Completeness in differential approximation classes
(Extended abstract)

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Abstract. We study completeness in differential approximability classes. In differential approximation, the quality of an approximation algorithm is the measure of both how far is the solution computed from a worst one and how close is it to an optimal one. The main classes considered are **DAPX**, the differential counterpart of **APX**, including the **NP** optimization problems approximable in polynomial time within constant differential approximation ratio and the **DGLO**, the differential counterpart of **GLO**, including problems for which their local optima guarantee constant differential approximation ratio. We define natural approximation preserving reductions and prove completeness results for the class of the **NP** optimization problems (class **NPO**), as well as for **DAPX** and for a natural subclass of **DGLO**. We also define class **0-APX** of the **NPO** problems that are not differentially approximable within any ratio strictly greater than 0 unless **P** = **NP**. This class is very natural for differential approximation, although has no sense for the standard one. Finally, we prove the existence of hard problems for a subclass of **DPTAS**, the differential counterpart of **PTAS**, the class of **NPO** problems solvable by polynomial time differential approximation schemata.

1 Preliminaries

An **NP** optimization problem Π is defined as a four-tuple $(\mathcal{I}, \text{sol}, m, \text{opt})$ such that: $\mathcal{I}$ is the set of instances of Π and it can be recognized in polynomial time; given $x \in \mathcal{I}$, $\text{sol}(x)$ denotes the set of feasible solutions of x; for every $y \in \text{sol}(x)$, $|y|$ is polynomial in $|x|$; given any x and any y polynomial in $|x|$, one can decide in polynomial time if $y \in \text{sol}(x)$; given $x \in \mathcal{I}$ and $y \in \text{sol}(x)$, $m(x, y)$ denotes the value of y for x; m is polynomially computable and is commonly called feasible value, or objective value; finally, $\text{opt} \in \{\max, \min\}$. The set of **NP** optimization problems forms the class **NPO**. An **NPO** problem Π is said to be *polynomially bounded*, if, for any instance x of Π, the value of the optimum solution of x is bounded by a polynomial in $|x|$. The set of polynomially bounded problems of **NPO** forms the class **NPO-PB**. In what follows, given an instance x of Π and a feasible solution y for x, we denote by $\text{opt}(x)$ the value of an optimal solution of x and by $\omega(x)$ the value of a worst solution of x; $\omega(x)$ is the value of the optimum solution for x with respect to the **NPO** problem $\Pi' = (\mathcal{I}, \mathrm{sol}, m, \mathrm{opt}')$ where $\mathrm{opt}' = \max$, if $\mathrm{opt} = \min$ and $\mathrm{opt}' = \min$, if $\mathrm{opt} = \max$.

Polynomial approximation deals with polynomial computation of "good", with respect to a predefined criterion, feasible solutions for hard **NPO** problems. Two main such criteria have been used until now: the *standard approximation ratio* and the *differential approximation ratio*. For an approximation algorithm $\mathtt{A}$ computing a feasible solution y for x with value $m_{\mathtt{A}}(x, y)$, its standard approximation ratio is defined as $\gamma_{\Pi}^{\mathtt{A}}(x, y) = m_{\mathtt{A}}(x, y)/\mathrm{opt}(x)$ and its differential one as $\delta_{\Pi}^{\mathtt{A}}(x, y) = |\omega(x) - m_{\mathtt{A}}(x, y)|/|\omega(x) - \mathrm{opt}(x)|$. In what follows, whenever it is understood, reference to problem Π will be dropped. Finally note that, for any problem Π and for any algorithm $\mathtt{A}$, $0 \leqslant \delta_{\Pi}^{\mathtt{A}} \leqslant 1$. An approximation measure μ is called *cost-respecting* ([1]) if given two solutions y_1 and y_2 for an instance x of an optimization problem Π, the fact that y_1 is worse than y_2 implies that $\mu(y_1)$ is worse than $\mu(y_2)$. Obviously, both standard and differential approximation ratios are cost-respecting measures.

Regarding the type of approximation results, **NPO** problems can be classified with respect to the approximation ratios known for them. The main approximability classes are: **APX** (**DAPX**), the class of **NPO** problems polynomially approximable within constant standard (differential) approximation ratio; **PTAS** (**DPTAS**), the class of problems polynomially approximable by standard (differential) polynomial time approximation schemata, i.e., within standard (differential) ratios arbitrarily close to 1; **FPTAS** (**DFPTAS**), the class of problems approximable by standard (differential) fully polynomial time approximation schemata, i.e., within ratios arbitrarily close to 1 in time polynomial in both the size of their instances and in $1/\epsilon$.

Since the beginning of the 80's, researchers have been highly interested in providing a structure in standard approximation by defining suitable approximation preserving reductions in order to study completeness in approximability classes. Pioneering works in this direction, used in this paper, are, among others, the ones in [2, 1, 3]. In [1] several natural minimization problems have been shown to be **NPO**-complete under an approximation preserving reduction called strict-reduction, dealing with any cost-respecting approximation measure r. Throughout the paper, for any reduction R, we will denote by $\Pi \leq_{\mathsf{R}} \Pi'$ the fact that Π R-reduces to Π'. In [3], the subclass **MAX-SNP** of **APX** has been introduced and complete problems have been provided for it, under L-reduction. In [2], a polynomial time approximation schema preserving reduction, called P-reduction there, has been introduced and the existence of **APX**-complete problems has been shown. In what follows, we borrow the term PTAS from [4, 5] and we will use it instead of P. Furthermore, another reduction called F has been defined in [2] by means of which **PTAS**-complete problems have been provided.

Surprisingly enough, differential approximation, although introduced in [6] since 1977, has not been systematically used until the mid-90's when a formal framework for it and a more systematic use started to be drawn ([7]). In any case, no structural approach to the study of differential approximability has been developed until now. This is the main objective of this paper.

In Section 2, we show the existence of **NPO**-complete problems in the framework of the differential approximation. We then introduce a subclass of **NPO**, called **0-DAPX**, for the problems of which no polynomial time algorithm can guarantee that any solution computed will be even slightly far from a worst one, unless **P** = **NP**; in other words, the differential ratio of any polynomial time algorithm is equal to 0. We prove that under the strict-reduction **NPO**-complete = **0-DAPX**-complete ⊆ **0-DAPX** ⊆ **NPO**. In Section 3, we tackle the question of the existence of complete problems for **DAPX**. We define a suitable reduction, called DPTAS-reduction and show that under it many natural **NPO** problems are **DAPX**-complete. In section 4, we devise an appropriate reduction and show the existence of hard problems for a natural subclass of **DPTAS**.

Besides **PTAS**, the two most notable classes of **APX** in the literature are **MAX-SNP** and **GLO**. The first one, introduced, as we have already mentioned in [3], is defined in logical terms and, furthermore, independently on any approximability property of its members; henceforth, **MAX-SNP** is notorious for differential approximation also without need of defining any differential counterpart for it. The latter one, **GLO**, is, roughly speaking, the class of the **NPO-PB** problems whose all locally optimal solutions (with respect to a suitable neighborhood) guarantee constant standard approximation ratio. It is introduced in [8] where a local optima preserving (LOP) reduction, which is a special case of L-reduction provided with some suitable local optimality properties, is also defined. In Section 5, we devise a local optima preserving reduction strongly inspired from the LOP-reduction of [8] and, under this new reduction we prove the existence of natural complete problems for a natural subclass of **DGLO** (the differential counterpart of **GLO**).

The definitions of the **NPO** problems mentioned and/or discussed in this paper can be found in [9]. Also, results are given without detailed proofs which can be found in [10].

2 Differential NPO-completeness

We study in this section **NPO**-completeness with respect to differential approximation. Based upon the generic strict-reduction of [1], we define a particular strict-reduction, called D-reduction, which we use in the sequel for proving **NPO**-completeness.

Definition 1. *A D-reduction is a strict-reduction dealing with differential ratio. Two optimization problems Π and Π' are* D-equivalent *if Π D-reduces to Π' and Π' D-reduces to Π.*

Theorem 1. MAX WSAT *and* MIN WSAT *are D-equivalent.*

As usually, ([11, 1]), we denote by **Max NPO** and **Min NPO**, the classes of maximization and minimization **NPO** problems, respectively.

Theorem 2. MAX WSAT *is **Max NPO**-complete and* MIN WSAT *is **Min NPO**-complete under $\leq_{\mathsf{D}}$. **Max NPO**-hard and **Min NPO**-hard (under $\leq_{\mathsf{D}}$) coincide and form the class of **NPO**-hard problems.*

In a completely analogous way, one can prove the D-equivalence of MIN $\{0,1\}$ INTEGER PROGRAMMING and MAX $\{0,1\}$ INTEGER PROGRAMMING. In other words MIN $\{0,1\}$ INTEGER PROGRAMMING *and* MAX $\{0,1\}$ INTEGER PROGRAMMING *are* ***NPO**-complete, under* $\leq_{\mathsf{D}}$.

We note here that, the result of [1] about the **Min NPO**-completeness of MIN TSP (Theorem 3.3) can be erroneously seen as in "glaring contradiction" to a result of [12, 13] where it is proved that MIN TSP on graphs with polynomially bounded edge-distances is in **DAPX**. In fact, there is no contradiction at all. Solution *triv* for MIN TSP adopted in [1], is considered as a tour containing exclusively edges of maximum distance. But such a solution is not always feasible for any instance of MIN TSP (the worst-value solution for this problem is an optimal solution of MAX TSP); hence the strict reduction of Theorem 3.3 in [1] is not a D-one.

We now introduce an approximation class, called **0-DAPX** in what follows, that seems very natural for differential approximation while has no sense in the standard case.

Definition 2. ***0-DAPX*** *is the class of* ***NPO*** *problems Π for which approximation within any differential approximation ratio $\delta > 0$ would entail $\boldsymbol{P} = \boldsymbol{NP}$. A problem Π is said to be* ***0-DAPX****-hard, if approximation of Π within any strictly positive differential approximation ratio would imply approximation of any other* ***0-DAPX*** *problem within strictly positive approximation ratios.*

Remark that inclusion in **0-DAPX** is rather a negative than a positive approximation result. This seems quite natural since 0-approximability represents the worst intractability level for an **NPO** problem in the differential approach.

In [14] it is proved that if $\mathbf{P} \neq \mathbf{NP}$, then, for any decreasing $\delta : \mathbb{N} \to (0, 1)$, MIN INDEPENDENT DOMINATING SET is not differential δ-approximable in polynomial time. By analogous reductions, it is proved in [15] that for any $k > 3$, polynomially bounded MAX Wk-SAT-B as well as the general minimization and maximization versions of integer-linear programming are in **0-DAPX**.

Theorem 3. *Under $\leq_{\mathsf{D}}$,* ***NPO****-complete =* ***0-DAPX****-complete $\subseteq$* ***0-DAPX****.*

A natural question rising from the above is: *what is the relation between* ***NPO****-complete and* **0-DAPX***?* Taking into consideration the fact that **0-DAPX** is the hardest differential approximability class in **NPO**, one might guess that **NPO**-complete $\equiv$ **0-DAPX**, but in order to prove it we need a stronger reducibility. We show in [10] that defining a special a kind of Turing-reduction, one can prove that **NPO**-complete = **0-DAPX**-complete = **0-DAPX**.

3 Differential APX-completeness

Let us now address the problem of completeness in the class **DAPX**. Note first that a careful reading of the proof of the standard **APX**-completeness of MAX WSAT-B given in [2] establishes also the following proposition which will be used in what follows.

Proposition 1. *Let $\Pi \in$ **APX**. There exist 3 polynomially computable functions f, g and c_ρ $:]0,1[\cap\mathbb{Q} \to]0,1[\cap\mathbb{Q}$ such that $\forall x \in \mathcal{I}_\Pi$, $\forall z \in \mathrm{sol}_\Pi(x)$, $\forall \rho \in]0,1[$: (1) $f(x,z,\rho) = (\phi_{x,z,\rho}, W_{x,z,\rho}, w_{x,z,\rho})$ with $(\phi_{x,z,\rho}, w_{x,z,\rho}) \in \mathcal{I}_{\text{MAX WSAT}}$; (2) $\forall y \in \mathrm{sol}_{\text{MAX WSAT}}(f(x,z,\rho))$, $g(x,z,\rho,y) \in \mathrm{sol}_\Pi(x)$; (3) if $\gamma_\Pi(x,z) \geqslant \rho$, then $f(x,z,\rho)$ is an instance of MAX WSAT-B and, for any solution y of $f(x,z,\rho)$, if $\gamma_{\text{MAX WSAT-}B}(f(x,z,\rho),y) \geqslant 1 - c_\rho(\epsilon)$, then $\gamma_\Pi(x, g(x,z,\rho,y)) \geqslant 1 - \epsilon$.*

We now define a notion of polynomial time differential approximation schemata preserving reducibility, called DPTAS-reduction in what follows.

Definition 3. *Let $\Pi, \Pi' \in$ **NPO**. Then, $\Pi \leq_{\mathsf{DPTAS}} \Pi'$ if there exist two functions f, g and a function $c :]0,1[\cap\mathbb{Q} \to]0,1[\cap\mathbb{Q}$, all computable in polynomial time, such that: (i) $\forall x \in \mathcal{I}_\Pi$, $\forall \epsilon \in]0,1[\cap\mathbb{Q}$, $f(x,\epsilon) \in \mathcal{I}_{\Pi'}$;* f is possibly multi-valued*; (ii) $\forall x \in \mathcal{I}_\Pi$, $\forall \epsilon \in]0,1[\cap\mathbb{Q}$, $\forall y \in \mathrm{sol}_{\Pi'}(f(x,\epsilon))$, $g(x,y,\epsilon) \in \mathrm{sol}_\Pi(x)$; (iii) $\forall x \in \mathcal{I}_\Pi$, $\forall \epsilon \in]0,1[\cap\mathbb{Q}$, $\forall y \in \mathrm{sol}_{\Pi'}(f(x,\epsilon))$, $\delta_{\Pi'}(f(x,\epsilon),y) \geqslant 1 - c(\epsilon) \Rightarrow \delta_\Pi(x, g(x,y,\epsilon)) \geqslant 1 - \epsilon$; if f is multi-valued, i.e., $f = (f_1, \ldots, f_i)$, for some i polynomial in $|x|$, then, the former implication becomes: $\forall x \in \mathcal{I}_\Pi$, $\forall \epsilon \in]0,1[\cap\mathbb{Q}$, $\forall y \in \mathrm{sol}_{\Pi'}((f_1,\ldots,f_i)(x,\epsilon))$, $\exists j \leqslant i$ such that $\delta_{\Pi'}(f_j(x,\epsilon),y) \geqslant 1 - c(\epsilon) \Rightarrow \delta_\Pi(x, g(x,y,\epsilon)) \geqslant 1 - \epsilon$.*

It is easy to see that given two **NPO** problems Π and Π', if $\Pi \leq_{\mathsf{DPTAS}} \Pi'$ and $\Pi' \in$ **DAPX**, then $\Pi \in$ **DAPX**.

Let $\Pi \in$ **DAPX** and let $\mathtt{T}$ be a differential ρ-approximation algorithm for Π, with $\rho \in]0,1[$. There exists a polynomial p such that $\forall x \in \mathcal{I}_\Pi$, $|\omega(x) - \mathrm{opt}(x)| \leqslant 2^{p(|x|)}$. An instance $x \in \mathcal{I}_\Pi$ can be written in terms of an integer linear program as: $x : \mathrm{opt}\, v(y)$ subject to $y \in C_x$, where C_x is the constraint-set of x. For any $i \in \{0, \ldots, p(|x|)\}$ and for any $l \in \mathbb{N}$, we define $x_{i,l}$ by: $x_{i,l} : \max[v_{i,l}(y) = \lfloor v(y)/2^i \rfloor - l]$ subject to $y \in C_x$, if Π is a maximization problem, or $x_{i,l} : \min[v_{i,l}(y) = l - \lfloor v(y)/2^i \rfloor]$ subject to $y \in C_x$, if Π is a minimization problem. Any $x_{i,l}$ can be considered as an instance of an **NPO** problem denoted by $\Pi_{i,l}$. Then, the following proposition holds.

Proposition 2. *Let $\epsilon < \min\{\rho, 1/2\}$, $x \in \mathcal{I}_\Pi$ and $(i,l) \in \{1, \ldots, p(|x|)\} \times \mathbb{N}$ be such that $2^i \leqslant \epsilon|\mathrm{opt}(x) - \omega(x)| \leqslant 2^{i+1}$ and set $l = \lfloor \omega(x)/2^i \rfloor$. Then, for any $y \in \mathrm{sol}_\Pi(x) = \mathrm{sol}_{\Pi_{i,l}}(x_{i,l})$: (1) $\delta_{\Pi_{i,l}}(x_{i,l}, y) \geqslant (1-\epsilon) \Longrightarrow \delta_\Pi(x,y) \geqslant 1 - 3\epsilon$; (2) $\delta_\Pi(x,y) \geqslant \rho \Longrightarrow \delta_{\Pi_{i,l}}(x_{i,l}, y) \geqslant (\rho - \epsilon)/(1+\epsilon)$.*

The proof of the existence of a **DAPX**-complete problem is performed along the following schema. We first prove that any **DAPX** problem Π is reducible to MAX WSAT-B by a reduction transforming a **PTAS** for MAX WSAT-B into a **DPTAS** for Π; we denote it by $\leq_{\mathsf{S}}^{\mathsf{D}}$. Next, we consider a particular **APX**-complete problem Π', say MAX INDEPENDENT SET-B; MAX WSAT-B that is in **APX** is PTAS-reducible to MAX INDEPENDENT SET-B. MAX INDEPENDENT SET-B is both in **APX** and in **DAPX** and, moreover, standard and differential approximation ratios coincide for it; this coincidence draws a trivial reduction called ID-reduction; it trivially transforms a differential polynomial time approximation schema into a standard polynomial time approximation schema. In other words, we prove that

$$\Pi \leq_{\mathsf{S}}^{\mathsf{D}} \text{MAX WSAT-}B \leq_{\mathsf{PTAS}} \text{MAX INDEPENDENT SET-}B$$
$$\leq_{\mathsf{ID}} \text{MAX INDEPENDENT SET-}B$$

The composition of the three reductions, i.e., the one from Π to MAX WSAT-B, the one from MAX WSAT-B to MAX INDEPENDENT SET-B and the ID-reduction, is a DPTAS reduction transforming a differential polynomial time approximation schema for MAX INDEPENDENT SET-B into a differential polynomial time approximation schema for Π, i.e., MAX INDEPENDENT SET-$B \in$ **DAPX**-complete.

Theorem 4. MAX INDEPENDENT SET-B *is* ***DAPX****-complete.*

Proof. We sketch here the part $\forall \Pi \in \mathbf{DAPX}$, $\Pi \leq_{\mathsf{S}}^{\mathsf{D}}$ MAX WSAT-B (we assume integer valued problems; extension to the case of rational values is immediate).

Remark that given a formula ϕ, a variable-weight system $\boldsymbol{w}$ and a constant B, one can decide in polynomial time if $(\phi, B, \boldsymbol{w}) \in \mathcal{I}_{\text{MAX WSAT-}B}$. Since Π is in **DAPX**, let $\mathtt{T}$ be a polynomial algorithm that guarantees differential ratio $\rho \in]0,1[$. Let $\epsilon < \min\{\rho, 1/2\}$.

For any $\zeta > 0$, we denote by $\mathtt{O}_\zeta$ an oracle that, for any instance x of MAX WSAT-B, computes a feasible solution $\mathtt{O}_\zeta(x) \in \text{sol}_{\text{MAX WSAT-}B}$ guaranteeing $\gamma_{\text{MAX WSAT-}B}(x, \mathtt{O}_\zeta) \geqslant 1 - \zeta$. We construct an algorithm $\mathtt{A}$ (this is the component of $\leq_{\mathsf{S}}^{\mathsf{D}}$ transforming solutions for MAX WSAT-B into solutions for Π) using this oracle such that: $\mathtt{A}$ guarantees differential approximation ratio $1 - \epsilon$ for Π and, in the case where $\mathtt{O}_\zeta$ is polynomial (in other words, $\mathtt{O}_\zeta$ can be seen as a polynomial time approximation schema), $\mathtt{A}$ is also polynomial.

The $\leq_{\mathsf{S}}^{\mathsf{D}}$-reduction claimed is based upon the construction of a family $\mathcal{F}$ of instances $x_{i,l}$: $\mathcal{F} = \{x_{i,l} : (i,l) \in F\}$, where F is of polynomial size and contains a pair (i_o, l_o) such that: either $i_0 \neq 0$, $2^{i_0} \leqslant \epsilon |\operatorname{opt}(x) - \omega(x)| \leqslant 2^{i_0+1}$ and $l_0 = \lfloor \omega(x)/2^{i_0} \rfloor$, or $i_0 = 0$, $\epsilon|\operatorname{opt}(x) - \omega(x)| \leqslant 2$ and $l_0 = \omega(x)$.

For instance x_{i_0,l_0} the worst value is 0; henceforth standard and differential ratios coincide. In other words, $\delta_{\Pi_{i_0,l_0}}(x_{i_0,l_0}, z) = \gamma_{\Pi_{i_0,l_0}}(x_{i_0,l_0}, z)$, for all feasible z. Moreover, for $i_0 = 0$, $\delta_\Pi(x,z) = \delta_{\Pi_{0,\omega(x)}}(x_{0,\omega(x)}, z) = \gamma_{\Pi_{0,\omega(x)}}(x_{0,\omega(x)}, z)$. We first suppose that F can be constructed in polynomial time. For each $(i,l) \in F$, we consider the three functions $g_{i,l}$, $f_{i,l}$ and $c_{i,l}$ (Proposition 1) for the instance $x_{i,l}$. We set $\epsilon' = \min\{(c_{i,l})_\rho(\epsilon), (c_{i,l})_{(\rho-\epsilon)/(1+\epsilon)}(\epsilon/3) : (i,l) \in F\}$ and define, for $(i,l) \in F$, $\eta = \rho$ if $i = 0$; otherwise, $\eta = (\rho - \epsilon)/(1+\epsilon)$. Let $z = \mathtt{T}(x)$; then, for any $(i,l) \in F$, we set $z_{i,l} = g_{i,l}(x_{i,l}, z, \eta, \mathtt{O}_{\epsilon'}(f_{i,l}(x_{i,l}, z, \eta)))$, if $f_{i,l}(x_{i,l}, z, \eta)$ is an instance of MAX WSAT-B; otherwise we set $z_{i,l} = z$. Remark that $z_{i,l}$ is a feasible solution for $x_{i,l}$ and, consequently, for x. In all, *$\mathtt{A}$ constructs $z_{i,l}$ for each $(i,l) \in F$ and selects the best among them as solution for x.*

Next, we prove that $\mathtt{A}$ achieves differential approximation ratio $1 - \epsilon$. Using Propositions 1 and 2, we can show that $\delta_\Pi(x, z_{i_0,l_0}) \geqslant 1 - \epsilon$. Since $(i_0, l_0) \in F$, $\mathtt{A}$ has already computed the solution z_{i_0,l_0}. By taking into account that the solution finally returned by $\mathtt{A}$ is the best among the computed ones, we immediately conclude that it is at least as good as z_{i_0,l_0}. Therefore, it guarantees ratio $1 - \epsilon$. Finally, we prove that F can be constructed in polynomial time. Steps sketched just above show that $\forall \Pi \in \mathbf{DAPX}$, $\Pi \leq_{\mathsf{S}}^{\mathsf{D}}$ MAX WSAT-B.

Theorem 5. MIN VERTEX COVER-B, MAX SET PACKING-B, MIN SET COVER-B, *are* ***DAPX****-complete under DPTAS-reductions. Furthermore,* MAX INDEPENDENT SET, MIN VERTEX COVER, MAX SET PACKING, MIN SET COVER, MAX CLIQUE *and* MAX ℓ-COLORABLE INDUCED SUBGRAPH, *are* ***DAPX****-hard under DPTAS-reductions.*

4 Differential PTAS-hardness

In this section, we will take into consideration the class **DPTAS** and we will address the problem of completeness in such class.

Consider the following reduction preserving fully polynomial time differential approximation schemata, denoted by DFPTAS-reduction in what follows.

Definition 4. *Assume two* ***NPO*** *problems Π and Π'. Then, $\Pi \leq_{\mathsf{DFPTAS}} \Pi'$, if there exist three functions f, g and c such that: (i) f and g are as for PTAS-reduction (Section 1; (ii) $c : (]0,1[\cap\mathbb{Q}) \times \mathcal{I}_\Pi \to]0,1[\cap\mathbb{Q}$; its time complexity and its value are polynomial in both $|x|$ and $1/\epsilon$; (iii) $\forall x \in \mathcal{I}_\Pi$, $\forall \epsilon \in]0,1[\cap\mathbb{Q}$, $\forall y \in \mathrm{sol}_{\Pi'}(f(x,\epsilon))$, $\delta_{\Pi'}(f(x,\epsilon),y) \geqslant 1 - c(\epsilon,x) \Rightarrow \delta_\Pi(x,g(x,y,\epsilon)) \geqslant 1-\epsilon$.*

Obviously, given two **NPO** problems Π and Π', if $\Pi \leq_{\mathsf{DFPTAS}} \Pi'$ and $\Pi' \in$ **DPTAS**, then $\Pi \in$ **DPTAS**.

In the following we study completeness not for the whole class **DPTAS** but for a subclass $\mathbf{DPTAS_p}$ mainly consisting of the maximization problems of **PTAS** the worst-value of which is computable in polynomial time (this class includes, in particular, maximization problems with worst value 0). Recall that, the first problem proved **PTAS**-complete (under FPTAS reduction) is MAX LINEAR WSAT-B ([2]).

Consider two problems $\Pi \in \mathbf{DPTAS_p}$ and Π', instances of which $x \in \mathcal{I}_\Pi$ and $x' \in \mathcal{I}_{\Pi'}$, respectively, are expressed, in terms of an integer linear programs as: $x : \mathrm{opt}\, v(y)$ subject to $y \in C_x$, $x' : \mathrm{opt}\, v(y') - \omega(x)$ subject to: $y' \in C_{x'}$ and $C_x \equiv C_{x'}$.

Obviously, $\delta_\Pi(x,y) = \delta_{\Pi'}(x',y') = \gamma_{\Pi'}(x',y')$ and, moreover, Π and Π' belong to $\mathbf{DPTAS_p}$; also, $\Pi' \in$ **PTAS** and $\Pi' \leq_{\mathsf{FPTAS}}$ MAX LINEAR WSAT-B. So, for any $\Pi \in \mathbf{DPTAS_p}$, $\Pi \equiv_{\mathsf{D}} \Pi' \leq_{\mathsf{FPTAS}}$ MAX LINEAR WSAT-B; reduction $\equiv_{\mathsf{D}} \circ \leq_{\mathsf{FPTAS}}$ is a DFPTAS-reduction.

Consider now the closure $\overline{\mathbf{DPTAS_p}}^{\mathbf{AF}}$ of $\mathbf{DPTAS_p}$ under affine transformations of objective functions of its problems. MIN VERTEX COVER in planar graphs is in $\overline{\mathbf{DPTAS_p}}^{\mathbf{AF}} \setminus \mathbf{DPTAS_p}$.

Let any $\Pi'' \in \overline{\mathbf{DPTAS_p}}^{\mathbf{AF}}$ and Π its "affine mate" in $\mathbf{DPTAS_p}$. Then, $\Pi'' \leq_{\mathsf{AF}} \Pi \equiv_{\mathsf{D}} \Pi' \leq_{\mathsf{FPTAS}}$ MAX LINEAR WSAT-B and since, obviously, the reduction $\leq_{\mathsf{AF}} \circ \equiv_{\mathsf{D}} \circ \leq_{\mathsf{FPTAS}}$ is a DFPTAS-one, the following proposition holds.

Proposition 3. MAX LINEAR WSAT-B *is* $\overline{\mathbf{DPTAS_p}}^{\mathbf{AF}}$*-hard, under* $\leq_{\mathsf{DFPTAS}}$.

5 MAX-SNP and differential GLO

In the theory of approximability of optimization problems based upon the standard approximation ratio interesting results have been obtained by studying the behavior of *local search* heuristics and the degree of approximation that such heuristics can achieve. In particular, in [8, 16], the class **GLO** is defined as the class of **NPO-PB** problems whose local optima have a guaranteed quality with respect to the global optima.

Of course, the differential counterpart of **GLO**, called **DGLO** in what follows, can be defined analogously. In [17] it is shown that MAX CUT, MIN DOMINATING SET-B, MAX INDEPENDENT SET-B, MIN VERTEX COVER-B, MAX SET PACKING-B, MIN COLORING, MIN SET COVER-B MIN SET W(K)COVER-B, MIN FEEDBACK EDGE SET, MIN FEEDBACK VERTEX SET-B and MIN MULTIPROCESSOR SCHEDULING, are included in **DGLO**. Furthermore in [18] it is proved that both MIN and MAX TSP on graphs with polynomially bounded edge-distances are also included in **DGLO**.

Let us now consider the relationship of **DGLO** with respect to the differential approximability class **DAPX**. Let $\overline{\mathbf{DGLO}}^{\mathsf{DPTAS}}$ be the closure of **DGLO** under $\leq_{\mathsf{DPTAS}}$. Analogously $\overline{\mathbf{GLO}}^{\mathsf{PTAS}}$ is defined in [16] where it is also proved that $\overline{\mathbf{GLO}}^{\mathsf{PTAS}} = \mathbf{APX}$. It is easy to show that the same holds for differential approximation.

Proposition 4. $\boldsymbol{DAPX} = \overline{\boldsymbol{DGLO}}^{\mathsf{DPTAS}}$.

Among other interesting properties of the class **GLO**, in [8] it is proved that MAX 3-SAT is complete in $\mathbf{GLO} \cap \mathbf{MAX\text{-}SNP}$ with respect to LOP-reduction. A related result in [19] shows that $\mathbf{MAX\text{-}SNP} \subseteq$ **Non-Oblivious GLO**, a variant of the class **GLO** defined by means of local search algorithms that are allowed to use more general kinds of objective functions, rather than the natural objective function of the given problem, for improving the quality of the solution.

In what follows, we show the existence of complete problems for a large, natural subclass of **DGLO**. As one can see from the definition of LOP-reduction in Section 1, the local optimality preserving properties do not depend on the approximation measure adopted. Hence, in an analogous way, we define here a reduction called DLOP which is a DPTAS-one with the same local optimality preserving properties as the ones of a LOP-reduction (Section 1).

Definition 5. *A DLOP-reduction is a DPTAS-reduction with the same* surjectivity, partial monotonicity, locality *and* dominance *properties as an LOP-reduction.*

Obviously, given two **NPO** problems Π and Π', if $\Pi \leq_{\mathsf{DLOP}} \Pi'$ and $\Pi' \in$ **DGLO**, then $\Pi \in$ **DGLO**.

Let $\mathbf{DGLO_0}$ be the class of **MAX-SNP** maximization problems that belong to **DGLO** and for which the worst value 0 is feasible for any instance (MAX INDEPENDENT SET-B, for example, is such a problem). Note that for the problems of $\mathbf{DGLO_0}$, the standard and differential approximation ratios coincide. Now let us consider the closure of $\mathbf{DGLO_0}$ under affine transformations. This leads to the following definition.

Definition 6. *Let Π be a polynomially bounded* ***NPO*** *problem. Then, $\Pi \in$ $\mathbf{DGLO'}$ if (i) it belongs to $\mathbf{DGLO_0}$, or (ii) it can be transformed into a problem in $\mathbf{DGLO_0}$ by means of an affine transformation; in other words, $\mathbf{DGLO'} = \overline{\mathbf{DGLO_0}}^{\mathbf{AF}}$.*

Theorem 6. *$\forall \Pi \in \mathbf{DGLO'}$, $\Pi \leq_{\mathsf{DLOP}}$ MAX INDEPENDENT SET-B.*

Proof. Assume $\Pi \in \mathbf{DGLO'}$. We then have the following two cases: (i) $\Pi \in \mathbf{DGLO_0}$ or (ii) Π can be transformed into a problem in $\mathbf{DGLO_0}$ by means of an affine transformation.

Dealing with case (i), note that for $\mathbf{DGLO_0}$, an LOP-reduction is also a DLOP-one and that the L-reduction of any $\pi \in \mathbf{GLO}$ (hence in $\mathbf{DGLO_0}$) is an LOP-reduction ([8]). We can show that both L-reductions in [3] from MAX 3-SAT to MAX 3-SAT-B and from MAX 3-SAT-B to MAX INDEPENDENT SET-B are also LOP-ones. So, the result follows.

Dealing with case (ii), since an affine transformation is a DLOP-reduction, $\Pi \leq_{\mathsf{DLOP}} \Pi'$ and by case (i), $\Pi' \leq_{\mathsf{DLOP}}$ MAX INDEPENDENT SET-B.

Proposition 5. *MAX CUT, MIN VERTEX COVER-B, MAX SET PACKING-B, MIN SET COVER-B are* ***DGLO'****-complete, under DLOP-reductions.*

Note that MIN MULTIPROCESSOR SCHEDULING, or even MIN and MAX TSP on graphs with polynomially bounded edge-distances belong to **DGLO** ([17, 18]) but neither to **GLO**, nor to $\mathbf{DGLO'}$. On the other hand, MIN VERTEX COVER-B belongs to $\mathbf{DGLO'}$ but not to **MAX-SNP**.

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