The β-ranking and the β-measure for directed networks: axiomatic characterizations

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Abstract

The β-measure has been introduced and characterized by R. van den Brink and R. P. Gilles (Measuring domination in directed networks, *Social Networks*, 22:141–157, 2000) for measuring the domination in directed networks. The present paper characterizes the ranking induced by the β-measure. It also provides an alternative characterization of the β-measure.

1 Introduction

Among the many different types of centrality concepts defined for social networks, one finds domination and the corresponding relational power measures as introduced by van den Brink and Gilles [2000] (henceforth referenced as vdBG2000) for directed networks[1]. vdBG2000 characterize two relational power measures: the β-measure and the score measure. Their characterization of the β-measure helps us understand the key properties of the β-measure and to compare it with other relational power measures[2]; for instance with the outdegree (called score-measure by vdBG2000 or score function by Herings et al. [2005]).

A couple of years later, van den Brink and Gilles [2003] (henceforth referenced as vdBG2003) characterized the ranking induced by the score measure (they call this ranking the ranking by outdegree). So far, no characterization of the ranking induced by the β-measure (we will call it the β-ranking) has been published. There is therefore a gap in the literature because it is not possible to compare the β-ranking with the ranking by outdegree from an axiomatic perspective. The aim of the present paper is to fill this gap.

Since our main emphasis is on the characterization of the β-ranking, we do not expand further on the motivation for the concept of domination and for relational

[1] van den Brink et al. [2008] consider the case of undirected networks.

[2] Relational power measures are called power functions by Herings et al. [2005]

power measures; we refer the reader to vdBG2000 and, e.g., Herings et al. [2005] and Tabak et al. [2010].

In the next section, we present the notation and the main definitions. Section 3 presents the characterization of the β-ranking and compares it with the characterization of the ranking by outdegree in vdBG2003. In Section 4, we present a new characterization of the β-measure. This alternative characterization of the β-measure mainly uses the same axioms as the characterization of the β-ranking. This allows us to have a unified view of the ranking and the measure.

2 Notation and definitions

Unlike vdBG2003, we consider a set of nodes (agents) that is not fixed, i.e., the set X of nodes is a subset of a countably infinite universe Ω and X will be allowed to vary in our analysis. The motivation for allowing X to vary will be detailed in Section 3.5.

A directed network[3] N is a pair (X, D) where D is a binary relation on X, i.e., a subset of $X \times X$. Examples of such directed networks are networks of papers and citations or Twitter accounts and followers. We limit our analysis to finite irreflexive directed networks, that is, networks $N = (X, D)$ such that X is finite and $(a, a) \notin D$ for all $a \in X$. Irreflexivity is assumed because it is coherent with the concept of domination, but it could easily be dispensed with. The set of all logically possible finite irreflexive directed networks is denoted by $\mathcal{N}$.

For every $N = (X, D) \in \mathcal{N}$ and $a \in X$, we define the set of successors of a as $S_N(a) = \{b \in X : (a, b) \in D\}$ and the set of predecessors of a as $P_N(a) = \{b \in X : (b, a) \in D\}$. The cardinalities of these two sets are respectively called outdegree and indegree of node a and are denoted by $s_N(a)$ and $p_N(a)$.

A relational power measure f (in vdBG2000's terminology) is a function mapping every network $N = (X, D) \in \mathcal{N}$ to a vector in $\mathbb{R}^X$ such that $f_N(a)$ is the relational power of node a in network N. For instance, the outdegree s_N is the score-measure characterized in vdBG2000. A relational power ranking $\succsim$ is a function mapping every network $N = (X, D) \in \mathcal{N}$ to a weak order[4] on X denoted $\succsim_N$. For instance, the ranking by outdegree (vdBG2003) is defined by $a \succsim_N^s b$ iff $s_N(a) \geq s_N(b)$.

For every node a in network $N = (X, D)$, the β-measure is the relational power measure defined by

$$\beta_N(a) = \sum_{b \in S_N(a)} \frac{1}{p_N(b)},$$

[3]Also often called directed graph or digraph.

[4]A weak order on a set A is a complete ($a \succsim b$ or $b \succsim a$ for every a, b in A) and transitive ($a \succsim b$ and $b \succsim c$ imply $a \succsim c$ for every a, b, c in A) binary relation.

where, by convention, the sum is equal to zero whenever $S_N(a)$ is empty. It induces the β-ranking $\succsim^\beta$ defined by $a \succsim^\beta_N b$ iff $\beta_N(a) \geq \beta_N(b)$. This paper will characterize the β-ranking and the β-measure.

The intuition behind this measure is the following: if node a dominates node b (there is an arc from a to b), then we have an argument for increasing the measure of a (there is a corresponding term in the sum), but this argument is weaker if b is dominated by many other nodes: the strength of the argument is equal to $1/p_N(b)$. A similar idea (but a different operationalisation) underlies the Elo ranking for chess players: when a player defeats a strong opponent, he earns more points than when he defeats a weak one [Elo, 1978].

A much related idea can also be found in the literature on bibliometrics. The "fractional counting of citations" proposed by Leydesdorff and Opthof [2010a,b] and Glänzel et al. [2011] and axiomatized by Bouyssou and Marchant [2016] is indeed quite reminiscent of the β-measure with links having a dual interpretation: if a paper p is cited by paper q, this raises the index of p by a factor that is inversely proportional to the number of papers citing q. Another related index is the PageRank index [Page and Brin, 1998, Altman and Tennenholtz, 2005], of which the fractional counting of citations is in some sense a non-recursive version.

3 The β-ranking

3.1 Axioms

We present some conditions satisfied by the β-ranking. The first one imposes that the labelling of the nodes be immaterial. Before presenting it, we need a new piece of notation. For every network $N = (X, D) \in \mathcal{N}$ and every permutation π of X, we define D^π by $(\pi(a), \pi(b)) \in D^\pi \iff (a, b) \in D$ and we define $N^\pi = (X, D^\pi)$.

A 1 Anonymity. *For every permutation π of X and every $N = (X, D) \in \mathcal{N}$, we have, for all $a, b \in X$ $\pi(a) \succsim_{N^\pi} \pi(b) \iff a \succsim_N b$.*

This condition is identical to Anonymity in vdBG2003. It is clear that Anonymity is necessary for the β-ranking since the labels of the nodes do not play any role in the definition of the β-ranking; only the binary relation D matters.

We also need a monotonicity condition guaranteeing that adding an arc never hurts the origin.

A 2 Positive Responsiveness. *For every $N = (X, D)$ and $N' = (X, D')$ in $\mathcal{N}$ and every $a, b, c \in X$ with $a \neq b$, if $a \succsim_N b$, $(a, c) \notin D$ and $D' = D \cup \{(a, c)\}$, then $a \succ_{N'} b$.*

Notice that this is a strict monotonicity condition. Indeed, after the addition of the arc (a, c), if a was strictly above b, this strict preference is preserved and, moreover, if a was indifferent to b, this indifference is transformed into a strict preference.

This condition is exactly Positive Responsiveness, as defined by vdBG2003 for characterizing the outdegree.

Let us show the necessity of this condition. When we add an arc (a, c), we add a successor to $S_N(a)$ and, for all other successors of a, the number of predecessors does not vary. Hence $\beta_{N'}(a) = \beta_N(a) + 1/p_N(c)$ and, so, the β-measure of a increases. At the same time, when we add an arc (a, c), the number of successors of b does not vary and the number of predecessors of a successor of b can possibly increase by 1. Put differently, the β-measure of b remains unchanged or decreases. Consequently, if a was at least as good as b in the network N, then a is strictly better than b in N'.

Our next condition says that the addition of nodes that are not linked to any other node has no influence on the ranking. Before presenting it, we need a new definition. The restriction of a weak order $\succsim$ on the set X to a subset $X' \subset X$ is the weak order $\succsim'$ defined for all $a, b \in X'$ by $a \succsim' b$ iff $a \succsim b$.

A 3 Node Addition. *For every network $N = (X, D) \in \mathcal{N}$ and $N' = (X \cup \{a\}, D)$, the ranking $\succsim_{N'}$ restricted to X is equal to $\succsim_N$.*

This condition is not related (at least not in a simple way) to any of the conditions in vdBG2003. The Node Addition condition is satisfied by the β-ranking. Indeed, adding an isolated node (i.e., a node that is not linked to any other node) does not change the number of successors or predecessors of any other node. The β-measure of all other nodes therefore remains constant.

Our next condition formalizes the idea that some arcs are irrelevant for comparing a and b.

A 4 Independence of Irrelevant Arcs. *Let $N = (X, D)$ and $N = (X, D')$ be two networks in $\mathcal{N}$ such that*

- $(c, d) \notin D$, $D' = D \cup \{(c, d)\}$,
- $c \notin \{a, b\}$, $d \notin S_N(a) \cup S_N(b)$.

Then $a \succsim_{N'} b$ iff $a \succsim_N b$.

This condition is strictly weaker than Independence of Non-dominated Arcs in vdBG2003. From the definition of the β-measure, it is clear that an arc influences the β-measure of node a only if the origin of the arc is a or if the destination of the arc is a successor of a. Independence of Irrelevant Arcs is therefore satisfied by the β-ranking.

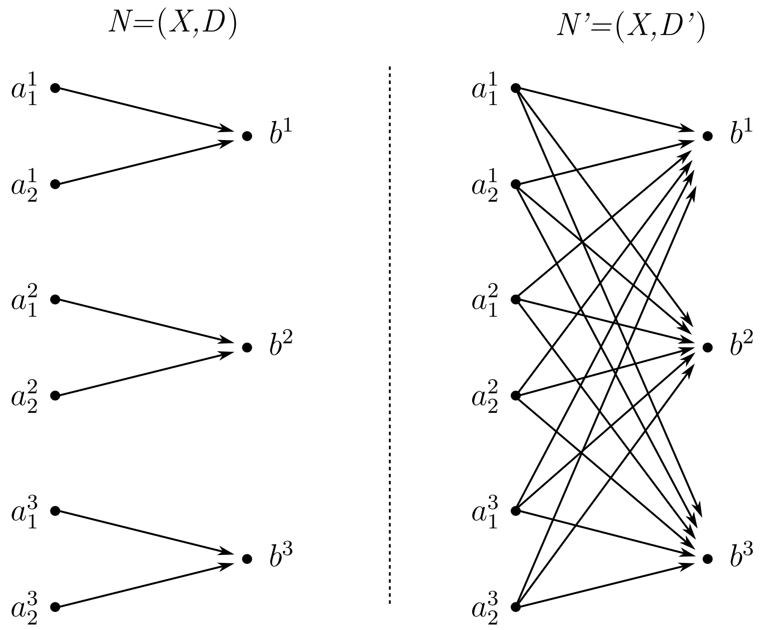

Figure 1: Independence of Local Density: left, the network $N = (X, D)$ and right, the network $N' = (X, D')$, with $l = 3$ and $n = 2$.

Consider two nodes a and b such that a dominates many nodes while b dominates few ones. At first sight, we may be tempted to conclude that a has more power than b. Suppose in addition that a is in a very dense region of the network, i.e., a region where nodes have many predecessors, while b is in a region with a low density. This may lead us to temper our previous conclusion. Our last condition is based on this idea. It says that increasing the number of successors of node a and simultaneously increasing their number of predecessors, in the same proportion, does not improve or worsen the position of a.

A 5 Independence of Local Density. *Consider two networks $N = (X, D)$ and $N' = (X, D')$ in $\mathcal{N}$ and $l, n \in \mathbb{N}$. Let $a_1^k, \ldots, a_n^k, b^k$ for $k \in \{1, \ldots, l\}$ be distinct nodes in X such that*

1. *$P_N(b^k) = \{a_1^k, \ldots, a_n^k\}$, for $k \in \{1, \ldots, l\}$,*
2. *$(a_i^k, b^{k'}) \notin D$, for $k \neq k' \in \{1, \ldots, l\}$ and $i \in \{1, \ldots, n\}$,*
3. *$D' = D \cup \{(a_i^k, b^{k'}) : k \neq k' \in \{1, \ldots, l\}, i \in \{1, \ldots, n\}\}$.*

Then $\succsim_N = \succsim_{N'}$.

This condition, illustrated in Figure 1, is not related to any of the conditions in vdBG2003. We establish the necessity of this condition. Consider any node $d \in X$. We must show that $\beta_N(d) = \beta_{N'}(d)$. We distinguish several cases.

- $d \notin \{a_1^k, \ldots, a_n^k, b^k\}$ for $k \in \{1, \ldots, l\}$. Then the number of terms in the sum defining the β-measure does not change when going from N to N', i.e., when adding arcs of the form $(a_i^k, b^{k'})$. The denominator of each term also remains the same because d is not the predecessor of any $b^{k'}$.

- $d = a_i^k$. Then the term $(1/n)$ corresponding to the arc (d, b^k) is divided by l because of all the new arcs of the form $(a_i^{k'}, b^k)$. The number of terms in the sum also changes: node d has $l - 1$ new successors, because of the arcs of the form $(d, b^{k'})$, and each corresponding term in the new sum is equal to $1/(n \times l)$. The β-measure of d therefore remains unchanged since $1/n = 1/nl + (l - 1)/nl$.

- $d = b^k$. There is clearly no new term in the sum. Since we have assumed in the statement of the condition that $a_1^k, \ldots, a_n^k, b^k$ for $k \in \{1, \ldots, l\}$ are distinct, we know that none of the new arcs $(a_i^k, b^{k'})$ has a successor of d as destination. Each term of the sum therefore remains unchanged.

Independence of Local Density is a kind of normalization condition[5], in the sense that it permits comparisons across networks with different densities or to compare nodes located in different parts of a single network, with different local densities. Another example of a normalized index is the density of a network (number of arcs divided by $n(n-1)/2$); it is normalized in the sense that it is a ratio, and it is thereby independent of the size of the network. Comparisons across networks of different size are thus possible. Notice that such a normalization has nothing to do with units of measurement[6]. Indeed, even if we change the measurement unit, i.e., we multiply the standard density index (resp. the β-measure) by 2, by 10 or by 100, the scaled up density index (resp. the scaled up β-measure) remains normalized and comparisons across networks remain possible.

3.2 Result

We are now ready to state our first result, characterizing the ranking induced by the β-measure.

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[5]This concept of "normalization" is often used in the literature on networks [e.g., Ruhnau, 2000, Koschützki et al., 2005].

[6]Contrary to the Dominance Normalization condition used in vdBG2000, as explained later.

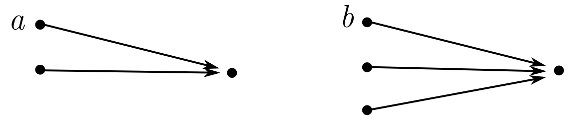

Figure 2: Before step 1. The nodes a, b, their successors and the predecessors of their successors in the network $N = (X, D)$.

Theorem 1 *A relational power ranking satisfies (i) Anonymity, (ii) Positive Responsiveness, (iii) Node Addition, (iv) Independence of Irrelevant Arcs, and (v) Independence of Local Density if and only if it is the β-ranking.*

Before proving this theorem, we state and prove a lemma.

Lemma 1 (Transfer) *Let $N = (X, D)$ and $N' = (X, D')$ be two networks in $\mathcal{N}$ and $n \in \mathbb{N}$ be such that*

1. $P_N(b) = \{a_1, \ldots, a_n\}$, $P_N(b') = \{a'_1, \ldots, a'_n\}$,
2. $P_{N'}(b) = \{a'_1, a_2, \ldots, a_n\}$, $P_{N'}(b') = \{a_1, a'_2, \ldots, a'_n\}$,
3. $(a_i, b') \notin D$ *and* $(a'_i, b) \notin D$ *for all* $i \in \{1, \ldots, n\}$,
4. $D \Delta D' = \{(a_1, b), (a'_1, b'), (a_1, b'), (a'_1, b)\}$.

If $\succsim$ satisfies Independence of Local Density, then $\succsim_N = \succsim_{N'}$.

Proof. Consider the network $N'' = (X, D'')$ such that

$$D'' = D \cup \{(a_i, b'), (a'_i, b) : i \in \{1, \ldots, n\}\}.$$

By Independence of Local Density, $\succsim_{N''} = \succsim_N$. By Independence of Local Density as well, $\succsim_{N''} = \succsim_{N'}$. Hence $\succsim_N = \succsim_{N'}$. □

Proof of Theorem 1. The necessity of our conditions has been shown above. We now prove the sufficiency. Choose any two nodes a, b in X. In a number of steps, we will transform the network N into N_1, N_2, etc. in such a way that (1) the β-measure of a and b will not change and (2) the way a and b compare to each other will also not change. At each step, the obtained network will be simpler. When we will reach N_4, it will be so simple, that, using Anonymity and Positive Responsiveness, it will be easy to know how a and b compare to each other. Figures 2–6 illustrate the proof with a simple example. The initial network $N = (X, D)$ is displayed in Figure 2.

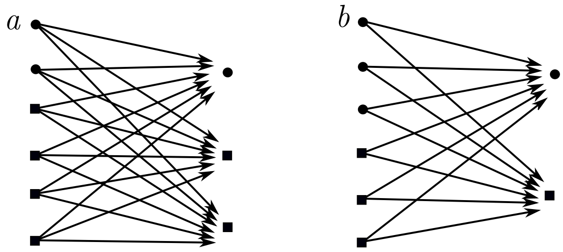

Figure 3: Step 1. The network $N_1 = (X_1, D_1)$. The nodes added during step 1 are represented by a square. The least common multiple m is 6.

Step 1. Let us construct $N_1 = (X_1, D_1)$ as follows. Let m be the least common multiple of $p_N(c)$ for all $c \in S_N(a) \cup S_N(b)$. For each successor c of a, we do the following (we will do the same with the successors of b). Let $n = m/p_N(c)$. We relabel a as d_1^1, c as c_1 and the other predecessors of c as $d_1^2, \ldots, d_1^{p_N(c)}$. We add $n-1$ new nodes $c_2, \ldots, c_n$ and $(n-1)p_N(c)$ new nodes d_i^j with $i \in \{2, \ldots, n\}, j \in \{1, \ldots, p_N(c)\}$. Thanks to Node Addition, the new nodes have no influence on the comparison between a and b. We then add $(n-1)p_N(c)$ new arcs from c_i to d_i^j for $i \in \{2, \ldots, n\}, j \in \{1, \ldots, p_N(c)\}$. Thanks to Independence of Irrelevant Arcs, the new arcs have no influence on the comparison between a and b. We then add an arc from each d_i^j to each $c_{i'}$ for all $i \neq i' \in \{1, \ldots n\}$ and all $j \in \{1, \ldots, p_N(c)\}$. Thanks to Independence of Local Density, $a \succsim_{N_1} b$ iff $a \succsim_N b$. As a result of this step (see Fig.3), all successors of a and b have the same indegree m. Notice that $\beta_{N_1}(a) = \beta_N(a)$ and $\beta_{N_1}(b) = \beta_N(b)$.

Step 2. Let us construct $N_2 = (X_2, D_2)$ as follows. For each successor c of a, we add $1 + m$ new nodes: $c', d_1, \ldots, d_m$.Thanks to Node Addition, the new nodes have no influence on the comparison between a and b. We add a new arc from each $d_1, \ldots, d_m$ to c'. Thanks to Independence of Irrelevant Arcs, the new arcs have no influence on the comparison between a and b. We add two new arcs $(a, c'), (d_1, c)$ and we remove the arcs $(a, c), (d_1, c')$.

Thanks to Transfer, $\succsim_{N_2} = \succsim_{N_1}$ and, therefore, $a \succsim_{N_2} b$ iff $a \succsim_{N_1} b$. Hence $a \succsim_{N_2} b$ iff $a \succsim_N b$. Notice that $\beta_{N_2}(a) = \beta_N(a)$ and $\beta_{N_2}(b) = \beta_N(b)$. As a result of this step (see Fig. 4), all successors of a have no successor; each predecessor (unless it is a) of a successor of a has no predecessor and has only one successor.

Step 3. This step is similar to the previous one; it handles the successors of b. Let us construct $N_3 = (X_3, D_3)$ as follows. For each successor c of b, we add $1 + m$ new nodes: $c', d_1, \ldots, d_m$.Thanks to Node Addition, the new nodes have no influence on the comparison between a and b. We add a new arc from each $d_1, \ldots, d_m$ to c'. Thanks to Independence of Irrelevant Arcs, the new arcs have no influence on the comparison between a and b. We add two new arcs $(b, c'), (d_1, c)$

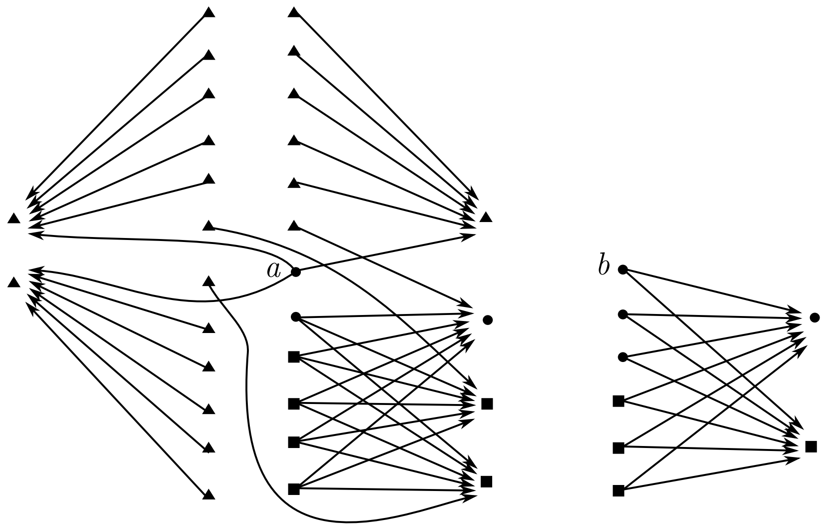

Figure 4: Step 2. The network $N_2 = (X_2, D_2)$. The nodes added during step 2 (3 groups of 7 nodes) are represented by a triangle.

and we remove the arcs $(b, c), (d_1, c')$. Thanks to Transfer, $a \succsim_{N_3} b$ iff $a \succsim_{N_2} b$. Hence $a \succsim_{N_3} b$ iff $a \succsim_N b$. Notice that $\beta_{N_3}(a) = \beta_N(a)$ and $\beta_{N_3}(b) = \beta_N(b)$. As a result of this step (see Fig. 5), all successors of a and b have no successor and they all have the same indegree; each predecessor (unless it is a or b) of a successor of a or b has no predecessor and has only one successor; a and b have no successor in common.

Step 4. Let $N_4 = (X_4, D_4)$ be such that $X_4 = X_3$ and $D_4 = D_3 \setminus \{(c, d) : d \notin S_{N_3}(a) \cup S_{N_3}(b)\}$. Thanks to Independence of Irrelevant Arcs, $a \succsim_{N_4} b$ iff $a \succsim_{N_3} b$. Hence $a \succsim_{N_4} b$ iff $a \succsim_N b$. Notice that $\beta_{N_4}(a) = \beta_N(a)$ and $\beta_{N_4}(b) = \beta_N(b)$. As a result of this step (see Fig. 6), all arcs have a successor of a or b as destination; only a and b can have an outdegree larger than 1; only successors of a or b have a positive indegree; a and b have no successor in common; all successors of a and b have no successor and they all have the same indegree.

We now consider three cases.

1. $\beta_N(a) = \beta_N(b)$. In this case, it is easy to see that a and b have the same outdegree in N_4. Because of the high symmetry of N_4, there is clearly a permutation π of X_4 such that $\pi(a) = b, \pi(b) = a$ and $N_4^\pi = N_4$ (there are actually many such permutations). Because of Anonymity, we have $b \succsim_{N_4^\pi} a \iff a \succsim_{N_4} b$. Since $N_4^\pi = N_4$, this implies $a \sim_{N_4} b$ and, hence, $a \sim_N b$.

2. $\beta_N(a) > \beta_N(b)$. In this case, the outdegree of a in N_4 is larger than that of b. By removing some arcs leaving a, we can construct a network N_5 (see

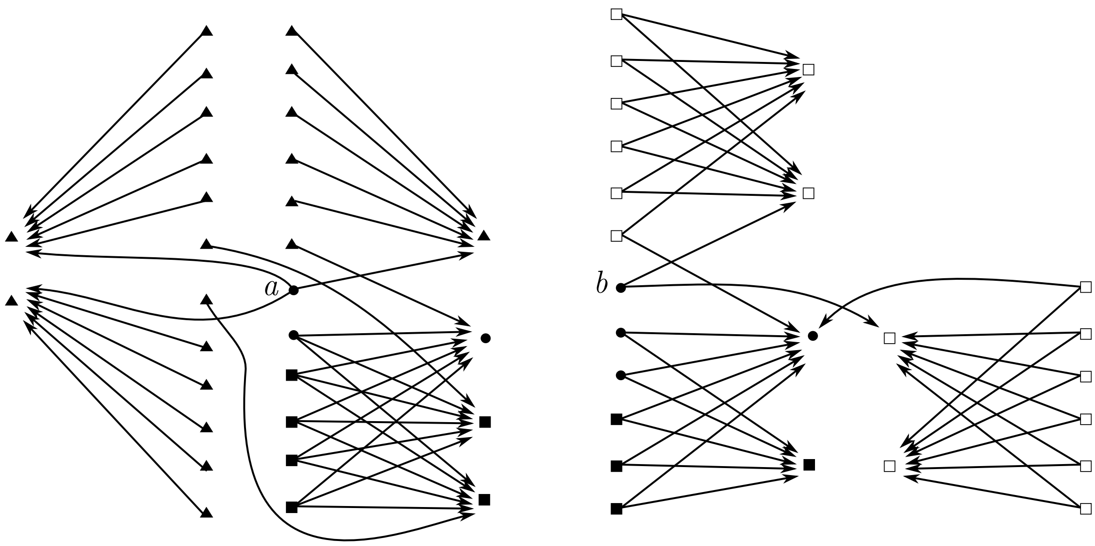

Figure 5: Step 3. The network $N_3 = (X_3, D_3)$. The nodes added during step 3 (2 groups of 7 nodes) are represented by a empty square.

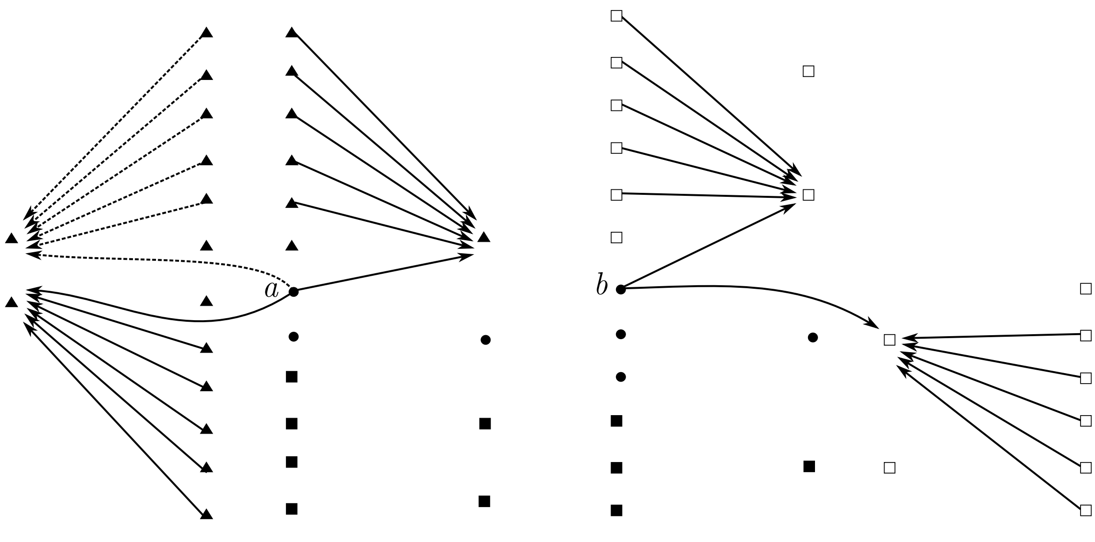

Figure 6: Step 4. The network $N_4 = (X_4, D_4)$ contains all arcs in this figure. The network $N_5 = (X_5, D_5)$ contains only the solid arcs.

Fig.6) in which $\beta_{N_5}(a) = \beta_{N_5}(b)$. From case 1, we know that $a \sim_{N_5} b$ and by Positive Responsiveness, we conclude that $a \succ_{N_4} b$. Hence $a \succ_N b$.

3. $\beta_N(a) < \beta_N(b)$. This case is handled as the previous one.

□

3.3 Independence of the conditions in Theorem 1

For each of the five conditions invoked in Theorem 1, we provide an example of a relational power ranking satisfying four conditions but one. This proves that our result cannot be improved by dropping one of the five conditions.

Example 1 Anonymity. *Choose any $a \in \Omega$. For all $N = (X, D) \in \mathcal{N}$ and all $b \in X$, define the relational power measure f by*

$$f_N(b) = \begin{cases} 2\beta_N(b), & \text{if } b = a, \\ \beta_N(b), & \text{otherwise.} \end{cases}$$

Define then the relational power ranking $\succsim$ by $b \succsim_N c$ iff $f_N(b) \geq f_N(c)$, for all $b, c \in X$.

It is simple to check that $\succsim$ violates Anonymity but satisfies Node Addition, Positive Responsiveness, Independence of Irrelevant Arcs and Independence of Local Density.

Example 2 Positive Responsiveness. *For all $N = (X, D) \in \mathcal{N}$, define the relational power ranking $\succsim$ by $\succsim_N = X^2$.*

It is simple to check that $\succsim$ violates Positive Responsiveness but satisfies Anonymity, Node Addition, Independence of Irrelevant Arcs and Independence of Local Density.

Example 3 Independence of Irrelevant Arcs. *For all $N = (X, D) \in \mathcal{N}$ and all $a \in X$, define the relational power measure f by $f_N(a) = \beta_N(a) - \sum_{c \in P_N(a)} \beta_N(c)$. Define then the relational power ranking $\succsim$ by $a \succsim_N b$ iff $f_N(a) \geq f_N(b)$ for all $a, b \in X$.*

Adding a node to the set X does not affect the β-measure of any node and Node Addition is therefore satisfied. Anonymity is clearly satisfied. Positive Responsiveness holds because, when we add an arc from node a to any other node d, the measure $f_N(a)$ strictly increases. Indeed, $\beta_N(a)$ strictly increases and, for all $c \in P_N(a)$, $\beta_N(c)$ remains constant except if $d \in S_N(c)$. In that case, $\beta_N(c)$ decreases. The overall effect on $f_N(a)$ is thus an increase.

We now show that Independence of Irrelevant Arcs does not hold. Consider the network $N = (X, D)$ *with* $X = \{a, b, c, d, e\}$ *and* $D = \{(c, a), (e, b)\}$*. Then* $a \sim_N b$ *because* $f_N(a) = -1 = f_N(b)$*. We now add the arc* (c, d) *to this network and we obtain* $N' = (X, D')$ *with* $D' = \{(c, a), (e, b), (c, d)\}$*. We have* $a \prec_N b$ *because* $f_N(a) = -2$ *and* $f_N(b) = -1$*, contrary to what Independence of Irrelevant Arcs imposes.*

Independence of Local Density holds because $f_N(a)$ *is a combination of the* β*-measures of some nodes and we have shown in Section 3.1 that the* β*-measure of all nodes remains unchanged when we increase the local density as in the statement of Independence of Local Density.*

Example 4 Independence of Local Density. *For all* $N = (X, D) \in \mathcal{N}$ *and all* $a \in X$*, define the relational power measure* f *by* $f_N(a) = s_N(a)$*. Define then the relational power ranking* $\succsim$ *by* $a \succsim_N b$ *iff* $f_N(a) \geq f_N(b)$ *for all* $a, b \in X$*.*

It is simple to check that $\succsim$ *violates Independence of Local Density but satisfies Anonymity, Node Addition, Positive Responsiveness and Independence of Irrelevant Arcs.*

Example 5 Node Addition. *For all* $N = (X, D) \in \mathcal{N}$ *and all* $a \in X$*, define the relational power measure* f *by*

$$f_N(a) = \begin{cases} s_N(a), & \textit{if } \#X \leq 3, \\ \beta_N(a), & \textit{otherwise.} \end{cases}$$

Define then the relational power ranking $\succsim$ *by* $a \succsim_N b$ *iff* $f_N(a) \geq f_N(b)$ *for all* $a, b \in X$*.*

To see that Node Addition is not satisfied, use the following example. Consider the network $N = (X, D)$ *with* $X = \{a, b, c\}$ *and* $D = \{(a, b), (b, a), (b, c), (c, b), (c, a)\}$*. We have* $f_N(a) = s_N(a) = 1$*. Similarly,* $f_N(b) = 2$ *and* $f_N(c) = 2$*. Therefore,* $b \sim_N c \succ_N a$*. Consider now the network* $N' = (X', D)$ *with* $X' = \{a, b, c, d\}$*. We have* $f_{N'}(a) = \beta_{N'}(a) = 1/2$*. Similarly,* $f_{N'}(b) = 3/2$ *and* $f_{N'}(c) = 1$*. Therefore,* $b \succ_{N'} c \succ_{N'} a$*. Hence* $\succsim_N \neq \succsim_{N'}$ *contrary to what Node Addition imposes.*

It is clear that Anonymity, Positive Responsiveness, Independence of Irrelevant Arcs and Independence of Local density are satisfied when $\#X > 3$*. It is also clear that Anonymity, Positive Responsiveness and Independence of Irrelevant Arcs hold when* $\#X \leq 3$*. Finally, Independence of Local density holds when* $\#X \leq 3$ *because Independence of Local Density is vacuous (the premise is never true) when* $\#X \leq 3$*.*

These five examples formally prove the logical independence of our conditions. Yet, the last one, although formally correct, is not fully satisfactory. Indeed, if we would state Theorem 1 for sets containing at least four nodes, Example 5 would no

Theorem 2.4 in vdBG2003		Our Theorem 1
Anonymity	$\Leftrightarrow$	Anonymity
Positive responsiveness	$\Leftrightarrow$	Positive responsiveness
(Node Addition)	$\Leftrightarrow$	Node Addition
Independence of Non-Dominated Arcs	$\Rightarrow$	Independence of Irrelevant Arcs
		Independence of Local density

Table 1: Comparison of the axioms in Theorem 2.4 in vdBG2003 and in our Theorem 1. Node Addition is satisfied by the ranking by outdegree, but is not used in its characterization.

longer work. Finding a better example or slightly weakening some of our conditions so as to obtain conditions that are logically independent even if the result is stated for sets containing at least k alternatives (with $k > 3$) is left as an open problem.

3.4 Comparison with the characterization of the ranking by outdegree

Theorem 2.4 in vdBG2003 characterizes the ranking by outdegree by means of three conditions: Anonymity, Positive Responsiveness and Independence of Non-Dominated Arcs. Obviously, the same result holds in a framework where X is allowed to vary: no additional condition is needed to characterize the outdegree. So, when we compare their Theorem 2.4 (adapted to a framework with X varying) and our Theorem 1, we find the following (see Table 1). The ranking by outdegree and the β-ranking have two characterizing axioms in common: Anonymity and Positive Responsiveness. They both satisify Node Addition, but this condition is not used in the characterization result for the outdegree. The fact that the ranking based on the outdegree can be characterized without appealing to Node Addition is also the sign that the measure on which this ranking is based is somewhat simpler than the β-measure. What are then the differences? The ranking by outdegree satisfies Independence of Non-Dominated Arcs while the β-ranking satisfies a weaker independence condition, namely Independence of Irrelevant Arcs. Besides, the β-ranking satisfies Independence of Local Density.

3.5 Why do we let X vary?

Given a fixed, finite set of nodes, the number of possible networks is also finite. And the set of all different values taken by the β-measure is also finite. Any axiom stated for rankings imposes an ordering constraint or an equality constraint on some pairs of values. When the set of all different values is finite, an axiom imposes a finite number of such constraints. With a finite set of axioms, the number of ordering or equality constraints remains finite. Obviously, with a finite number of ordering or equality constraints imposed on a finite number of pairs of values, it is impossible to completely determine what these values are, unless they are equally-spaced. For instance, the set of all values taken by the outdegree is equally-spaced. It is the set $\{0, 1, 2, ..., n-1\}$, where n is the size of the set of nodes. But the set of values taken by the β-measure on a fixed set of nodes is not equally-spaced. For instance, with 4 alternatives, the set of values that can possibly be taken by the β-measure is $\{0, 1/3, 1/2, \ldots\}$. We therefore think it is impossible to characterize the β-ranking with a fixed set of nodes.

By considering a variable set of nodes and imposing a finite number of axioms, we actually impose infinitely many constraints. This way, it is possible to isolate the β-ranking among all possible rankings.

4 The β-measure

In this section, we are interested in the β-measure itself, and no longer in the ranking induced by the β-measure. A characterization of the β-measure has already been published by vdBG2000. The alternative characterization we will propose will use many of the conditions already introduced for the characterization of the β-ranking presented in the previous section. It therefore offers a unified view of the ranking and the measure and it allows us to clearly identify the difference between the ranking and the measure in terms of axioms.

Since any relational power measure induces a ranking and since we already characterized the β-ranking in Theorem 1, a first way to impose conditions on a relational power measure f is to impose the conditions of Theorem 1 on the ranking induced by f. This way, we are sure that f is a numerical representation of the β-ranking in the sense that $f_N(a) \geq f_N(b)$ iff $a \succsim_N^\beta b$. Yet, these conditions will not be strong enough to characterize the β-measure because many different measures induce the same ranking. Indeed, any strictly increasing transformation of f induces the same ranking as the one induced by f. So we need to add some new conditions and/or to reinforce some of the conditions used in Theorem 1. We will actually use two additional conditions and reinforce the Node Addition condition.

4.1 Axioms

We begin with an additivity condition satisfied by the β-measure. Suppose two networks have the same set of nodes while their sets of arcs have almost nothing in common. Then, if we merge the two networks, the relational power measure of any node is the sum of the relational power measure in the two original networks. It is

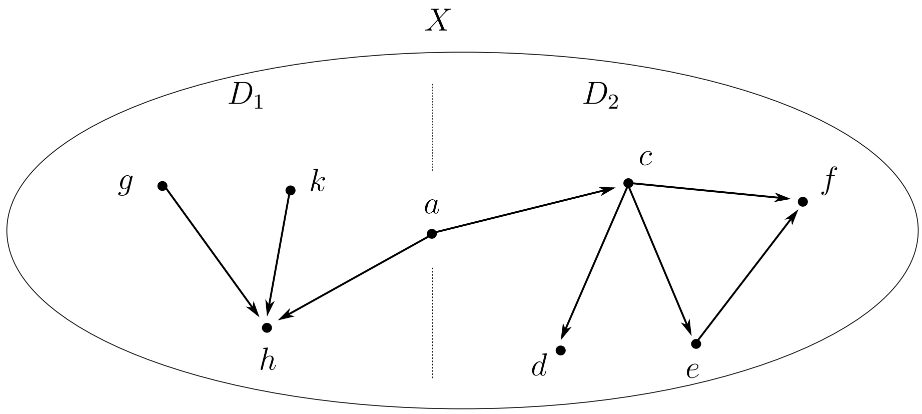

Figure 7: Additivity. The network $N = (X, D_1 \cup D_2)$ with two components connected via a.

difficult to motivate this condition on purely normative grounds. Why the sum and not another binary operation? Or more generally, why should the measure in the merged network be a combination of the measures in the original networks? There is no clear normative reason for this. Yet such a condition is necessary if we want to characterize the measure rather than the ranking. We can nevertheless motivate this condition by some operational arguments: the fact that the measure in the merged network is the sum of the measures in the original networks guarantees that the measure will be easy to calculate and that many optimization problems that we may want to define and to solve (in terms of the relational power measure), will have some nice computational properties.

Before we formally define our additivity condition, we need an extra piece of notation: for $N = (X, D) \in \mathcal{N}$, define $I_N = \{a \in X : (b, a)$ or $(a, b) \in D$ for some $b \in X\}$.

A 6 Additivity. *Let $X \subset \Omega$. Consider three networks $N_1 = (X, D_1), N_2 = (X, D_2)$ and $N = (X, D_1 \cup D_2) \in \mathcal{N}$ such that there is some $a \in X$ with $I_{N_1} \cap I_{N_2} = \{a\}$ and $P_N(a) = \emptyset$. Then $f_N(b) = f_{N_1}(b) + f_{N_2}(b)$ for all nodes $b \in X$.*

This condition, illustrated in Fig. 4.1, is clearly similar in spirit to the Additivity over Independent Partitions condition used in vdBG2000.

Let us show the necessity of this condition for the β-measure. Choose any node $b \in X$ (except a). If it has successors in N_1, then it has no successors in N_2 (and vice versa). Besides, its successors in N are the same as in N_1. And the predecessors of the successors of b are the same in N as in N_1. Hence $\beta_N(b) = \beta_{N_1}(b) + 0 = \beta_{N_1}(b) + \beta_{N_2}(b)$. We now consider the node a. It has successors in N_1 and in N_2, but the predecessors of the successors of a in N_1 are distinct of the predecessors of the successors of a in N_2. Furthermore the merging of N_1 and N_2 does not change the set of predecessors of any node. Hence $\beta_N(a) = \beta_{N_1}(a) + \beta_{N_2}(a)$.

The second additional condition says that a node that has no successor nor predecessor has a relational power measure equal to zero.

A 7 Isolated Node. *Let $N = (X, D) \in \mathcal{N}$ and $a \in X$. If $S_N(a) = P_N(a) = \emptyset$, then $f_N(a) = 0$.*

This is very close in spirit to the Dummy Node Condition of vdBG2000, but our condition is weaker because vdBG2000 impose $f_N(a) = 0$ whenever a node has no successor. It is clearly satisfied by the β-measure.

We now reinforce the Node Addition condition: adding an isolated node (that has no successor nor predecessor) has no effect on the relational power measure.

A 8 Node Addition*. *For all networks $N = (X, D) \in \mathcal{N}$ and all $a \in \Omega \setminus X$, if $N' = (X \cup \{a\}, D)$, then $f_N(b) = f_{N'}(b)$ for all $b \in X$.*

Node Addition* is not related to the Dummy Node Condition of vdBG2000. The necessity of Node Addition* for the β-measure is obvious. The following example shows that Node Addition does not imply Node Addition*.

Example 6 *For all $N = (X, D) \in \mathcal{N}$, define the relational power measure f by*

$$f_N(a) = \frac{\beta_N(a)}{\#X}, \ \forall a \in X.$$

Node Addition is clearly violated while Node Addition is satisfied.*

4.2 Result

Theorem 2 *A relational power measure f satisfies (i) Additivity, (ii) Isolated Node and (iii) Node Addition* and induces a ranking $\succsim$ satisfying (iv) Anonymity, (v) Positive Responsiveness, (vi) Independence of Irrelevant Arcs, and (vii) Independence of Local Density if and only if $f = k\beta$ for some positive real number k.*

Instead of imposing Anonymity, Positive Responsiveness, Independence of Local Density, and Independence of Irrelevant Arcs on the ranking induced by the relational power measure f, we could alternatively redefine those conditions for relational power measures and impose them directly on f. For instance, Anonymity would become

A 9 f-Anonymity. *For every permutation π of X and every $N = (X, D) \in \mathcal{N}$, we have, for all $a, b \in X$ $f_{N^\pi}(\pi(a)) \geq f_{N^\pi}(\pi(b)) \iff f_N(a) \geq f_N(b)$.*

The statement of Theorem 2 would then read "A relational power measure f satisfies f-Anonymity, f-Positive Responsiveness, f-Independence of Local Density, f-Independence of Irrelevant Arcs, Additivity, Isolated Node and Node Addition* if and only if $f = k\beta$ for some positive real number k." We prefer the former statement because it makes clear which conditions are specifically tailored for the index (the cardinal conditions), as opposed to the ordinal conditions.

Proof of Theorem 2. The necessity of our conditions has been shown above. We now prove the sufficiency. Clearly, if f satisfies Node Addition*, then $\succsim$ satisfies Node Addition. Hence, thanks to Theorem 1, $\succsim = \succsim^\beta$ (defined in Section 2). Consider two networks $N = (X, D)$ and $M = (Y, B) \in \mathcal{N}$ with $a \in X \cap Y$. The proof will consist of three parts: i) the index $f_N(b) = k\beta_N(b)$, for any node $b \in X$; ii) the index $f_M(b) = k\beta_M(b)$, for any $b \in Y$ (with the same constant k); iii) parts i and ii hold even if $X \cap Y = \emptyset$.

By construction, for any $c \in X$ or $d \in Y$, $\beta_N(c)$ and $\beta_M(d)$ are rational numbers. There exist therefore

- a natural number z,
- $\#X$ natural numbers x_c, $\forall c \in X$, such that $\beta_N(c) = x_c/z$ and
- $\#Y$ natural numbers y_c, $\forall c \in Y$, such that $\beta_M(c) = y_c/z$.

Let us choose any $b \in X$ and construct a new network $N' = (X', D')$ with $D' = D$ and $X' = X \cup \{b', c_1, \ldots, c_{x_b}, e_{1,1}, \ldots, e_{x_b,z}\}$. By Node Addition*, $f_{N'}(b) = f_N(b)$. By construction, $\beta_{N'}(b) = \beta_N(b)$. By Isolated Node, $f_{N'}(c) = 0$ for all $c \in X' \setminus X$. We now construct a series of networks $N_1, N_2, \ldots$ in a number of steps.

Step 1. Define $N_1 = (X', D_1)$ by

$$D_1 = D' \cup \{(b', c_1)\} \cup \{(e_{1,j}, c_1) : j = 1, \ldots, z-1\}.$$

By construction, $\beta_{N_1}(b) = \beta_N(b)$ and $\beta_{N_1}(b') = 1/z$. By Additivity, $f_{N_1}(b) = f_N(b)$ and $f_{N_1}(b') = f_N(b')$. Let $\gamma = f_{N_1}(b')$; thanks to Anonymity, it is independent of b' (and thus of b) because it only depends upon the structure of the subgraph $\{(b', c_1)\} \cup \{(e_{1,j}, c_1) : j = 1, \ldots, z-1\}$.

Step i, for $i \in \{2, \ldots, x_b\}$. Define $N_i = (X', D_i)$ by

$$D_i = D_{i-1} \cup \{(b', c_i)\} \cup \{(e_{i,j}, c_i) : j = 1, \ldots, z-1\}.$$

By construction, $\beta_{N_i}(b) = \beta_N(b)$ and $\beta_{N_i}(b') = i/z$. By Additivity, $f_{N_i}(b) = f_N(b)$. Anonymity and Additivity imply $f_{N_i}(b') = i\gamma$. Notice that, at the end of Step x_b, $f_{N_{x_b}}(b') = x_b\gamma$. It is also clear that $\beta_{N_{x_b}}(b) = \beta_{N_{x_b}}(b')$. This and Theorem 1 imply $f_{N_{x_b}}(b) = f_{N_{x_b}}(b')$. As a consequence, $f_N(b) = f_{N_{x_b}}(b) = f_{N_{x_b}}(b') = x_b\gamma$. So,

$$\frac{f_N(b)}{\beta_N(b)} = \frac{x_b\gamma}{x_b/z} = \gamma z. \tag{1}$$

Remember we have shown that γ does not depend on b. The ratio $f_N(b)/\beta_N(b)$ is therefore constant for all $b \in X$. This concludes the proof of part i.

Notice also that the same reasoning applies to the network M and any $b \in Y$, that is,

$$\frac{f_M(b)}{\beta_M(b)} = \frac{y_b\gamma}{y_b/z} = \gamma z \tag{2}$$

for any $b \in Y$. By construction, z is the same number in (1) and (2). The same holds for γ. The ratio $f_N(b)/\beta_N(b)$ is therefore constant for all $N = (X, D) \in \mathcal{N}$ that share a common agent a and for all $b \in X$. This concludes the proof of part ii.

We now turn to part iii. Consider two networks $N = (X, D)$ and $M = (Y, B)$ with $X \cap Y = \emptyset$. We can easily construct a third network $O = (Z, C)$ with $X \cap Z \neq \emptyset$ and $Y \cap Z \neq \emptyset$. Applying the above reasoning to the pair (N, O) and to the pair (M, O) yields $f_N(b)/\beta_N(b) = f_O(b)/\beta_O(b) = f_M(b)/\beta_M(b)$. The ratio $f_N(b)/\beta_N(b)$ is therefore constant for all $N = (X, D) \in \mathcal{N}$ and for all $b \in X$.

In order to complete the proof, define $k = \gamma z$ and notice that, by Positive Responsiveness, $k > 0$. $\square$

4.3 Independence of the conditions in Theorem 2

Examples 1, 3 and 4 in Section 3 are stated for rankings but are all induced by a relational power measure. They can therefore be reused for showing that none of Anonymity, Independence of Irrelevant Arcs or Independence of Local Density is implied by the other conditions of Theorem 2.

In order to prove that Node Addition* is not implied by the other conditions of Theorem 2, we can use f_N as defined in Example 5. Yet, we prefer to use Example 6. It is simpler than Example 5 and it cannot be used in place of Example 5 for showing the independence of the conditions of Theorem 1 because it induces a ranking that satisfies the Node Addition condition. Besides, Example 6 does not make a distinction between sets with at most three nodes and sets with more than three nodes. The logical independence of the axioms in Theorem 2 is thus more strongly established that for Theorem 1.

For the other conditions, we need some additional examples.

Example 7 Positive Responsiveness. *For all $N = (X, D) \in \mathcal{N}$, define the relational power measure f by*

$$f_N(a) = 0, \ \forall a \in X.$$

The ranking induced by f violates Positive Responsiveness. All other conditions are satisfied. This example is essentially identical to Example 2, but stated in terms of relational power measure.

Example 8 Additivity. *For all $N = (X, D) \in \mathcal{N}$, define the relational power measure f by*

$$f_N(a) = (\beta_N(a))^2, \ \forall a \in X.$$

It obviously violates Additivity. That is satisfies Isolated Node and Node Addition is clear as well. Since f is a strictly increasing transform of β, it is also a numerical representation of the ranking $\succsim^\beta$ and it therefore satisfies Anonymity, Positive Responsiveness, Independence of Local Density, and Independence of Irrelevant Arcs.*

Example 9 Isolated Node. *For all $N = (X, D) \in \mathcal{N}$, define the relational power measure f by*

$$f_N(a) = \#D + \beta_N(a), \ \forall a \in X.$$

It clearly violates the Isolated Node condition. It satisfies Additivity because each of $\#D$ and $\beta_N(a)$ are additive measures. Node Addition is easy to check since the addition of an isolated node does not change $\#D$. Anonymity is obvious. Positive Responsiveness holds because, when we add an arc, all measures increase by 1, but the measure of the origin of the arc increases by more than 1. Independence of Local Density is satisfied. Indeed, when we transform a network as in the statement of the condition, the measure of all nodes (without exception) increases by exactly 1. A similar reasoning shows that Independence of Irrelevant Arcs holds.*

4.4 The characterization of van den Brink and Gilles

In this section, we present vdBG2000's characterization of the β-measure and we compare it with ours. In their characterization of the β-measure, vdBG2000 use the following four axioms.

A 10 Dominance normalization. *For every network $N = (X, D) \in \mathcal{N}$,*

$$\sum_{a \in X} f_N(a) = \#\{a \in X : P_N(a) \neq \emptyset\}.$$

A 11 Dummy Node Property. *For every network $N = (X, D) \in \mathcal{N}$ and every a in X with $S_N(a) = \emptyset$, it holds $f_N(a) = 0$.*

A 12 Symmetry. *For every network $N = (X, D) \in \mathcal{N}$ and every a, b in X such that $S_N(a) = S_N(b)$ and $P_N(a) = P_N(b)$, it holds $f_N(a) = f_N(b)$.*

For the fourth axiom, we need a new definition. A collection $\{D_1, \ldots, D_m\}$ of binary relations on X is an independent partition of $N = (X, D)$ if

- the union of $\{D_1, \ldots, D_m\}$ is equal to D,
- all relations $\{D_1, \ldots, D_m\}$ are mutually disjoint,
- each node has no predecessor in D or has predecessors in only one of the relations $\{D_1, \ldots, D_m\}$.

A 13 Additivity over Independent Partitions. *For every network $N = (X, D) \in \mathcal{N}$, if the collection $\{D_1, \ldots, D_m\}$ of binary relations on X is an independent partition of $N = (X, D)$, then*

$$f_N(a) = \sum_{i=1}^{m} f_{(X,D_i)}(a),$$

for all $a \in X$.

Their characterization result is then:

Theorem 3 *[vdBG2000, Th. 2.7, p. 145] Suppose X is given. A relational power measure satisfies Dominance Normalization, Dummy Node Property, Symmetry and Additivity over Independent Partitions if and only if it is the β-measure.*

The most salient difference between vdBG2000's result and our Theorem 2 is that vdBG2000 exactly characterize the β-measure while we characterize it up to a multiplicative constant. We do so because we think there is no need for normalization in the sense that there is no need for setting *a unit of measurement* (in vdBG2000's words, p.144). To make our point clear, we consider a simple example in geometry. Suppose we want to characterize the Euclidean distance. Shall we impose an axiom saying that the distance be measured in meters? Or in yards? Definitely not. We want to characterize the Euclidean distance up to a multiplication by some positive real number. This is perfectly sufficient for understanding the Euclidean distance. The same is true for the β-measure: the relational power measure f defined by $f_N(a) = 10\beta_N(a)$ is for all purposes as good as $\beta_N(a)$. We therefore want to characterize the β-measure up to a multiplication by some positive real number.

Since vdBG2000 want to exactly characterize the β-measure, they need to use a normalization condition for setting a unit of measurement: this is their Dominance Normalization. Unfortunately this is a complex condition and it does more than normalizing (in the sense of setting a unit of measurement for) the relational power measure. If it were just a normalization condition (in the sense of setting a unit of measurement), then the three other conditions (namely Dummy Node Property, Symmetry and Additivity over independent partitions) would characterize the β-measure up to a multiplication by some positive real number. This is clearly not the case since it is easy to find other relational power measures satisfying these three conditions. One such measure is the outdegree, i.e., the relational power measure f defined by $f_N(a) = s_N(a)$ for all $N = (X, D)$ and all $a \in X$.

A pure normalization axiom (in the sense of setting a unit of measurement) would be much simpler than Dominance Normalization. It could for instance be stated as follows: for every network $N = (X, D) \in \mathcal{N}$ such that $D = \{(a, b)\}$, we have $\beta_N(a) = 1$. Such a condition does nothing else but setting the unit of measurement. If we add it to the conditions of our Theorem 2, we exactly characterize the β-measure. Without this condition, we characterize the β-measure up to a multiplication by a positive real number.

Theorem 2.7 in vdBG2000 continues to hold if we restate it in a framework where X is allowed to vary: no additional condition is needed to characterize the β-measure. So, we can compare Theorem 2.7 in vdBG2000 to our Theorem 2. Table 2 summarizes the axioms used in both theorems; it has five rows delimited by dotted lines: one for each axiom of vdBG2000 plus an extra row for Node Addition*. In a given row, the axioms in the right hand side column are similar in spirit (or identical) to those in the left hand side column. In the right hand side column, the last row has several axioms, thereby showing that vdBG2000's Dominance Normalization has been split in several weaker axioms, that all have a clear normative content. Hence, in our view, the alternative characterization of the β-mesure proposed in Theorem 2 does not compare unfavorably with respect to Theorem 2.7 in vdBG2000.

The only condition in our Theorem 2 that has no clear normative content is Additivity (similar to Additivity over Independent Partitions in vdBG2000). We do not see a way to avoid such a condition, unless one is willing to work with a ranking rather than with an index. Indeed, our Theorem 1 does not invoke any additivity condition and uses all conditions with a clear normative content.

Acknowledgements

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Theorem 2.7 in vdBG2000		Our Theorem 2
Dummy Node Property	$\Rightarrow$	Isolated Node
Symmetry	$\Leftarrow$	Anonymity
Additivity over Independent Partitions		Additivity
(Node Addition*)	$\Leftrightarrow$	Node Addition*
Dominance Normalization		Positive Responsiveness Independence of Irrelevant Arcs Independence of Local Density

Table 2: Comparison of the axioms in Theorem 2.7 in vdBG2000 and in our Theorem 2. Node Addition is satisfied by the ranking by outdegree, but is not used in its characterization.

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