Acyclic fuzzy preferences and the Orlovsky choice function: A note

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Abstract

This note corrects and extends a recent axiomatic characterization of the Orlovsky choice function for a particular class of transitive fuzzy relations. The interest of characterizing choice functions for "well-behaved" fuzzy relations is then discussed.

Keywords: Decision Making, Fuzzy Preferences, Choice functions.

I- Introduction

Following the classic work of Orlovsky [10], much attention has been devoted to the so-called "Orlovsky choice function" in the literature on "rational choice" based on fuzzy preferences. Barrett *et al.* [2] have shown that the Orlovsky choice function possesses many interesting desirable properties when applied to max-min transitive fuzzy relations. Recently Banerjee [1] proposed an axiomatic characterization of this choice function for a particular class of transitive fuzzy relations using a notion of transitivity distinct from max-min transitivity. This characterization proves incorrect however and a recent citation in [9] of Banerjee's alleged result motivated this note. After having recalled a few useful definitions and notations, we propose various characterizations of the Orlovsky choice function for several classes of "well-behaved" fuzzy preference relations. These characterizations heavily rest on classical results concerning crisp relations and we conclude with some remarks concerning the interest of characterizing choice functions for "well-behaved" fuzzy relations.

II- Notations and Definitions

Let X be a finite set of objects called "alternatives" with at least three elements. We denote by $\mathcal{P}$(X) the set of all nonempty subsets of X.

Definition 2.1. A *fuzzy* (binary) *relation* on X is a function R associating with each ordered pair of alternatives (x, y) $\in$ X^2 an element of [0, 1]. Without loss of generality for our purposes, we shall suppose throughout this note that all fuzzy relations are reflexive, *i.e.* such that R(x, x) = 1, for all x $\in$ X. We denote by $\mathcal{R}$(X) the set of all fuzzy reflexive relations on X. A fuzzy relation R is said to be *crisp* if R(x, y) $\in$ {0, 1}, for all x, y $\in$ X. When R is crisp, we often write x R y instead of R(x, y) = 1.

Definition 2.2. Let R be a crisp relation on X. We say that R is:

- reflexive if x R x,
- complete if x R y or y R x,
- asymmetric if x R y implies Not (y R x),
- transitive if x R y and y R z imply x R z.

for all x, y , z $\in$ X.

We denote by α(R) the asymmetric part of R, *i.e.* the crisp relation on X such that, for all x, y $\in\supseteq$X, [x α(R) y] $\Leftrightarrow$ [x R y and Not(y R x)]. We say that R is acyclic if, for all k $\geq$ 1 and all $x_1, x_2, ..., x_k \in$ X, [x_1 α(R) x_2, x_2 α(R) x_3, ..., x_{k-1} α(R) x_k] $\Rightarrow$ [Not x_k α(R) x_1].

Definition 2.3. A choice function (cf) C on X is a function from $\mathcal{P}$(X) to $\mathcal{P}$(X), such that C(A) $\subseteq$ A, for all A $\in$ $\mathcal{P}$(X), *i.e.*, a function associating with each nonempty subset A of X a nonempty choice set C(A) included in A which we interpret as the set of the chosen alternatives in A. The base relation R_C of a cf C is the crisp relation on X such that x R_C y $\Leftrightarrow$ x $\in$ C({x, y}), for all x, y $\in$ X. Observe that the base relation of a cf is always complete.

Definition 2.4. A cf C on X satisfies:

Condition α if [x $\in$ A $\subseteq$ B] $\Rightarrow$ [x $\in$ C(B) $\Rightarrow$ x $\in$ C(A)].

Condition $\alpha2$ if [x $\in$ C(A)] $\Rightarrow$ [x $\in$ C({x, y}) for all y $\in$ A].

Condition $\gamma2$ if [x $\in$ C({x, y}) for all y $\in$ A] $\Rightarrow$ [x $\in$ C(A)].

Condition γ if [B = $A_1 \cup A_2$, x $\in$ C(A_1), x $\in$ C(A_2)] $\Rightarrow$ [x $\in$ C(B)],

Condition β if [x, y $\in$ C(A) and A $\subseteq$ B] $\Rightarrow$ [x $\in$ C(B) $\Leftrightarrow$ y $\in$ C(B)],

for all A, B, A_1, A_2 $\in$ $\mathcal{P}$(X).

Definition 2.5. A cf C is said to be "crisp-rationalizable" if there is a crisp binary relation R on X such that, for all A $\in$ $\mathcal{P}$(X), C(A) = {a $\in$ A: a R b for all b $\in$ A}, in which case we say that R crisp-rationalizes C. It is easy to see that if a cf is crisp-rationalizable, it is crisp-rationalizable by a unique crisp relation which is necessarily complete and acyclic.

The following lemma recalls a number of well-known facts from the literature.

Lemma 2.6. For any cf C on a finite set X,

(a) $\alpha \Rightarrow \alpha2$,

(b) $\gamma \Rightarrow \gamma2$,

(c) [$\alpha2$ and $\gamma2$] $\Leftrightarrow$ [α and γ] $\Leftrightarrow$ [C is crisp-rationalizable] $\Leftrightarrow$ [R_C crisp-rationalizes C]

(d) [$\alpha2$ and β] $\Rightarrow \gamma$,

(e) [$\alpha2$ and β] $\Leftrightarrow$ [C is crisp-rationalizable by a transitive relation].

Proof. See, *e.g.*, [13].

Definition 2.7. An $\mathcal{R}$−preference-based choice function (pbcf) on X is a function $\mathcal{C}$: $\mathcal{P}$(X) x $\mathcal{R}$(X) $\rightarrow$ $\mathcal{P}$(X), such that $\mathcal{C}$(A, R) $\subseteq$ A, for all R $\in$ $\mathcal{R}$(X) and all A $\in$ $\mathcal{P}$(X), *i.e.*, a function associating with each nonempty subset A of X and each fuzzy relation R $\in$ $\mathcal{R}$(X) a nonempty choice set $\mathcal{C}$(A, R) included in A which we interpret as the set of the chosen alternatives in A given the relation R. Observe that any $\mathcal{R}$−pbcf $\mathcal{C}$ defines a cf C in an obvious way and properties of cf are easily extended to $\mathcal{R}$−pbcf.

Remark 2.8. In the preceding definition, when $\mathcal{R}$(X) is replaced by a subset $\mathcal{H}$ $\subseteq$ $\mathcal{R}$(X) of "admissible" fuzzy relations we speak of an $\mathcal{H}$−pbcf. Properties of $\mathcal{R}$−pbcf obviously extend to $\mathcal{H}$−pbcf.

Remark 2.9. Let $A \in \mathcal{P}(X)$ and $R \in \mathcal{R}(X)$. A simple way to obtain an $\mathcal{R}$–pbcf is to associate a score $W(x, A, R)$ with each alternative $x \in A$ based on the behavior of R on A and to include in the choice set $\mathcal{C}(A, R)$ the alternatives with the highest score, *i.e.*

$$\mathcal{C}(A, R) = \{a \in A : W(a, A, R) \geq W(b, A, R) \text{ for all } b \in A\}. \quad (1)$$

Definition 2.10. Let $R \in \mathcal{R}(X)$ and $A \in \mathcal{P}(X)$. We respectively define the *Orlovsky score* and the *Min Difference score* of an alternative $x \in A$ given the relation $R \in \mathcal{R}(X)$ by:

$$W_{OV}(x, A, R) = \min_{y \in A} \operatorname{Min} [1 - R(y, x) + R(x, y) ; 1]. \quad (2)$$

and

$$W_{mD}(x, A, R) = \min_{y \in A} [R(x, y) - R(y, x)]. \quad (3)$$

Equations (1) and (2) (resp. (1) and (3)) define the Orlovsky $\mathcal{R}$–pbcf $\mathcal{C}_{OV}$ (resp. the Min Difference $\mathcal{R}$–pbcf $\mathcal{C}_{mD}$).

Lemma 2.11. For all $R \in \mathcal{R}(X)$ and all $A \in \mathcal{P}(X)$, $\mathcal{C}_{OV}(A, R) = \mathcal{C}_{mD}(A, R)$.

Proof. See [2, proposition 3.10].

Definition 2.12. For all $R \in \mathcal{R}(X)$ and $A \in \mathcal{P}(X)$, the set of *pairwise dominant* alternatives in A given R is defined by:

$$PD(A, R) = \{x \in A : R(x, y) \geq R(y, x), \text{ for all } y \in A\}. \quad (4)$$

The set of pairwise dominant alternatives does not define a pbcf since this set may well be empty for some $R \in \mathcal{R}(X)$ and some $A \in \mathcal{P}(X)$.

Definition 2.13. Let $R \in \mathcal{R}(X)$. We say that R is:

– *max-min transitive* if $R(x, z) \geq \min[R(x, y) ; R(y, z)]$,

– *Banerjee-transitive* if $[R(x, y) \geq R(y, x) \text{ and } R(y, z) \geq R(z, y)] \Rightarrow [R(x, z) \geq R(z, x)]$,

– *Acyclic* if $[R(x_1, x_2) > R(x_2, x_1), R(x_2, x_3) > R(x_3, x_2), \ldots, R(x_{n-1}, x_n) > R(x_n, x_{n-1})] \Rightarrow [R(x_1, x_n) \geq R(x_n, x_1)]$,

for all $x, y, z, x_1, x_2, \ldots, x_n \in X$.

Remark 2.14. When R is a crisp relation, it is clear that the definition of acyclicity in definitions 2.2 and 2.13 coincide. Furthermore, for a crisp relation, max-min transitivity and Banerjee-transitivity are equivalent and coincide with the transitivity property of definition 2.2. The version of Banerjee-transitivity given here is slightly different from the one given in Banerjee's paper. They are nevertheless easily seen to be equivalent. It is well-known that a fuzzy relation R is max-min transitive if and only if all of its λ-cuts are transitive. Banerjee-transitivity and acyclicity focus, instead of λ-cuts, on the crisp relation R^* derived from a fuzzy relation R letting, for all $x, y \in X$,

$$x R^* y \Leftrightarrow R(x, y) \geq R(y, x). \quad (5)$$

It is easily seen that a fuzzy relation R is acyclic (resp. Banerjee-transitive) if and only if R^* is acyclic (resp. transitive). It should be noticed that R^* is always complete.

The following lemmas recall a number of simple and well-known facts from the literature.

Lemma 2.15. For all $R \in \mathcal{R}(X)$,

(a) R is max-min transitive $\Rightarrow$ R is acyclic,

(b) R is Banerjee-transitive $\Rightarrow$ R is acyclic.

Proof. Part (b) is obvious. For part (a) see, *e.g.* [8].

Lemma 2.16. For all R $\in \mathcal{R}$(X) and all A $\in \mathcal{P}$(X),

(a) [PD(A, R) $\neq \emptyset$] $\Rightarrow$ [PD(A, R) = $\mathcal{C}_{mD}$(A, R) = $\mathcal{C}_{OV}$(A, R)],

(b) [PD(B, R) $\neq \emptyset$, for all B $\in \mathcal{P}$(X)] $\Leftrightarrow$ [R is acyclic].

Proof. For part (a) see [2, proposition 3.10], for part (b) see [8, corollary 1].

Remark 2.17. Since (4) can be rewritten as:

PD(A, R) = {x $\in$ A : x R* y, for all y $\in$ A}, (6)

it is clear, given remark 2.14, that part (b) of lemma 2.16 is nothing but a reformulation of a classical result for crisp relations (see *e.g.* [12, 13]).

III- Results and Discussion

Definition 3.1. An $\mathcal{R}$–pbcf $\mathcal{C}$ is said to satisfy:

Pairwise Strict Dominance (PSD) if [R(x, y) > R(y, x)] $\Rightarrow$ [$\mathcal{C}$({x, y}, R) = {x}],

Pairwise Weak Dominance (PWD) if [R(x, y) $\geq$ R(y, x)] $\Rightarrow$ [x $\in \mathcal{C}$({x, y}, R)],

Reward for Pairwise Weak Dominance (RPWD) if, for all A $\in \mathcal{P}$(X) and all x $\in$ A, [R(x, y) $\geq$ R(y, x), for all y $\in$ A] $\Rightarrow$ x $\in \mathcal{C}$(A, R),

for all R $\in \mathcal{R}$(X), all x, y $\in$ X and all A, B $\in \mathcal{P}$(X).

Remark 3.2. PSD and PWD are fairly obvious conditions on the choice from a two-element set. In spite of names, it is clear that PWD and PSD are independent conditions. Together they imply that R* is identical to the base relation $R_{\mathcal{C}}$ of $\mathcal{C}$. Condition RPWD, introduced in [2], generalizes PWD for sets with more than two elements. It implies that PD(A, R) $\subseteq \mathcal{C}$(A, R), *i.e.* that the choice set contains all pairwise dominant alternatives.

Remark 3.3. Banerjee's [1] characterization of $\mathcal{C}_{OV}$ can be stated as follows: "Let $\mathcal{H}$ be a set of (reflexive) Banerjee-transitive fuzzy relations on X. An $\mathcal{H}$–pbcf satisfies conditions PSD, PWD and β *if and only if* it coincides with the Orlovsky pbcf". The fact that the Orlovsky pbcf satisfies conditions PSD and PWD is obvious. Verification of condition β follows from lemmas 2.6 and 2.16 together with (6) since Banerjee-transitivity implies that R* is transitive. The characterization result is incorrect however as shown by the following example. Let $\mathcal{H}$ be a set of Banerjee-transitive fuzzy relations and let $\mathcal{C}$ be the $\mathcal{H}$–pbcf such that:

$\mathcal{C}$(A, R) = $\mathcal{C}_{OV}$(A, R) if |A| = 2 and $\mathcal{C}$(A, R) = A otherwise.

It is clear that $\mathcal{C}$ satisfies PSD, PWD. Condition β is an "expansion" condition and, thus, is also obviously satisfied. The problem in Banerjee's proof is that it incorrectly assumes a "contraction" condition (his condition 5.5 defined here as condition α) to hold for any pbcf.

Lemma 3.4. Let $\mathcal{H} \subseteq \mathcal{R}$(X) and $\mathcal{C}$ be an $\mathcal{H}$–pbcf.We have:

[$\mathcal{C}$ satisfies PWD and γ2] $\Rightarrow$ [PD(A, R) $\subseteq \mathcal{C}$(A, R), for all R $\in \mathcal{H}$ and all A $\in \mathcal{P}$(X)].

Proof. By definition, $[x \in PD(A, R)] \Leftrightarrow [x \in PD(\{x, y\}, R)$ for all $y \in A]$. From PWD we know that $[x \in PD(\{x, y\}, R)] \Rightarrow [x \in \mathcal{C}(\{x, y\}, R)]$ and using $\gamma 2$ completes the proof.

Proposition 3.5. Let $\mathcal{H}$ be a nonempty set of acyclic fuzzy relations on X.

(a) The $\mathcal{H}$–pbcf $\mathcal{C}_{OV}$ is the smallest $\mathcal{H}$–pbcf (w.r.t. inclusion) satisfying PWD and $\gamma 2$.

(b) The $\mathcal{H}$–pbcf $\mathcal{C}_{OV}$ is the smallest $\mathcal{H}$–pbcf (w.r.t. inclusion) satisfying RPWD.

(c) An $\mathcal{H}$–pbcf $\mathcal{C}$ satisfies PSD, PWD, $\alpha 2$ and $\gamma 2$ if and only if it coincides with $\mathcal{C}_{OV}$.

Proof.

(a) Combine lemmas 2.16 and 3.4.

(b) Obvious combining lemma 2.16 and remark 3.2.

(c) By lemma 2.16, we have $\mathcal{C}_{OV}(A, R) = PD(A, R)$, for all $A \in \mathcal{P}(X)$ and all $R \in \mathcal{H}$. Thus, it is clear that $\mathcal{C}_{OV}$ satisfies PSD, PWD, $\alpha 2$ and $\gamma 2$. Let us show that any $\mathcal{H}$–pbcf satisfying PSD, PWD, $\alpha 2$ and $\gamma 2$ coincides with $\mathcal{C}_{OV}$. Given PWD and $\gamma 2$, we know from lemma 3.4 that $PD(A, R) \subseteq \mathcal{C}(A, R)$, for all $A \in \mathcal{P}(X)$ and all $R \in \mathcal{H}$. Suppose now that $x \in \mathcal{C}(A, R)$. If $x \notin PD(A, R)$ then $R(y, x) > R(x, y)$ for some $y \in A$ and, from PSD, it follows that $\mathcal{C}(\{x, y\}, R) = \{y\}$. Thus, $x \in \mathcal{C}(A, R)$ contradicts $\alpha 2$.

Remark 3.6. From lemma 2.15, we know that Banerjee-transitivity as well as max-min transitivity implies acyclicity. Thus part (c) of proposition 3.5 offers a characterization of $\mathcal{C}_{OV}$ in the set of Banerjee transitive relations or max-min transitive relations. Proposition 3.7 offers an alternative one which is closer from Banerjee's alleged result. Simple examples show that in proposition 3.7, it is not possible to replace "Banerjee-transitive" by "acyclic" or by "max-min transitive".

Proposition 3.7. Let $\mathcal{H}$ be a set of Banerjee-transitive fuzzy relations on X. An $\mathcal{H}$-pbcf PSD, PWD, $\alpha 2$ and β if and only if it coincides with $\mathcal{C}_{OV}$.

Proof.

Since Banerjee-transitivity implies acyclicity, we know from corollary 3.6 that $\mathcal{C}_{OV}$ satisfies PSD, PWD and $\alpha 2$. We already observed – see remark 3.3 – that $\mathcal{C}_{OV}$ satisfies β when applied to a Banerjee-transitive relation. The sufficiency part follows from proposition 3.5 and lemma 2.6.

Remark 3.8. From lemma 2.16 we know that $\mathcal{C}_{OV}$ coincides with the set PD of pairwise dominant alternatives when attention is restricted to acyclic relations and, thus, makes little use of the fuzziness of R. The characterization of $\mathcal{C}_{OV}$ in this particular case mainly boils down to rephrasing in terms of R well-known properties of the set of the greatest elements of a crisp relation (the crisp relation R* defined by (5)). This explains why $\mathcal{C}_{OV}$ behaves so nicely when applied to reflexive and max-min transitive – and, thus, acyclic – fuzzy relations, see [2].

Remark 3.9. In many cases, *e.g.* in Social Choice Theory or in Multiple Criteria Decision Making, preference are modelled using fuzzy relations that are not necessarily acyclic (see, *e.g.*, [11] or [4]). Even in this case, $\mathcal{C}_{OV}$ intuitively seems to be a very reasonable choice function. Obtaining a nice characterization of $\mathcal{C}_{OV}$ in the general case is an open and interesting problem.

Contrary to the situation with acyclic relation, its solution is likely to lead to "truly fuzzy" results. As already observed in [7] it should be noticed that $\mathcal{C}_{OV}$ satisfies γ2 whatever the properties of the fuzzy relation to which it is applied.

Remark 3.10. Let C be a cf on X. The problem of rationalizing C in terms of fuzzy preferences consists in finding a pbcf $\mathcal{C}$ and a fuzzy relation R ∈ $\mathcal{R}$(X) such that C(A) = $\mathcal{C}$(A, R), for all A ∈ $\mathcal{P}$(X) – on this problem, see, *e.g.* [3], [5] or [6]. Many pbcf can be envisaged to rationalize a "non preference-based" cf. The ones that have been proposed so far in the literature all satisfy condition PWD. Thus, proposition 3.5 (a) shows that if we insist on using acyclic relations and pbcf satisfying γ2, very few gains in terms of explanatory power will be obtained with the fuzzy preference framework compared to the crisp one. Since PWD seems rather undisputable, if one is to show that the fuzzy preference framework is more flexible than the crisp one in order to rationalize cf then either acyclicity – *e.g.* using a form of transitivity that is independent from acyclicity such as the so-called "sum minus one transitivity", see Jain's T_5 transitivity in [6], or no transitivity condition at all, see [3] – or γ2 must be dropped. Examples of pbcf violating γ2 even when coupled with max-min transitive relations can be found in [2].

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