Approximation with a fixed number of solutions of some biobjective maximization problems*

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Abstract. We investigate the problem of approximating the Pareto set of biobjective optimization problems with a given number of solutions. This task is relevant for two reasons: (*i*) Pareto sets are often computationally hard so approximation is a necessary tradeoff to allow polynomial time algorithms; (*ii*) limiting explicitly the size of the approximation allows the decision maker to control the expected accuracy of approximation and prevents him to be overwhelmed with too many alternatives. Our purpose is to exploit general properties that many well studied problems satisfy. We derive existence and constructive approximation results for the biobjective versions of Max Bisection, Max Partition, Max Set Splitting and Max Matching.

1 Introduction

In multiobjective combinatorial optimization a solution is evaluated considering several objective functions and a major challenge in this context is to generate the set of efficient solutions or the Pareto set (see [8] about multiobjective combinatorial optimization). However, it is usually difficult to identify the efficient set mainly due to the fact that the number of efficient solutions can be exponential in the size of the input and moreover the associated decision problem is NP-complete even if the underlying single-objective problem can be solved in polynomial time. To handle these two difficulties, researchers have been interested in developing approximation algorithms with an a priori provable guarantee such as polynomial time constant approximation algorithms. Considering that all objectives have to be maximized, and for a positive $\rho \leq 1$, a ρ-approximation of Pareto set is a set of solutions that includes, for each efficient solution, a solution that is at least at a factor ρ on all objective values. Intuitively, the larger the size of the approximation set, the more accurate it can be.

It has been pointed out by Papadimitriou and Yannakakis [18] that, under certain general assumptions, there always exists a $(1-\varepsilon)$-approximation, with

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any given accuracy $\varepsilon > 0$, whose size is polynomial both in the size of the instance and in $1/\varepsilon$ but exponential in the number of criteria. In this result, the accuracy $\varepsilon > 0$ is given explicitly but the size of the approximation set is not given explicitly. When the number of solutions in the approximation set is limited, not every level of accuracy is possible. So, once the number of solutions is fixed in the approximation set of a multiobjective problem, the following questions are raised: What is the accuracy for which an approximation is guarantee to exist? Which accuracy can be obtained in polynomial time?

In this paper we are interested in establishing for biobjective maximization problems the best approximation ratio of the set of efficient solutions when the size of the approximation set is given explicitly. We give two approaches that deal with biobjective problems that allow us to obtain approximations of the set of efficient solutions with one or several solutions. More precisely, in a first approach, we consider a general maximization problem (denoted by Π_1 in the following) and establish a sufficient condition that guarantees the construction of a constant approximation of the Pareto set with an explicitly given number of solutions. As a corollary, we can construct a $(1-\varepsilon)$-approximation of the Pareto set with $O(\frac{1}{\varepsilon})$ solutions. In a second approach, we establish a necessary and sufficient condition for the construction of a constant approximation of the Pareto set with one solution.

Properties defined in these two approaches apply to several problems previously studied in single-objective approximation. Then we derive polynomial time constant approximations with one solution for Biobjective Max Bisection, Biobjective Max Partition, Biobjective Max Cut, Biobjective Max Set Splitting, Biobjective Max Matching. Some instances show that the given biobjective appromixation ratios are the best we can expect. In addition Biobjective Max Partition, Biobjective Max Cut, Biobjective Max Set Splitting admit a $(1-\varepsilon)$-approximation of the Pareto set with $O(\frac{1}{\varepsilon})$ solutions.

Several results exist in the literature on the approximation of multiobjective combinatorial optimization problems. One can mention the existence of fully polynomial time approximation schemes for biobjective shortest path [12, 22, 21], knapsack [9, 5], minimum spanning tree [18], scheduling problems [4], randomized fully polynomial time approximation scheme for matching [18], and polynomial time constant approximation for max cut [2], a biobjective scheduling problem [20] and the traveling salesman problem [3, 16]. Note that [2] and [20] are approximations with a single solution.

This article is organized as follows. In Section 2, we introduce basic concepts about multiobjective optimization and approximation. Section 3 is devoted to an approach for approximating some biobjective problems with one or several solutions. Section 4 presents a necessary and sufficient condition for approximating within a constant factor some biobjective problems with one solution. Conclusions are provided in a final section. Due to space limitation, some proofs are omitted.

2 Preliminaries on multi-objective optimization and approximation

Consider an instance of a multi-objective optimization problem with k criteria or objectives where X denotes the finite set of feasible solutions. Each solution $x \in X$ is represented in the objective space by its corresponding objective vector $w(x) = (w_1(x), \ldots, w_k(x))$. We assume that each objective has to be maximized.

From these k objectives, the dominance relation defined on X states that a feasible solution x dominates a feasible solution x' if and only if $w_i(x) \geq w_i(x')$ for $i = 1, \ldots, k$ with at least one strict inequality. A solution x is *efficient* if and only if there is no other feasible solution $x' \in X$ such that x' dominates x, and its corresponding objective vector is said to be *non-dominated*. Usually, we are interested in finding a solution corresponding to each non-dominated objective vector, set that is called Pareto set.

For any $0 < \rho \leq 1$, a solution x is called a ρ-approximation of a solution x' if $w_i(x) \geq \rho \cdot w_i(x')$ for $i = 1, \ldots, k$. A set of feasible solutions X' is called a ρ-approximation of a set of efficient solutions if, for every feasible solution $x \in X$, X' contains a feasible solution x' that is a ρ-approximation of x. If such a set exists, we say that the multi-objective problem admits a ρ-approximate Pareto set with $|X'|$ solutions.

An algorithm that outputs a ρ-approximation of a set of efficient solutions in polynomial time in the size of the input is called a ρ-approximation algorithm. In this case we say that the multi-objective problem admits a polynomial time ρ-approximate Pareto set.

Consider in the following a single-objective maximization problem P defined on a ground set $\mathcal{U}$. Every element $e \in \mathcal{U}$ has a non negative weight $w(e)$. The goal is to find a feasible solution (subset of $\mathcal{U}$) with maximum weight. The weight of a solution S must satisfy the following scaling hypothesis: if $opt(I)$ denotes the optimum value of I, then $opt(I') = t \cdot opt(I)$, where I' is the same instance as I except that $w'(e) = t \cdot w(e)$. For example, the hypothesis holds when the weight of S is defined as the sum of its elements' weights, or $\min w(e) : e \in S$, etc.

In the biobjective version, called biobjective P, every element $e \in \mathcal{U}$ has two non negative weights $w_1(e), w_2(e)$ and the goal is to find a Pareto set within the set of feasible solutions. Given an instance I of biobjective P, we denote by $opt_i(I)$ (or simply opt_i) the optimum value of I restricted to objective i, $i = 1, 2$. Here, the objective function on objective 1 is not necessarily the same as on objective 2, but both satisfy the scaling hypothesis.

3 Approximation with a given number of solutions

Papadimitriou and Yannakakis [18] proved the existence of at least one $(1-\varepsilon)$-approximation of size polynomial in the size of the instance and $\frac{1}{\varepsilon}$. In this result, the accuracy $\varepsilon > 0$ is given explicitly but the size of the approximation set is not given explicitly. In this section we consider a general maximization problem Π_1 and establish a sufficient condition that guarantees the construction of a constant approximation of the Pareto set with an explicitly given number of solutions for Π_1. This result allows to construct a $(1-\varepsilon)$-approximation of the Pareto set with $O(\frac{1}{\varepsilon})$ solutions but not necessarily in polynomial time. Moreover, if the single objective problem is polynomial time constant approximable and the sufficient condition is strengthened then the biobjective version is also polynomial time constant approximable with one solution. Thus we obtain constant approximations and polynomial time constant approximations with one solution for Biobjective Max Partition, Biobjective Max Cut, Biobjective Max Set Splitting, Biobjective Max Matching.

In the following, we are interested in particular cases of biobjective maximization problems, Biobjective Π_1, which satisfy the following property.

Property 1. Given any two feasible solutions S_1 and S_2, and any real α satisfying $0 < \alpha \leq 1$, if $w_2(S_1) < \alpha w_2(S_2)$ and $w_1(S_2) < \alpha w_1(S_1)$ then there exists a feasible solution S_3 which satisfies $w_1(S_3) > (1-\alpha) w_1(S_1)$ and $w_2(S_3) > (1-\alpha) w_2(S_2)$.

We say that *Biobjective Π_1 satisfies polynomially Property 1* if S_3 can be constructed in polynomial time.

Property 1 means that if S_1 is not an α-approximation of S_2 and S_2 is not an α-approximation of S_1 for both objective functions w_1 and w_2, then there exists a feasible solution S_3 which simultaneously approximates S_1 and S_2 with performance guarantee $1-\alpha$.

Given a positive integer ℓ, consider the equations $x^{2\ell} = 1 - x^{\ell}$ and $x^{2\ell-1} = 1 - x^{\ell}$. Denote by α_ℓ and β_ℓ their respective solutions in the interval $[0, 1)$. Remark that $\alpha_\ell = \left(\frac{\sqrt{5}-1}{2}\right)^{1/\ell}$ and $\alpha_\ell < \beta_{\ell+1} < \alpha_{\ell+1}, \ell \geq 1$.

Theorem 1. *If Biobjective Π_1 satisfies Property 1, then it admits a β_ℓ-approximate Pareto set (resp. an α_ℓ-approximate Pareto set) containing at most p solutions, where p is a positive odd integer such that $p = 2\ell - 1$ (resp. a positive even integer such that $p = 2\ell$).*

Proof. Let S_1 (resp. S_2) be a solution optimal for the first objective (resp. second one). In the following, *opt* denotes the optimal value on the first objective and also on the second objective. This can be assumed without loss of generality because a simple rescaling can make the optimal values coincide (e.g. we can always assume that $opt_2 \neq 0$, thus by multiplying each weight $w_2(e)$ by $\frac{opt_1}{opt_2}$ we are done). Then $w_1(S_1) = w_2(S_2) = opt$. If p is odd then $\rho = \beta_\ell$ with $p = 2\ell - 1$, otherwise $\rho = \alpha_\ell$ with $p = 2\ell$. Subdivide the bidimensionnal value space with coordinates $\{0\} \cup \{\rho^i opt : 0 \leq i \leq p\}$. See Figure 1 for an illustration.

Given $i, 1 \leq i \leq p$, the *strip* $s(i, .)$ is the part of the space containing all couples (w_1, w_2) satisfying $\rho^i opt < w_1 \leq \rho^{i-1} opt$ and $0 \leq w_2 \leq opt$. The strip $s(p+1, .)$ is the part of the space containing all couples (w_1, w_2) satisfying $0 \leq w_1 \leq \rho^p opt$ and $0 \leq w_2 \leq opt$. Given $j, 1 \leq j \leq p$, the *strip* $s(., j)$ is the part of the space containing all couples (w_1, w_2) satisfying $\rho^j opt < w_2 \leq \rho^{j-1} opt$ and

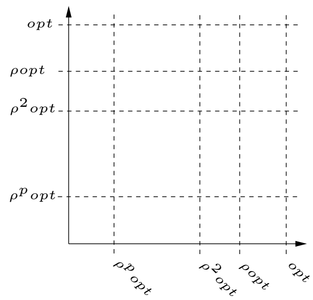

Fig. 1. Illustration of Theorem 1

$0 \leq w_1 \leq opt$. The strip $s(.,p+1)$ is the part of the space containing all couples (w_1, w_2) satisfying $0 \leq w_2 \leq \rho^p opt$ and $0 \leq w_1 \leq opt$.

Suppose that $w_2(S_1) < \rho^p opt$ and $w_1(S_2) < \rho^p opt$. In other words $S_1 \in s(1,.) \cap s(.,p+1)$ and $S_2 \in s(.,1) \cap s(p+1,.)$. Using Property 1 there exists a solution S_3 satisfying $w_1(S_3) > (1-\rho^p)opt$ and $w_2(S_3) > (1-\rho^p)opt$. For the case $\rho = \beta_\ell$ and $p = 2\ell - 1$, we get that $1 - \rho^p = 1 - \beta_\ell^{2\ell-1} = \beta_\ell^\ell = \rho^\ell$. For the case $\rho = \alpha_\ell$ and $p = 2\ell$, we get that $1 - \rho^p = 1 - \alpha_\ell^{2\ell} = \alpha_\ell^\ell = \rho^\ell$. Then S_3 is a ρ-approximation of any solution S satisfying $\max\{w_1(S), w_2(S)\} \leq \rho^{\ell-1} opt$.

One can construct a ρ-approximate Pareto set P as follows: $P = \{S_3\}$ at the beginning and for $j = \ell - 1$ down to 1, pick a feasible solution s with maximum weight w_1 in $s(.,j)$ (if $s(.,j)$ contains at least one value of a feasible solution) and set $P = P \cup \{S\}$. Afterwards, for $i = \ell - 1$ down to 1, pick a feasible solution S with maximum weight w_2 in $s(i,.)$ (if $s(i,.)$ contains at least one value of a feasible solution) and set $P = P \cup \{S\}$. For every strip the algorithm selects a solution which ρ-approximates (on both objective functions) any other solution in the strip. Since the solutions of P approximate the whole bidimensionnal space, P is a ρ-approximate Pareto set containing at most $2\ell - 1$ solutions. Here $2\ell - 1$ is equal to p when p is odd, otherwise it is equal to $p - 1$.

Now suppose that $w_2(S_1) \geq \rho^p opt$ (the case $w_1(S_2) \geq \rho^p opt$ is treated similarly). Solution S_1 must be in $s(.,j^*)$ for $1 \leq j^* \leq p$. Since $w_1(S_1) = opt$, S_1 is a ρ-approximation of any solution S in $s(.,p) \cup s(.,p+1)$. One can build an ρ-approximate Pareto set P as follows: $P = \{S_1\}$ at the beginning and for $j = j^* - 1$ down to 1, pick a feasible solution S with maximum weight w_1 in $s(.,j)$ (if $s(.,j)$ contains at least one value of a feasible solution) and set $P = P \cup \{S\}$. Since the strips form a partition of the space, the algorithm returns an ρ-approximate Pareto set containing at most p solutions. □

Corollary 1. *If Biobjective Π_1 satisfies Property 1, then it admits a $(1-\varepsilon)$-approximate Pareto set containing $O(\frac{1}{\varepsilon})$ solutions.*

Property 1 can be relaxed in the following way:

Property 2. We are given two feasible solutions S_1 and S_2, and a real α satisfying $0 < \alpha \leq 1$. If $w_2(S_1) < \alpha w_2(S_2)$ and $w_1(S_2) < \alpha w_1(S_1)$ then there exists a feasible solution S_3 which satisfies $w_1(S_3) > (c-\alpha)w_1(S_1)$ and $w_2(S_3) > (c-\alpha)w_2(S_2)$, where $0 < c \leq 1$.

We define similarly that Biobjective Π_1 satisfies polynomially Property 2.

Given a positive integer ℓ, consider the equations $x^{2\ell} = c - x^\ell$ and $x^{2\ell-1} = c - x^\ell$. Denote by γ_ℓ and δ_ℓ their respective solutions in the interval $[0,1)$. Remark that $\gamma_\ell = (\frac{\sqrt{1+4c}-1}{2})^{1/\ell}$ and $\gamma_\ell < \delta_\ell < \gamma_{\ell+1}, \ell \geq 1$.

Theorem 2. *If Biobjective Π_1 satisfies Property 2, then it admits a δ_ℓ-approximate Pareto set (resp. an γ_ℓ-approximate Pareto set) containing at most p solutions, where p is a positive odd integer such that $p = 2\ell - 1$ (resp. a positive even integer such that $p = 2\ell$).*

Proof. The proof is similar with the proof of Theorem 1. Suppose that $w_2(S_1) < \rho^p opt$ and $w_1(S_2) < \rho^p opt$. Using Property 2 there exists a solution S_3 satisfying $w_1(S_3) > (c-\rho^p)opt$ and $w_2(S_3) > (c-\rho^p)opt$. For the case $\rho = \delta_\ell$ and $p = 2\ell-1$, we get that $c - \rho^p = c - \delta_\ell^{2\ell-1} = \delta_\ell^\ell = \rho^\ell$. For the case $\rho = \gamma_\ell$ and $p = 2\ell$, we get that $c - \rho^p = c - \gamma_\ell^{2\ell} = \gamma_\ell^\ell = \rho^\ell$. Then S_3 is a ρ-approximation of any solution S satisfying $\max\{w_1(S), w_2(S)\} \leq \rho^{\ell-1} opt$. □

The previous results of this section consider the construction, not necessarily in polynomial time, of an approximate Pareto set with a fixed number of solutions. We give in the following some conditions on the construction in polynomial time of an approximate Pareto set with one solution.

Proposition 1. *If Π_1 is polynomial time ρ-approximable and Biobjective Π_1 satisfies polynomially Property 1 (resp. 2), then Biobjective Π_1 is polynomial time $\frac{\rho}{2}$-approximable (resp. $\frac{c\rho}{2}$-approximable) with one solution.*

Proof. Let S_1 (resp. S_2) be a polynomial time ρ-approximation solution for the first objective (resp. second one). In the following, opt_1 (resp. opt_2) denotes the optimal value on the first objective (resp. second one). If $w_2(S_1) \geq \frac{w_2(S_2)}{2}$ then $w_2(S_1) \geq \frac{\rho}{2} opt_2$ and thus S_1 is a $\frac{\rho}{2}$-approximate Pareto set. If $w_1(S_2) \geq \frac{w_1(S_1)}{2}$ then $w_1(S_2) \geq \frac{\rho}{2} opt_1$ and thus S_2 is a $\frac{\rho}{2}$-approximate Pareto set. Otherwise, $w_2(S_1) < \frac{w_2(S_2)}{2}$ and $w_1(S_2) < \frac{w_1(S_1)}{2}$ and since Biobjective Π_2 satisfies polynomially Property 1, we can construct in polynomial time a feasible solution S_3 which satisfies $w_1(S_3) \geq \frac{w_1(S_1)}{2}$ and $w_2(S_3) \geq \frac{w_2(S_2)}{2}$, that is a $\frac{\rho}{2}$-approximate Pareto set. □

We consider in Sections 3.1, 3.2, and 3.3 several examples of problems Π_1 that satisfy the scaling hypothesis and such that Biobjective Π_1 satisfy Property 1 or Property 2.

3.1 Max Pos NAE

The Max Pos NAE problem consists of a set of clauses $\mathcal{C}$ defined on a set of boolean variables $x_1, \ldots, x_n$. The clauses are composed of two or more positive variables and they are endowed with a non negative weight. The Max Pos NAE problem consists of finding an assignment of the variables such that the total weight of the clauses that are satisfied is maximum, where a positive clause is satisfied by an assignment if it contains at least a true variable and at least a false variable. Max Pos NAE generalizes Max Cut and so it is NP-hard and 0.7499-approximable [24]. Max Pos NAE is also known under the name Max Set Splitting or Max Hypergraph Cut [24].

Lemma 1. *Biobjective* Max Pos NAE *satisfies polynomially Property 1.*

Proof. Let $\alpha \in (0,1]$ and S_1, S_2 two solutions of an instance of biobjective Max Pos NAE satisfying the inequalities: $w_2(S_1) < \alpha w_2(S_2)$ and $w_1(S_2) < \alpha w_1(S_1)$. Consider $S_3 = (S_1 \setminus S_2) \cup (S_2 \setminus S_1)$. Let $c(S)$ be the set of clauses satisfied by assigning variables from S to true and those from $\bar{S}$ to false. Clearly $c(S) = \{C_i = x_{i_1} \vee \ldots \vee x_{i_t} : \exists x_{i_j} \in S, \exists x_{i_\ell} \in \bar{S}\}$. In the following a clause C_i is identified by the set of variables that it contains $\{x_{i_1}, \ldots, x_{i_t}\}$. Then $c(S_1) \setminus c(S_2) = \{C : C \cap S_1 \neq \emptyset \text{ and } C \cap \bar{S}_1 \neq \emptyset\} \cap \{C : C \subseteq S_2 \text{ or } C \subseteq \bar{S}_2\}$. Let $C \in c(S_1) \setminus c(S_2)$. If $C \subseteq S_2$ then since $C \cap \bar{S}_1 \neq \emptyset$ we have $\emptyset \neq C \cap (S_2 \setminus S_1) \subseteq C \cap S_3$. Moreover $C \cap \bar{S}_3 \neq \emptyset$ since $C \cap S_1 \cap S_2 \neq \emptyset$. Thus $C \in c(S_3)$. If $C \subseteq \bar{S}_2$ then since $C \cap S_1 \neq \emptyset$ we have $\emptyset \neq C \cap (S_1 \setminus S_2) \subseteq C \cap S_3$. Moreover $C \cap \bar{S}_3 \neq \emptyset$ since $C \cap \bar{S}_3 \subseteq C \cap \bar{S}_1 \cap \bar{S}_2 \neq \emptyset$. Thus $c(S_1) \setminus c(S_2) \subseteq C(S_3)$. In the similar way we can prove $c(S_2) \setminus c(S_1) \subseteq C(S_3)$. Thus, $c(S_1) \Delta c(S_2) = \big(c(S_1) \setminus c(S_2)\big) \cup \big(c(S_2) \setminus c(S_1)\big)$ is contained in $c(S_3)$.

The inequality $w_2(S_1) < \alpha w_2(S_2)$ can be rewritten as follows:

$$\sum_{C \in c(S_1)} w_2(C) < \alpha \sum_{C \in c(S_2)} w_2(C)$$

$$\sum_{C \in c(S_1) \setminus c(S_2)} w_2(C) + (1-\alpha) \sum_{C \in c(S_1) \cap c(S_2)} w_2(C) < \alpha \sum_{C \in c(S_2) \setminus c(S_1)} w_2(C)$$

We can use it to get

$$w_2(S_3) \geq \sum_{C \in c(S_1) \setminus c(S_2)} w_2(C) + \sum_{C \in c(S_2) \setminus c(S_1)} w_2(C) =$$

$$= \sum_{C \in c(S_1) \setminus c(S_2)} w_2(C) + \alpha \sum_{C \in c(S_2) \setminus c(S_1)} w_2(C) + (1-\alpha) \sum_{C \in c(S_2) \setminus c(S_1)} w_2(C) >$$

$$> 2 \sum_{C \in c(S_1) \setminus c(S_2)} w_2(C) + (1-\alpha) \sum_{C \in c(S_1) \cap c(S_2)} w_2(C) + (1-\alpha) \sum_{C \in c(S_2) \setminus c(S_1)} w_2(C) \geq$$

$$\geq (1-\alpha) \sum_{C \in c(S_2)} w_2(C) = (1-\alpha) w_2(S_2).$$

Using the same technique we can show that $w_1(S_3) > (1-\alpha) w_1(S_1)$. □

Corollary 2. *Biobjective* MAX POS NAE *admits a*

(i) β_ℓ*-approximate Pareto set (resp. an* α_ℓ*-approximate Pareto set) containing at most* p *solutions, where* $p = 2\ell - 1$ *(resp.* $p = 2\ell$*).*

(ii) $(1-\varepsilon)$*-approximate Pareto set containing* $O(\frac{1}{\varepsilon})$ *solutions.*

As indicated above, Corollary 2 deals with the possibility to reach some approximation bounds when the number of solutions in the Pareto set is fixed. We give in the following an approximation bound that we can obtain in polynomial time with one solution.

Corollary 3. *Biobjective* MAX POS NAE *admits a polynomial time* 0.374*-approximate Pareto set with one solution.*

Proof. The results follows from Lemma 1 and Proposition 1 with $\rho = 0.7499$. □

We consider in the following a particular case of MAX POS NAE in which every clause contains exactly k variables, denoted MAX POS kNAE. MAX POS 3NAE is 0.908-approximable [25]. For $k \geq 4$, MAX POS kNAE is $(1 - 2^{1-k})$-approximable [1, 14] and this is the best possible since it is hard to approximate within a factor of $1 - 2^{1-k} + \varepsilon$, for any constant $\varepsilon > 0$ [13].

Corollary 4. *Biobjective* MAX POS 3NAE *admits a polynomial time* 0.454*-approximate Pareto set with one solution. For* $k \geq 4$*,* MAX POS kNAE *admits a polynomial time* $1/2 - 2^{-k}$*-approximate Pareto set with one solution.*

Proof. The results follows from Lemma 1 and Proposition 1 with $\rho = 0.908$ and $\rho = 1 - 2^{1-k}$. □

We consider in the following another particular case of MAX POS NAE in which every clause contains exactly 2 variables, that is exactly MAX CUT which is 0.878-approximable [10].

Corollary 5. *Biobjective* MAX CUT *admits a*

(i) β_ℓ*-approximate Pareto set (resp. an* α_ℓ*-approximate Pareto set) containing at most* p *solutions, where* $p = 2\ell - 1$ *(resp.* $p = 2\ell$*).*

(ii) $(1-\varepsilon)$*-approximate Pareto set containing* $O(\frac{1}{\varepsilon})$ *solutions.*

Corollary 6. *Biobjective* MAX CUT *admits a polynomial time* 0.439*-approximate Pareto set with one solution.*

Proof. The results follows from Lemma 1 and Proposition 1 with $\rho = 0.878$ [10]. □

Clearly this last result is the same as the one given in [2] but we use a different method. We remark that Biobjective MAX CUT is not $(1/2 + \varepsilon)$-approximable with one solution [2], meaning that we are close to the best possible approximation result.

3.2 Max Partition

The Max Partition problem is defined as follows: given a set J of n items $1,\ldots,n$, each item j of positive weight $w(j)$, find a solution S that is a bipartition $J_1 \cup J_2$ of the n items such that $w(S) = \min\{\sum_{j\in J_1} w(j), \sum_{j\in J_2} w(j)\}$ is maximized. This NP-hard problem was also studied in the context of scheduling, where the number of partitions is not fixed, and consists of maximizing the earliest machine completion time [23].

Lemma 2. *Biobjective* Max Partition *satisfies polynomially Property 1.*

Corollary 7. *Biobjective* Max Partition *admits a*

(i) β_ℓ-approximate Pareto set (resp. an α_ℓ-approximate Pareto set) containing at most p solutions, where $p = 2\ell - 1$ (resp. $p = 2\ell$).

(ii) $(1-\varepsilon)$-approximate Pareto set containing $O(\frac{1}{\varepsilon})$ solutions.

Corollary 8. *Biobjective* Max Partition *admits a polynomial time $(1/2-\varepsilon)$-approximate Pareto set with one solution, for every $\varepsilon > 0$.*

Proof. Max Partition is a particular case of the Max Subset Sum problem. An input of Max Subset Sum is formed by a set J of n items $1,\ldots,n$, each item j has a positive weight $w(j)$, and an integer t. The problem consists of finding a subset S of J whose sum $w(S)$ is bounded by t and maximum. Max Subset Sum has a fptas [6]. We can obtain a fptas for Max Partition using the previous fptas for $t = \sum_{i=1}^{n} w(i)/2$.

The results follows from Lemma 2 and Proposition 1 with $\rho = 1 - 2\varepsilon$. □

Observe that Biobjective Max Partition is not $(1/2+\varepsilon)$-approximable with one solution. In order to see this, consider 3 items of weights $w_1(1) = 2, w_2(1) = 1, w_1(2) = 1, w_2(2) = 2, w_1(1) = 1, w_2(3) = 1$. The two efficient solutions S_i, $i = 1,2$ consists of placing i in a part and the other items in the other part and have weights $w_1(S_1) = 2, w_2(S_1) = 1, w_1(S_2) = 1, w_2(S_2) = 2$. Any other solution is either dominated by one of these two or has weights equal to 1 on both criteria.

3.3 Max Matching

Given a complete graph $G = (V,E)$ with non negative weights on the edges, the Max Matching problem is to find a matching of the graph of total weight maximum. Max Matching is solvable in polynomial time [7]. We study in this part the biobjective Max Matching problem and consider instances where the graph is a collection of complete graphs inside which the weights satisfy the triangle inequality, since otherwise the biobjective Max Matching problem is not at all approximable with one solution. In order to see this, consider a complete graph on 3 vertices with weights $(1,0),(0,1),(0,0)$. The optimum value on each objective is 1. Nevertheless, any solution has value 0 on at least one objective. Clearly Property 1 is not satisfied in this case.

Biobjective Max Matching problem is NP-hard [19]. It remains NP-hard even on instances where the graph is a collection of complete graphs inside which the weights satisfy the triangle inequality.

Lemma 3. *Biobjective* Max Matching *satisfies polynomially Property 2 with* $c = 1/3$.

Corollary 9. *Biobjective* Max Matching *admits a* δ_ℓ*-approximate Pareto set (resp. an* γ_ℓ*-approximate Pareto set) containing at most* p *solutions, where* $p = 2\ell - 1$ *(resp.* $p = 2\ell$*).*

Corollary 10. *Biobjective* Max Matching *admits a polynomial time* $\frac{1}{6}$*-approximate Pareto set with one solution.*

Proof. It follows from Lemma 3 and Proposition 1 considering $\rho = 1$. □

4 Approximation with one solution

In this section, we establish a necessary and sufficient condition for constructing, not necessarily in polynomial time, a constant approximation with one solution of the Pareto set for biobjective maximization problems. Moreover, if the condition is strengthened and the single-objective problem is polynomial time constant approximable, then the biobjective version is polynomial time constant approximable with one solution. Thus, using this condition, we establish a polynomial time 0.174-approximation with one solution for Biobjective Max Bisection.

In the following, we are interested in particular cases of biobjective maximization problems, Biobjective Π_2 which satisfy the following property.

Property 3. We can construct three solutions S_1, S_2, S_3 such that S_i is a ρ_i-approximation for problem Π_2 on objective i, $i = 1, 2$, and S_3 is such that $w_1(S_2) + w_1(S_3) \geq \alpha \cdot w_1(S_1)$ and $w_2(S_1) + w_2(S_3) \geq \alpha \cdot w_2(S_2)$ for some $\alpha \leq 1$.

We say that *Biobjective* Π_2 *satisfies polynomially Property 3* if S_1, S_2, S_3 can be constructed in polynomial time.

The aim of solution S_3 in Property 3 is to compensate the potential inefficiency of S_i on criterion $3 - i$, $i = 1, 2$.

Theorem 3. *Biobjective* Π_2 *is (resp. polynomial time) constant approximable with one solution if and only if it satisfies (resp. polynomially) Property 3. More precisely, if Biobjective* Π_2 *satisfies polynomially Property 3 such that* S_i *is a polynomial time* ρ_i*-approximation for problem* Π_2 *on objective* i, $i = 1, 2$*, then Biobjective* Π_2 *admits a polynomial time* $\alpha\frac{\min\{\rho_1,\rho_2\}}{2}$*-approximation algorithm with one solution.*

Proof. Suppose that Biobjective Π_2 is ρ-approximable with one solution. Let S_3 be this solution and S_1 and S_2 any two solutions. Then $w_1(S_3) \geq \rho \cdot opt_1 \geq \rho \cdot w_1(S_1)$ and thus by setting $\alpha = \rho$ we have $w_1(S_2) + w_1(S_3) \geq \alpha \cdot w_1(S_1)$. The second inequality holds also.

Suppose now that Biobjective Π_2 satisfies Property 3. Since S_i is a ρ_i-approximation for problem Π_2 on objective i, $i = 1, 2$, we have $w_1(S_1) \geq \rho_1 \cdot opt_1$ and $w_2(S_2) \geq \rho_2 \cdot opt_2$.

Since Property 3 is satisfied, we can construct S_3 such that

$$w_1(S_2) + w_1(S_3) \geq \alpha \cdot w_1(S_1) \tag{1}$$

and

$$w_2(S_1) + w_2(S_3) \geq \alpha \cdot w_2(S_2) \tag{2}$$

Now, we study different cases:

- If $w_1(S_2) \geq \frac{\alpha}{2} w_1(S_1)$, then we deduce that S_2 is a good approximation of the Pareto set. From the hypothesis, we have $w_1(S_2) \geq \frac{\alpha}{2} w_1(S_1) \geq \alpha \cdot \frac{\min\{\rho_1,\rho_2\}}{2} opt_1$. On the other hand, we also have $w_2(S_2) \geq \rho_2 \cdot opt_2 \geq \alpha \frac{\min\{\rho_1,\rho_2\}}{2} opt_2$.
- If $w_2(S_1) \geq \frac{\alpha}{2} w_2(S_2)$, then we deduce that S_1 is a good approximation of the Pareto set. From the hypothesis, we have $w_2(S_1) \geq \frac{\alpha}{2} w_2(S_2) \geq \alpha \cdot \frac{\min\{\rho_1,\rho_2\}}{2} opt_2$. On the other hand, by the construction of S_1 we also have $w_1(S_1) \geq \rho_1 \cdot opt_1 \geq \alpha \cdot \frac{\min\{\rho_1,\rho_2\}}{2} opt_1$.
- If $w_1(S_2) \leq \frac{\alpha}{2} w_1(S_1)$ and $w_2(S_1) \leq \frac{\alpha}{2} w_2(S_2)$, then it is S_3 which is a good approximation of the Pareto set. Indeed, from inequality (1), we deduce $w_1(S_3) \geq \frac{\alpha}{2} w_1(S_1) \geq \alpha \cdot \frac{\min\{\rho_1,\rho_2\}}{2} opt_1$ and on the other hand, from inequality (2), we also get $w_2(S_3) \geq \frac{\alpha}{2} w_2(S_2) \geq \alpha \cdot \frac{\min\{\rho_1,\rho_2\}}{2} opt_2$.

In any of these three cases, we obtain a $\alpha \cdot \frac{\min\{\rho_1,\rho_2\}}{2}$-approximation with one solution.

Clearly, if S_1, S_2, S_3 are computable in polynomial time, then Biobjective Π_2 is approximable in polynomial time. □

Remark that we can extend Theorem 3 to the case where ρ_i are not constant.

The interest of Property 3 is to find a simple method in order to construct a polynomial time constant approximation for Biobjective Π_2. This method does not allow us to obtain the best polynomial time constant approximation for Biobjective Π_2 with one solution, but only to prove the fact that the problem is polynomial time constant approximable with one solution.

In Lemma 1 we prove that if a problem Π is (resp. polynomial time) constant approximable and if Biobjective Π satisfies (resp. polynomially) Property 1, then Biobjective Π is (resp. polynomial time) constant approximable with one solution, and thus Biobjective Π satisfies (resp. polynomially) Property 3 by Theorem 3. Thus all problems studied in Section 3 satisfies Property 3.

There exist problems which are polynomial time constant approximable and thus satisfy Property 3 and do not satisfy Property 1. One example is Biobjective

TSP, which is polynomial time $\frac{7}{27}$-approximable with one solution [16, 17] and does not satisfy Property 1.

Proposition 2. *Biobjective TSP does not satisfy Property 1.*

Proof. Consider the complete graph K_5 where a fixed K_4 is decomposable into 2 Hamiltonian paths P_1 and P_2. For every edge $e \in E(K_5)$, set $w_1(e) = 1$ and $w_2(e) = 0$ if $e \in P_1$, $w_1(e) = 0$ and $w_2(e) = 1$ if $e \in P_2$ and $w_1(e) = 0$ and $w_2(e) = 0$ if $e \notin P_1 \cup P_2$. We can check that there are four non-dominated tours T_i, $i = 1, \ldots, 4$ with $w_1(T_1) = 3$, $w_2(T_1) = 0$, $w_1(T_2) = 0$, $w_2(T_2) = 3$, $w_1(T_3) = 2$, $w_2(T_3) = 1$ and $w_1(T_4) = 1$, $w_2(T_4) = 2$. Consider $S_i = T_i$, $i = 1, 2$ and $\alpha = 1/2$. Clearly $w_2(S_1) < \alpha w_2(S_2)$ and $w_1(S_2) < \alpha w_1(S_1)$. Moreover there is no solution S_3 such that $w_1(S_3) > (1-\alpha)w_1(S_1)$ and $w_2(S_3) > (1-\alpha)w_2(S_2)$. □

We consider in the following a problem that satisfies Property 3 and for which we are not able to prove that it satisfies Property 1.

4.1 Max Bisection

Given a graph $G = (V, E)$ with non negative weights on the edges, the Max Bisection problem consists of finding a bipartition of the vertex set V into two sets of equal size such that the total weight of the cut is maximum. We establish in this part a polynomial time $\frac{\rho}{4}$-approximation algorithm for Biobjective Max Bisection where ρ is any polynomial time approximation ratio given for Max Bisection. Max Bisection is NP-hard [15] and the best approximation ratio known for Max Bisection is $\rho = 0.701$ [11].

Lemma 4. *Biobjective* Max Bisection *satisfies polynomially Property 3 with* $\alpha = 1$ *and* $\rho_1 = \rho$ *and* $\rho_2 = \frac{\rho}{2}$*, where* ρ *is any polynomial time approximation ratio given for* Max Bisection*.*

Corollary 11. *Biobjective* Max Bisection *admits a polynomial time* 0.174*-approximate Pareto set with one solution.*

Proof. The results follows from Theorem 3 and Lemma 4 and using the polynomial time 0.701-approximation algorithm for Max Bisection [11]. □

5 Conclusion

In this paper, we established some sufficient conditions that allow to conclude on the existence of constant approximations of the Pareto set with an explicitly given number of solutions for several biobjective maximization problems. The results we obtained establish a *polynomial time* approximation when we ask for a single solution in the approximation set. A possible future work would be to give a polynomial time approximation for any explicitly given number of solutions. A necessary and sufficient condition is given for the construction of (polynomial time) constant approximation with one solution for biobjective maximization problems. It would be interesting to generalize this result to maximization problems with more than two objectives. Another interesting future work would be to establish lower bounds for any explicitly given number of solutions for multi-objective maximization problems.

Our approaches deal with maximization problems and they do not seem to apply to minimization problems. A possible explanation is that, in the maximization framework, adding elements to a partial solution rarely deteriorates it. Minimization problems rarely satisfy this property. Establishing constant approximation of the Pareto set with a given number of solutions or show that this is not possible for minimization problems is an interesting open question.

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