Max-Bisection*, Proceedings of the 36th Annual ACM Symposium on Theory of Computing (STOC 2004), 11–17.

[JS04] G. Jger, A. Srivastav, *Improved Approximation Algorithms for Maximum Graph Partitioning Problems*, Proceedings of the 24th International Conference Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2004), 348–359.

[KMSV94] S. Khanna, R. Motwani, M. Sudan and U. Vazirani, *On syntactic versus computational views of approximability*, Proceedings of the 35th Annual IEEE Annual Symposium on Foundations of Computer Science, 819–830, 1994, Also published in SIAM Journal on Computing, 28(1), 1999, 164–191.

[KS96] S. Khanna and M. Sudan, *The optimization complexity of constraint satisfaction problems*, Technical note STAN-CS-TN-96-29, Stanford University, CA, 1996.

[KST97] S. Khanna, M. Sudan and L. Trevisan, *Constraint Satisfaction: the Approximability of Minimization Problems*, Proceedings of 12th IEEE Computational Complexity, 1997, 282–296.

[KSTW01] S. Khanna, M. Sudan, L. Trevisan and D. P. Williamson, *The Approximability of Constraint Satisfaction Problems*, SIAM Journal of Computing, 30(6), 1863-1920 (2001).

[KSW97] S. Khanna, M. Sudan and D. Williamson, *A Complete Classification of the Approximability of Maximization Problems Derived from Boolean Constraint*

Satisfaction, Proceedings of 29th ACM Symposium on Theory of Computing (STOC 1997), 11–20.

[Kho04] S. Khot, *Ruling Out PTAS for Graph Min-Bisection, Densest Subgraph and Bipartite Clique*, Proceedings of the 45th Annual IEEE Annual Symposium on Foundations of Computer Science (FOCS 2004), 136–145.

[PY91] C. H. Papadimitriou and M. Yannakakis, *Optimization, approximation, and complexity classes*, Journal of Computing and System Science, 43 (1991), 425-440.

[Sch78] T. Schaefer, *The complexity of satisfiability problems*, In Conference Record of the 10th Annual ACM Symposium on Theory and Computing (STOC 1978), 216–226.

[Svi01] M. I. Sviridenko, *Best possible approximation algorithm for MAX SAT with cardinality constraint*, Algorithmica, 30(3), 2001, 398–405.