Some remarks on ELECTRE TRI[1]

Denis Bouyssou[2]
CNRS – LAMSADE &
Université Paris Dauphine

Thierry Marchant[3]
Ghent University

16 November 2013

Abstract

ELECTRE TRI is a method designed to sort alternatives evaluated on several criteria into ordered categories. The original method uses limiting profiles. A recently introduced variant uses central profiles. We study the relations between these two variants. We do so by investigating if an ordered partition obtained with one variant can also be obtained by the other variant, after a suitable redefinition of the profiles. We also investigate the transposition operation that was introduced by the authors of the variant using central profiles. We study the consequences of applying this operation to the variant using limiting profiles. This leads us to propose an alternative variant of ELECTRE TRI that uses limiting profiles. We show that this alternative variant may have some advantages over the original one.

Keywords: Decision with multiple attributes, Sorting models, ELECTRE TRI-B, ELECTRE TRI-C.

1 Introduction

This paper deals with ELECTRE TRI. It is the most recent method belonging to the ELECTRE family of methods (for overviews, see Roy and Bouyssou, 1993, Figueira, Mousseau, and Roy, 2005, Figueira, Greco, Roy, and Słowiński, 2013).

[1] Jose Figueira and Bernard Roy made helpful comments on an earlier draft of this text. We are fully responsible for all remaining problems or errors.

[2] CNRS, LAMSADE, UMR 7243 & Université Paris Dauphine, Place du Maréchal de Lattre de Tassigny, F-75775 Paris Cedex 16, France, tel: +33 1 44 05 48 98, fax: +33 1 44 05 40 91, e-mail: `bouyssou@lamsade.dauphine.fr`.

[3] Ghent University, H. Dunantlaan, 1, B-9000 Gent, Belgium tel: +32 9 264 63 73, fax: +32 9 264 64 87, e-mail: `thierry.marchant@UGent.be`.

ELECTRE TRI was introduced in the doctoral dissertation of Wei (1992) (supervised by B. Roy) and was detailed in Roy and Bouyssou (1993). It is designed to sort alternatives evaluated on multiple criteria into ordered categories defined by *limiting profiles* (see Roy and Bouyssou, 1993, Ch. 6, for a detailed analysis of the sorting problem formulation). This method has generated much interest. Indeed, sorting alternatives into ordered categories is a problem occurring in many real-world situations. Moreover, on a more technical level, the fact that the method only compares alternatives with a set of carefully selected limiting profiles that are linked by dominance greatly facilitates the exploitation of the outranking relation that is built. This limits the consequences of the fact that this relation is, in general, neither transitive nor complete (Bouyssou, 1996). Many techniques have been proposed for the elicitation of the parameters of the original variant of ELECTRE TRI (see Cailloux, Meyer, and Mousseau, 2012, Damart, Dias, and Mousseau, 2007, Dias and Clímaco, 2000, Dias and Mousseau, 2003, 2006, Dias, Mousseau, Figueira, and Clímaco, 2002, Leroy, Mousseau, and Pirlot, 2011, Mousseau and Dias, 2004, Mousseau and Słowiński, 1998, Mousseau, Słowiński, and Zielniewicz, 2000, Mousseau, Figueira, and Naux, 2001, Mousseau, Figueira, Dias, da Silva, and Clímaco, 2003, Mousseau, Dias, and Figueira, 2006, Ngo The and Mousseau, 2002, Zheng, Metchebon, Mousseau, and Pirlot, 2012). Most of them use mathematical programming tools to infer the parameters of the method on the basis of assignment examples. ELECTRE TRI has been applied to a large variety of real world problems (see the references at the end of Sect. 6 in Almeida-Dias, Figueira, and Roy, 2010). ELECTRE TRI has received a fairly complete axiomatic analysis in Bouyssou and Marchant (2007a,b). In a nutshell, ELECTRE TRI can be considered as a real success story within the ELECTRE family of methods.

A recent paper (Almeida-Dias et al., 2010) introduced a variant of ELECTRE TRI that uses *central profiles* instead of limiting profiles (Almeida-Dias et al., 2010 use the term "characteristic reference action" instead of central profiles). This is an interesting development since it seems intuitively easier to elicit central rather than limiting profiles (a related paper, Almeida-Dias, Figueira, and Roy, 2012, deals with the case of multiple central profiles. We do not study this more general case in the present paper).

The present paper was prompted by the analysis of this new variant of the method and its comparison with the original one. After having recalled the essential elements of both variants of ELECTRE TRI (Section 2), we investigate two main points. We first study the relations between these two variants of the method (Section 3). We do so by investigating if an ordered partition obtained with one variant of the method can also be obtained by the other variant, after a suitable redefinition of the profiles. Our main conclusion is that this proves difficult. This difficulty should not be interpreted as a criticism of ELECTRE

TRI but as the sign that the two variants of the method use, beyond surface, different principles. We then investigate (Section 4) the transposition operation used by Almeida-Dias et al. (2010) to justify their proposition of two versions of the variants using central profiles that they recommend to use conjointly. Since this transposition operation is fairly intuitive, we study the consequences of applying it to the original variant of the method. As first observed in Roy (2002), the original variant of the method is not fully consistent with this transposition operation: the pseudo-disjunctive version (also known as "optimistic") is not obtained from the pseudo-conjunctive version (also known as "pessimistic") via the transposition operation and vice versa. We detail this point that may explain why most of the elicitation techniques proposed so far only deal with the pseudo-conjunctive version (Zheng, 2012, Zheng et al., 2012, are exceptions) and why the axiomatic analysis conducted in Bouyssou and Marchant (2007a,b) is also limited to the pseudo-conjunctive version. This will lead us to propose a new variant of ELECTRE TRI using limiting profiles that is fully consistent with the transposition operation. We show that this new variant may have some advantages over the original variant of the method. A final section (Section 5) concludes with the indication of directions for future research[1].

2 ELECTRE TRI: a brief reminder

We consider a set of alternatives A. Each alternative $a \in A$ is supposed to be evaluated on a family of n real-valued criteria, i.e., n functions $g_1, g_2, \ldots, g_n$ from A into $\mathbb{R}$. Let us define $N = \{1, 2, \ldots, n\}$. We suppose, w.l.o.g., that increasing the performance on any criterion increases preference. The *dominance* relation Δ is defined letting, for all $a, b \in A$, $a \; \Delta \; b$ if $g_i(a) \geq g_i(b)$, for all $i \in N$. In such a case, we say that *a dominates b*. We say that *a strictly dominates b* if $a \; \Delta \; b$ and $Not[b \; \Delta \; a]$, which we denote by $a \; \Delta^a \; b$, since Δ^a is the asymmetric part of Δ.

2.1 Construction of the outranking relation

In all the examples that follow, discordance will play no rôle and, on all criteria, the indifference and preference thresholds will be equal (hence, it is not restrictive to take them constant, see Roy and Vincke, 1987). In order to keep things simple, we briefly recall here how the outranking relation is built in this particular case.

[1]In what follows, we will use the following terminology. ELECTRE TRI is a *method*. It comes in two *variants*, ELECTRE TRI-B and ELECTRE TRI-C. Each of these two variants has two versions: ELECTRE TRI-B, pseudo-conjunctive and ELECTRE TRI-B, pseudo-disjunctive one the one hand, and ELECTRE TRI-C, ascending version and ELECTRE TRI-C, descending version on the other hand.

Section A in Appendix presents the construction of the outranking relation in the general case, i.e., when indifference and preference thresholds may be unequal and may vary and when discordance plays a rôle. It is important to realize the definition of the outranking relation that we detail below, although simpler than the general definition (recalled in Appendix), is a particular case of the general one. All relations that can be obtained using the formulae in this section can be obtained using the more general formulae presented in Appendix.

We associate with each criterion $i \in N$ a nonnegative preference threshold $p_i \geq 0$. If the value $g_i(a) - g_i(b)$ is positive but less than p_i, it is supposed that this difference is insignificant given the way g_i has been built so that, on this criterion, the two alternatives should be considered indifferent.

The above information is used to define, on each criterion $i \in N$, a valued relation on A, i.e., a function from $A \times A$ into $[0, 1]$, called the partial concordance relation, such that:

$$c_i(a, b) = \begin{cases} 1 & \text{if } g_i(b) - g_i(a) \leq p_i, \\ 0 & \text{if } g_i(b) - g_i(a) > p_i, \end{cases}$$

(in the particular case studied here, the valued relations c_i can only take the values 0 or 1. In the general case dealt with in Appendix, they can take any value between 0 and 1).

Each criterion $i \in N$ is assigned a non-negative weight w_i. We suppose, w.l.o.g., that weights have been normalized so that $\sum_{i=1}^{n} w_i = 1$.

The valued relations c_i are aggregated into a single valued outranking relation s letting, for all $a, b \in A$,

$$s(a, b) = \sum_{i=1}^{n} w_i c_i(a, b).$$

On the basis of the valued relation s, a binary relation S_λ on A is defined letting:

$$a \; S_\lambda \; b \Leftrightarrow s(a, b) \geq \lambda,$$

where $\lambda \in [0, 1]$ is a cutting level (usually taken to be above 1/2). The relation S_λ is interpreted as an "at least as good" relation between alternatives. From S_λ, we derive the following relations:

$$\begin{aligned} a \; P_\lambda \; b &\Leftrightarrow [a \; S_\lambda \; b \text{ and } Not[b \; S_\lambda \; a]], \\ a \; I_\lambda \; b &\Leftrightarrow [a \; S_\lambda \; b \text{ and } b \; S_\lambda \; a], \\ a \; J_\lambda \; b &\Leftrightarrow [Not[a \; S_\lambda \; b] \text{ and } Not[b \; S_\lambda \; a]], \end{aligned}$$

that are respectively interpreted as "strictly better than", "indifferent to" and "incomparable to" relations between alternatives.

It is easy to check (Roy and Bouyssou, 1993, Ch. 5) that if $a \Delta b$ then $s(a,b)=1$ and, for all $c \in A$, $s(b,c) \leq s(a,c)$ and $s(c,a) \leq s(c,b)$. Hence, we have $a \ S_\lambda \ b$ and, for all $c \in A$,

$$b \ S_\lambda \ c \Rightarrow a \ S_\lambda \ c, \quad b \ P_\lambda \ c \Rightarrow a \ P_\lambda \ c,$$
$$c \ S_\lambda \ a \Rightarrow c \ S_\lambda \ b, \quad c \ P_\lambda \ a \Rightarrow c \ P_\lambda \ b.$$

The following proposition will be useful. It is taken from Roy and Bouyssou (1993, Ch. 6) and its validity is independent of the simplifying hypotheses we have done for the construction of the outranking relation.

Proposition 1
Let $c^1, c^2, \dots, c^t \in A$ be such that $c^{k+1} \ \Delta^a \ c^k$ for $k = 1, 2, \dots, t-1$. Suppose furthermore that, for all $a \in A \setminus \{c^t, c^1\}$, $c^t \ P_\lambda \ a$ and $a \ P_\lambda \ c^1$.

When an alternative $a \in A \setminus \{c^t, c^1\}$ is compared to the subset of alternatives $c^1, c^2, \dots, c^t$, three distinct situations may arise, for $t \geq \ell+1 \geq k+1 \geq 2$,

1. $c^t \ P_\lambda \ a, c^{t-1} \ P_\lambda \ a, \dots, c^{k+1} \ P_\lambda \ a, a \ P_\lambda \ c^k, a \ P_\lambda \ c^{k-1}, \dots, a \ P_\lambda \ c^1$,
2. $c^t \ P_\lambda \ a, c^{t-1} \ P_\lambda \ a, \dots, c^{\ell+1} \ P_\lambda \ a, a \ I_\lambda \ c^\ell, a \ I_\lambda \ c^{\ell-1}, \dots, a \ I_\lambda \ c^{k+1}, a \ P_\lambda \ c^k, \dots, a \ P_\lambda \ c^1$,
3. $c^t \ P_\lambda \ a, c^{t-1} \ P_\lambda \ a, \dots, c^{\ell+1} \ P_\lambda \ a, a \ J_\lambda \ c^\ell, a \ J_\lambda \ c^{\ell-1}, \dots, a \ J_\lambda \ c^{k+1}, a \ P_\lambda \ c^k, \dots, a \ P_\lambda \ c^1$,

The proof of the above proposition is easy (see Roy and Bouyssou, 1993, p. 392–393) and rests on the fact that the alternatives c^k are linked by strict dominance and that the outranking relation is compatible with dominance.

2.2 ELECTRE TRI-B

The original variant of ELECTRE TRI uses limiting profiles. It was renamed ELECTRE TRI-B by Almeida-Dias et al. (2010). We follow their naming conventions. ELECTRE TRI-B has two versions called "pessimistic" and "optimistic" in Roy and Bouyssou (1993). For reasons detailed in Roy (2002), it seems advisable to rename the pessimistic version of the ELECTRE TRI-B as "pseudo-conjunctive" and the optimistic version as "pseudo-disjunctive". We also follow this naming convention. We will sometimes abbreviate ELECTRE TRI-B, pseudo-conjunctive (resp. ELECTRE TRI-B, pseudo-conjunctive) as ETRI-B-pc (resp. ETRI-B-pd).

We consider the case of r ordered categories $C^1, C^2, \dots, C^r$, with C^r containing the most desirable alternatives. The category C^k is modelled using limiting profiles. The lower limiting profile of C^k is π^k. The upper limiting profile of C^k is π^{k+1}. Notice that the lower limiting profile π^k of C^k is also the upper limiting profile of C^{k-1}. Similarly, the upper limiting profile π^{k+1} of C^k is also the lower limiting profile of C^{k+1} (see Figure 1).

We suppose that the limiting profiles are such that π^{k+1} strictly dominates π^k. The profile π^1 (resp. π^{r+1}) is taken to be arbitrarily low (resp. high). It will be convenient to suppose that $\pi^k \in A$, for $k = 2, 3, \dots, r$, while $\pi^1, \pi^{r+1} \notin A$. With this convention, we have, for all $a \in A$, $a\ P_\lambda\ \pi^1$ and $\pi^{r+1}\ P_\lambda\ a$.

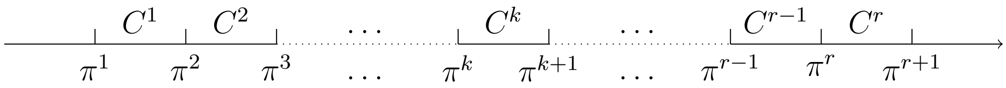

Figure 1: Sorting in ELECTRE TRI-B with r ordered categories

The two versions of ELECTRE TRI-B are defined as follows.

Definition 2 (ETRI-B-pc)
Decrease k from $r+1$ until the first value k such that $a\ S_\lambda\ \pi^k$. Assign alternative a to C^k.

Definition 3 (ETRI-B-pd)
Increase k from 1 to $r+1$ until the first value k such that $\pi^k\ P_\lambda\ a$. Assign alternative a to C^{k-1}.

Remark 4
Take any $a \in A$. We know that $a\ P_\lambda\ \pi^1$ and $\pi^{r+1}\ P_\lambda\ a$. This implies that it is not true that $a\ S_\lambda\ \pi^{r+1}$ but that $a\ S_\lambda\ \pi^1$ holds. This shows that, with ETRI-B-pc, each alternative $a \in A$ is assigned to one of the categories $C^1, C^2, \dots, C^r$.

Similarly, we know that $\pi^{r+1}\ P_\lambda\ a$ while we do not have $\pi^1\ P_\lambda\ a$. This shows that, with ETRI-B-pd, each alternative $a \in A$ is assigned to one of the categories $C^1, C^2, \dots, C^r$. •

Remark 5
Roy and Bouyssou (1993, Ch. 6) have shown that if $a \in A$ is assigned to category C^k by pseudo-conjunctive version of ELECTRE TRI-B and to category C^ℓ by pseudo-disjunctive version of ETRI-B, then $k \leq \ell$. This explains with ETRI-B-pc (resp. ETRI-B-pd) was initially called pessimistic (resp. optimistic).

Because of the definition of the profiles, we may apply Proposition 1 when comparing an alternative to the set of profiles $\pi^1, \pi^2, \dots, \pi^{r+1}$. It is easy to check that the two versions of ETRI-B lead to identical results in the first and second cases (a is assigned to C^k, where k is the highest index such that $a\ S_\lambda\ \pi^k$). When $\pi^{r+1}\ P_\lambda\ a, \dots, \pi^{\ell+1}\ P_\lambda\ a, a\ J_\lambda\ \pi^\ell, \dots, a\ J_\lambda\ \pi^{k+1}, a\ P_\lambda\ \pi^k, \dots, a\ P_\lambda\ \pi^1$, ETRI-B-pc assigns a to C^k while ETRI-B-pd assigns it to C^ℓ. •

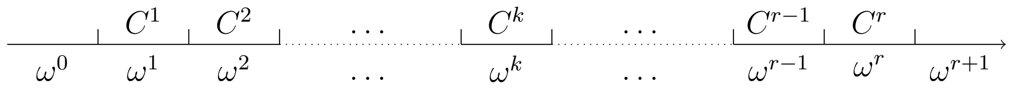

Figure 2: Sorting in ELECTRE TRI-C with r ordered categories

2.3 ELECTRE TRI-C

We still consider the case of r ordered categories $C^1, C^2, \dots, C^r$, with C^r containing the most desirable alternatives. The category C^k is modelled using a central profile ω^k (see Figure 2).

We suppose that the central profiles are such that ω^{k+1} strictly dominates ω^k. Moreover, we adjoin to $\omega^1, \omega^2, \dots, \omega^r$ two fictitious profiles ω^0 and ω^{r+1}. The profile ω^0 (resp. ω^{r+1}) is taken to be arbitrarily low (resp. high). It will be convenient to suppose that $\omega^k \in A$, for $k = 1, 2, \dots, r$, while $\omega^0, \omega^{r+1} \notin A$. With this convention, we have, for all $a \in A$, $a \; P_\lambda \; \omega^1$ and $\omega^{r+1} \; P_\lambda \; a$.

For all $a, b \in A$, let $\rho(a, b) = \min(s(a, b), s(b, a))$, ρ being called the *selecting function* in Almeida-Dias et al. (2010).

We define below the two versions of ELECTRE TRI-C, called ELECTRE TRI-C, ascending version and ELECTRE TRI-C, descending version, as introduced in Almeida-Dias et al. (2010). We will sometimes abbreviate ELECTRE TRI-C, ascending version (resp. ELECTRE TRI-C, descending version) as ETRI-C-a (resp. ETRI-C-d). Since Almeida-Dias et al. (2010) recommend to use these two versions conjointly, the word "component" might be preferred to the word "version" that we use for ELECTRE TRI-C (Roy, 2013). We have kept the term version for simplicity and this should not cause any confusion.

Definition 6 (ETRI-C-d)
Decrease k from $r+1$ until the first value k such that $s(a, \omega^k) \geq \lambda$,

a) for $k = r$, assign a to C^r,

b) for $1 \leq k \leq r-1$, assign a to C^k if $\rho(a, \omega^k) > \rho(a, \omega^{k+1})$, otherwise assign a to C^{k+1},

c) for $k = 0$, assign a to C^1.

Definition 7 (ETRI-C-a)
Increase k from 0 until the first value k such that $s(\omega^k, a) \geq \lambda$,

a) for $k = 1$, assign a to C^1,

b) for $2 \leq k \leq r$, assign a to C^k if $\rho(a, \omega^k) > \rho(a, \omega^{k-1})$, otherwise assign a to C^{k-1}, *c) for $k = r+1$, assign a to C^r.*

Almeida-Dias et al. (2010) recommend to use these two versions *conjointly*. This means that, if $a \in A$ is assigned to C^ℓ by one version of the method and to C^k by the other one, the result of ELECTRE TRI-C is that a is assigned to the interval of categories having C^ℓ and C^k as extremities. This interval may, of course, be reduced to a single category when $\ell = k$. Notice that, contrary to what is the case with ELECTRE TRI-B, it is not true here that one version of the method always gives an assignment that is at least as high as the assignment given by the other version.

Remark 8
We know that $a \; P_\lambda \; \omega^0$ and $\omega^{r+1} \; P_\lambda \; a$. It is impossible that $s(a, \omega^{r+1}) \geq \lambda$. Moreover, we have $a \; S_\lambda \; \omega^0$. Hence, each alternative $a \in A$ is assigned by ETRI-C-d to one of the categories $C^1, C^2, \ldots, C^r$.

Similarly, it is impossible that $s(\omega^0, a) \geq \lambda$. We also know that $\omega^{r+1} \; P_\lambda \; a$. Hence, each alternative $a \in A$ is assigned by ETRI-C-a to one of the categories $C^1, C^2, \ldots, C^r$. •

3 Relations between ELECTRE TRI-B and ELECTRE TRI-C

We have seen that Almeida-Dias et al. (2010) have introduced a variant of ELECTRE TRI, called ELECTRE TRI-C, that aims at sorting alternatives between ordered categories using, for each category, a profile that is supposed to be "central", as opposed to the limiting profiles used in ELECTRE TRI-B (Roy and Bouyssou, 1993, Wei, 1992).

Because we have a variant using limiting profiles (ELECTRE TRI-B) and a variant using central profiles (ELECTRE TRI-C), a natural question arises. Suppose that a result has been obtained using ELECTRE TRI-B with limiting profiles. In between two consecutive limiting profiles, is it possible to find a central profile so that using ELECTRE TRI-C with this family of central profiles, we obtain a similar result? Of course, the converse question may also be raised. Suppose that a result has been obtained using ELECTRE TRI-C with central profiles. In between two consecutive central profiles, is it possible to find a limiting profile so that using ELECTRE TRI-B with this family of limiting profiles, we obtain a similar result? These questions are not answered in Almeida-Dias et al. (2010) (Figueira, 2013, and Roy, 2013, indicate that this is because they felt that the two variants were really different. Almeida-Dias et al., 2010, sect. 6, study what happens if one applies ELECTRE TRI-B using the profiles defined for ELECTRE TRI-C keeping everything else unchanged).

Since there are two versions of each variant, there is a total of eight questions to be answered. For instance, starting with an ordered partition obtained with ETRI-C-a is it always possible to modify the profiles, while keeping all other parameters unchanged (i.e., the preference and indifference thresholds, the weights, the cutting level, the ordering and number of categories), so that applying ETRI-B-pc to the same problem leads to the same ordered partition? (the reader might be perplexed by the fact that in our statement of the problem we have apparently ignored the recommendation of Almeida-Dias et al. (2010) to use the two versions of ELECTRE TRI-C conjointly. We have nevertheless taken care of this difficulty. In all the examples detailed below, the two versions of ELECTRE TRI-C will lead to the same result).

It turns out that the answer to these eight questions is negative. As already mentioned, we do not view this fact as a criticism of ELECTRE TRI-C and/or ELECTRE TRI-B but as an indication that the relations between ELECTRE TRI-C and ELECTRE TRI-B are complex. Despite sharing a common name and apparently relying on similar principles, ELECTRE TRI-B and ELECTRE TRI-C seem to rest on somewhat different principles.

3.1 From ELECTRE TRI-C to ELECTRE TRI-B, pseudo-conjunctive

Because Bouyssou and Marchant (2007a,b) have given necessary and sufficient conditions for an ordered partition to be obtained with ETRI-B-pc, when preference and indifference thresholds are equal, this case is the easiest one. This also shows the power of axiomatic analysis for studying methods.

Our aim is to establish the following proposition.

Proposition 9
There are ordered partitions that can be obtained with ETRI-C-a and that cannot be obtained with ETRI-B-pc, after a suitable redefinition of the profiles. The same conclusion holds with ETRI-C-d instead of ETRI-C-a.

Proof
The proof consists in exhibiting suitable examples. Our first example has 6 criteria and two categories. For all criteria, in order to simplify the presentation, the preference and indifference thresholds are both null. The weights are denoted by w_i. We suppose that, on all criteria, the veto thresholds have been chosen so as to have no effect. The parameter λ is taken to be 0.6. H is the central profile of the High category ($\mathscr{H}$) and L is the central profile of the Low category ($\mathscr{L}$). The main parameters that are used are presented in Table 1. The evaluation of the alternatives that are to be sorted are given in Table 2. Applying ELECTRE TRI-C to this example leads to the valued relation in Table 3.

	g_1	g_2	g_3	g_4	g_5	g_6
w_i	0.16	0.17	0.19	0.16	0.16	0.16
H	75	75	75	75	75	75
L	25	25	25	25	25	25

Table 1: Main parameters used in the example

	g_1	g_2	g_3	g_4	g_5	g_6
a	100	75	100	25	25	25
b	50	100	100	50	25	25
c	50	75	100	25	25	25
d	0	100	100	50	25	25

Table 2: Alternatives to be sorted

Given that $\lambda = 0.6$, we obtain with ETRI-C-d that

$$a \in \mathscr{H}, b \in \mathscr{H}, c \in \mathscr{L}, d \in \mathscr{L}.$$

Indeed, we have $s(a, H) = 0.52 < \lambda = 0.6$ and $s(a, L) = 1.0 \geq \lambda = 0.6$. Alternative a will be assigned to $\mathscr{H}$ if $\rho(a, H) = \min(s(a, H), s(H, a)) > \rho(a, L) = \min(s(a, L), s(L, a))$. Using Table 3, we obtain $\rho(a, H) = 0.52 > \rho(a, L) = 0.48$.

Similarly, we obtain with ETRI-C-a that

$$a \in \mathscr{H}, b \in \mathscr{H}, c \in \mathscr{L}, d \in \mathscr{L}.$$

Indeed, we have $s(L, a) = 0.48 < \lambda = 0.6$ and $s(H, a) = 0.65 \geq \lambda = 0.6$. Alternative a will be assigned to $\mathscr{H}$ if $\rho(a, H) = \min(s(a, H), s(H, a)) > \rho(a, L) = \min(s(a, L), s(L, a))$. Using Table 3, we obtain $\rho(a, H) = 0.52 > \rho(a, L) = 0.48$.

Hence, in our example, the assignment of the four alternatives is identical using the ascending or the descending versions of ELECTRE TRI-C.

Using the analysis in Bouyssou and Marchant (2007a,b), it is now easy to see that these assignments cannot be obtained using ETRI-B-pc. Indeed, it is shown

	$s(\cdot, H)$	$s(H, \cdot)$	$\rho(\cdot, H)$	$s(\cdot, L)$	$s(L, \cdot)$	$\rho(\cdot, L)$
a	0.52	0.65	0.52	1.00	0.48	0.48
b	0.36	0.64	0.36	1.00	0.32	0.32
c	0.36	0.81	0.36	1.00	0.48	0.48
d	0.36	0.64	0.36	0.84	0.48	0.48

Table 3: Valued relation s

in these papers that a necessary condition for assignments to be obtained with ETRI-B-pc, when the preference and indifference thresholds are equal and there is no veto involved, is that:

$$\left.\begin{array}{c}(x_i,\alpha_{-i}) \in \mathscr{H} \\ \text{and} \\ (y_i,\beta_{-i}) \in \mathscr{H}\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(x_i,\beta_{-i}) \in \mathscr{H} \\ \text{or} \\ (z_i,\alpha_{-i}) \in \mathscr{H},\end{array}\right.$$

with the understanding that (x_i, α_{-i}) denotes the vector of evaluations of an alternative such that this vector is at x_i on criterion $i \in N$ and the evaluations α_{-i} on criteria other than i.

Taking $i = 1$, $x_i = 50$, $y_i = 100$, $z_i = 0$, $\alpha_{-i} = (100, 100, 50, 25, 25)$, $\beta_{-i} = (75, 100, 25, 25, 25)$, we obtain :

$$\begin{array}{c}b = (50, 100, 100, 50, 25, 25) \in \mathscr{H} \\ a = (100, 75, 100, 25, 25, 25) \in \mathscr{H}\end{array} \quad \text{and} \quad \begin{array}{c}c = (50, 75, 100, 25, 25, 25) \in \mathscr{L} \\ d = (0, 100, 100, 50, 25, 25) \in \mathscr{L}.\end{array}$$

This completes the proof. □

The above proposition shows that there are assignments that can be obtained with ETRI-C-a (or ETRI-C-d) that cannot be obtained with ETRI-B-pc after a suitable redefinition of the profiles. In fact the above proof shows more. Because the assignment obtained with ETRI-C-a (or ETRI-C-d) violates the necessary condition obtained in Bouyssou and Marchant (2007a,b), we could strengthen the above proposition allowing, not only for a change in the profiles, but also for a change of the indifference and preference thresholds (provided they remain equal, since the results in Bouyssou and Marchant (2007a,b) only cover this case), a change in the weights and a change in the cutting level λ. Indeed, in the conjoint measurement approach used in Bouyssou and Marchant (2007a,b), the condition exhibited above is necessary for a partition to be obtained with ETRI-B-pc, whatever the indifference and preference thresholds (provided they are equal), the weights, the cutting level and the limiting profile between $\mathscr{H}$ and $\mathscr{L}$.

Observe finally that, allowing for veto effects in ETRI-B-pc, would not change the conclusion of the above proposition, as long as we suppose, as in Bouyssou and Marchant (2007a,b), that veto effects occur in an all or nothing way, i.e., taking the valued relation to 0 as soon as there is a veto effect and keeping it unchanged otherwise. Indeed, in this case, a necessary condition for a partition to be obtained with ETRI-B-pc is that (using the notation presented in the above proof)

$$\left.\begin{array}{c}(x_i,\alpha_{-i}) \in \mathscr{H} \\ \text{and} \\ (z_i,\gamma_{-i}) \in \mathscr{H} \\ \text{and} \\ (y_i,\beta_{-i}) \in \mathscr{H}\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(x_i,\beta_{-i}) \in \mathscr{H} \\ \text{or} \\ (z_i,\alpha_{-i}) \in \mathscr{H}.\end{array}\right.$$

Observing that ETRI-C-a and ETRI-C-d with the parameters used above lead to assign the alternative $(0, 100, 100, 100, 100, 100)$ in category $\mathscr{H}$ shows that the impossibility to represent the information obtained using ETRI-B-pc is not due to the fact that an evaluation of 0 on g_1 prevents any alternative having such an evaluation from belonging to $\mathscr{H}$.

3.2 From ELECTRE TRI-C to ELECTRE TRI-B, pseudo-disjunctive

This is a more difficult case since the analysis in Bouyssou and Marchant (2007a,b) does not apply to ETRI-B-pd. Indeed, they have shown that the logic underlying ETRI-B-pd seems to be different from the one underlying ETRI-B-pc. This difference between the two versions of ELECTRE TRI-B has also been observed in many works dedicated to the elicitation of parameters in ELECTRE TRI-B that mainly concentrate on ETRI-B-pc.

Our aim is to establish the following proposition.

Proposition 10
There are ordered partitions that can be obtained with ETRI-C-a and that cannot be obtained with ETRI-B-pd, after a suitable redefinition of the profiles. The same conclusion holds with ETRI-C-d instead of ETRI-C-a.

Proof
We consider the same example as in the proof of Proposition 9. We have with both ETRI-C-a and ETRI-C-d,

$$a \in \mathscr{H}, b \in \mathscr{H}, c \in \mathscr{L}, d \in \mathscr{L}.$$

Let π be the limiting profile between $\mathscr{H}$ and $\mathscr{L}$ used in ETRI-B-pd. Let us show that the only way to make the above assignments compatible with ETRI-B-pd is to choose $g_1(\pi) = 50$.

With ETRI-B-pd, we have,

$$\begin{aligned} x \in \mathscr{H} &\Leftrightarrow Not[\pi\ P_\lambda\ x], \\ x \in \mathscr{L} &\Leftrightarrow \pi\ P_\lambda\ x. \end{aligned}$$

We have

$$\begin{array}{lcr} b = (50, 100, 100, 50, 25, 25) \in \mathscr{H} & \text{and} & c = (50, 75, 100, 25, 25, 25) \in \mathscr{L} \\ a = (100, 75, 100, 25, 25, 25) \in \mathscr{H} & & d = (0, 100, 100, 50, 25, 25) \in \mathscr{L}. \end{array}$$

Hence we know that

$$\begin{gathered} Not[\pi\ P_\lambda\ (x_i, \alpha_{-i})] \text{ and } Not[\pi\ P_\lambda\ (y_i, \beta_{-i})] \\ \pi\ P_\lambda\ (x_i, \beta_{-i}) \text{ and } \pi\ P_\lambda\ (z_i, \alpha_{-i}). \end{gathered}$$

together with

$$\pi\ P_\lambda\ (x_i, \beta_{-i}) \text{ and } \pi\ P_\lambda\ (z_i, \alpha_{-i})$$

Suppose that $Not[\pi\ S_\lambda\ (x_i, \alpha_{-i})]$ and $Not[\pi\ S_\lambda\ (y_i, \beta_{-i})]$. Because $\pi\ S_\lambda$ (z_i, α_{-i}), $Not[\pi\ S_\lambda\ (x_i, \alpha_{-i})]$ implies that π_i must be strictly below x_i and greater than or equal to z_i. Because $\pi\ S_\lambda\ (x_i, \beta_{-i})$, $Not[\pi\ S_\lambda\ (y_i, \beta_{-i})]$ implies that π_i must be strictly below y_i and greater than or equal to x_i. This is clearly impossible.

Suppose that $(x_i, \alpha_{-i})\ S_\lambda\ \pi$ and $(y_i, \beta_{-i})\ S_\lambda\ \pi$. Because $(x_i, \alpha_{-i})\ S_\lambda\ \pi$ and $Not[(z_i, \alpha_{-i})\ S_\lambda\ \pi]$, π_i must be strictly above z_i and less than or equal to x_i. Because $(y_i, \beta_{-i})\ S_\lambda\ \pi$ and $Not[(x_i, \beta_{-i})\ S_\lambda\ \pi]$, π_i must be strictly above x_i and less than or equal to y_i. This is clearly impossible.

Suppose that $Not[\pi\ S_\lambda\ (x_i, \alpha_{-i})]$ and $(y_i, \beta_{-i})\ S_\lambda\ \pi$. Because $\pi\ S_\lambda\ (z_i, \alpha_{-i})$ and $Not[\pi\ S_\lambda\ (x_i, \alpha_{-i})]$, π_i must be strictly below x_i and greater than or equal to z_i. Because $(y_i, \beta_{-i})\ S_\lambda\ \pi$ and $Not[(x_i, \beta_{-i})\ S_\lambda\ \pi]$, π_i must be strictly above x_i and less than or equal to z_i. This is clearly impossible.

Suppose finally that $(x_i, \alpha_{-i})\ S_\lambda\ \pi$ and $Not[\pi\ S_\lambda\ (y_i, \beta_{-i})]$ Because $(x_i, \alpha_{-i})\ S_\lambda$ π and $Not[(z_i, \alpha_{-i})\ S_\lambda\ \pi]$, π_i must be strictly above z_i and less than or equal to x_i. Because $\pi\ S_\lambda\ (x_i, \beta_{-i})$, $Not[\pi\ S_\lambda\ (y_i, \beta_{-i})]$ implies that π_i must be strictly below y_i and greater than or equal to x_i. This is the only possible case and we must have that π_i is equal to $x_i = 50$.

Let us now show that we must have $\pi_i = 50$, for all $i \in N$. We already know that $\pi_1 = 50$. Because the weight of criteria $w_1 = w_4 = w_5 = w_6 = 0.16$, it is clear, exchanging the rôles of g_1 and g_i with $i = 4, 5, 6$, that the above example also shows that we must have $\pi_1 = \pi_4 = \pi_5 = \pi_6 = 50$. The situation is slightly more difficult with g_2 and g_3 since their weights are different from the weight of g_1. Nevertheless the same example also works.

Indeed, for g_2, it suffices to consider the following alternatives:

	g_1	g_2	g_3	g_4	g_5	g_6
a'	75	100	100	25	25	25
b'	100	50	100	50	25	25
c'	75	50	100	25	25	25
d'	100	0	100	50	25	25

The reader will easily check that we have

$$a' \in \mathscr{H}, b' \in \mathscr{H}, c' \in \mathscr{L}, d' \in \mathscr{L},$$

with both ETRI-C-a and ETRI-C-d.

Similarly, for g_3, it suffices to consider the following alternatives:

	g_1	g_2	g_3	g_4	g_5	g_6
a''	100	75	100	25	25	25
b''	100	100	50	50	25	25
c''	100	75	50	25	25	25
d''	100	100	0	50	25	25

The reader will easily check that we have

$$a'' \in \mathscr{H}, b'' \in \mathscr{H}, c'' \in \mathscr{L}, d'' \in \mathscr{L},$$

with both ETRI-C-a and ETRI-C-d.

Hence, if the partition is to be represented using ETRI-B-pd, it must be true that $\pi = (50, 50, 50, 50, 50, 50)$.

Let us finally show that this leads to a contradiction. Consider an alternative e^i that has the evaluation 60 on all criteria except on a criterion i on which its evaluation is 25. The reader will easily check that all such alternatives e^i are assigned to $\mathscr{L}$ with both ETRI-C-a and ETRI-C-d. Since we now know the profile in ETRI-B-pd, it is easy to check that this leads to a contradiction. Indeed, we must have that $\pi \; P_\lambda \; e^i$, for all $i \in N$. This implies that, for all $i \in N$, we have:

$$w_i \geq \lambda \text{ and } \sum_{j \neq i} w_j < \lambda.$$

Because $\lambda \in [0, 1]$ and the weights sum up to 1, this is contradictory. This completes the proof. □

3.3 From ELECTRE TRI-B to ELECTRE TRI-C

This is also a difficult case since we do not have at hand an analysis similar to that of Bouyssou and Marchant (2007a,b) for ELECTRE TRI-C. In order to keep things simple, we only analyze this case under the following two assumptions.

1. If π is the limiting profile used in ELECTRE TRI-B to separate two categories, in ELECTRE TRI-C the central profiles of the higher category (ω^k) strictly dominates π which in turn strictly dominates the central profile of the lower category (ω^{k-1}). This seems an innocuous requirement that ensures a minimal semantic consistency between the profile that are manipulated.

2. We require that *hyper-strict separability* holds in the sense of Almeida-Dias et al. (2010, Condition 3), i.e., that, on all criteria, the central profile of the higher category is strictly preferred to the central profile of the lower category. Although this hypothesis is not completely necessary to prove the proposition below, it facilitates things greatly and appears to be quite reasonable.

Our aim is to establish the following proposition.

Proposition 11
Under the hypotheses made above, there are ordered partitions that can be obtained with ETRI-B-pc and that cannot be obtained with ETRI-C-a, after a suitable re-definition of the profiles. The same conclusion holds with ETRI-B-pd instead of ETRI-B-pc. The above two conclusions hold with ETRI-C-d, instead of ETRI-C-a.

PROOF
Our example has 11 criteria and two categories. For all criteria, in order to simplify the presentation, the preference and indifference threshold are both null. The weight of all criteria is equal to 1/11. We suppose that, on all criteria, the veto thresholds have been chosen so as to have no effect. The cutting level λ is taken to be 0.52.

The limiting profile π between the two categories is at 10 on all criteria. Consider now the alternative $a = (0, 0, 0, 0, 10 + \alpha, 10 + \alpha, 10 + \alpha, 10 + \alpha, 10 + \alpha, 10 - \varepsilon, 10 - \varepsilon)$ with $\alpha, \varepsilon > 0$. We have $s(\pi, a) = 0.55$ and $s(a, \pi) = 0.45$. This implies $\pi \ P_\lambda \ a$. Hence, alternative a is assigned to the lower category with either ETRI-B-pc or ETRI-B-pd.

We denote the central profile of the higher (resp. lower) category by H (resp. L). Our hypotheses imply that, for all $i \in N$, we have $g_i(H) > 10 > g_i(L)$. Hence, we may choose α, ε to be such that $10 + \alpha > g_i(H)$, $i = 5, 6, 7, 8, 9$, and $10 - \varepsilon > g_j(L)$, $j = 10, 11$. Whatever the choice of $g_i(H)$ and $g_i(L)$ compatible with our hypotheses, we obtain: $s(H, a) = 1.0$, $s(a, H) = 0.45$, $s(L, a) = 0.36$, $s(a, L) = 0.64$. Hence, with ETRI-C-a and with ETRI-C-d, alternative a is assigned to the higher category. This completes the proof. □

4 Transposition and the two versions of ELECTRE TRI-B

We have seen that ELECTRE TRI-C comes in two versions. Almeida-Dias et al. (2010) have proposed a very clever argument to justify the need for these two versions and the necessity to use theses two versions *conjointly*. It is based on the *transposition operation*. This operation consists in inverting the direction of preference on all criteria and in inverting the ordering of the categories. Clearly, the conclusions obtained after this transposition operation should be the same as the original conclusions (provided that these new conclusions are reinterpreted with the original ordering of categories). Technically, the effect of the transposition operation is to transform $s(a, b)$ into $s(b, a)$ and vice versa.

Almeida-Dias et al. (2010) have shown that the two versions of ELECTRE TRI-C correspond via the transposition operation. It is indeed easy to check that applying ETRI-C-a to a problem after it has been transposed amounts to applying ETRI-C-d, to the original problem. Similarly, applying ETRI-C-d to a problem after it has been transposed amounts to applying ETRI-C-a to the original problem. This gives very good grounds to justify the proposition of two versions of ELECTRE TRI-C and to require that they should be used conjointly.

When ELECTRE TRI-B was proposed (Roy and Bouyssou, 1993, Wei, 1992), its two versions (ETRI-B-pc and ETRI-B-pd) were not justified by appealing to the transposition operation. As their names suggest, the main argument was that when comparing an alternative to a profile one may wish to do in a more or less conjunctive or disjunctive fashion (it is not completely clear whether or not the two versions should or not be used conjointly: in the words of Roy and Bouyssou (1993, p. 390), ELECTRE TRI-B is the "jumelage" of ETRI-B-pc and ETRI-B-pd). The later literature on ELECTRE TRI-B has concentrated on ETRI-B-pc and the two versions of ETRI-B are, in general, not used conjointly. Most of the real-world applications of ELECTRE TRI-B use the pseudo-conjunctive version. It is the only one to have received an axiomatic analysis (Bouyssou and Marchant, 2007a,b). It is also the only one for which elicitation techniques have been widely developed (see Cailloux et al., 2012, Damart et al., 2007, Dias and Clímaco, 2000, Dias and Mousseau, 2003, 2006, Dias et al., 2002, Leroy et al., 2011, Mousseau and Dias, 2004, Mousseau and Słowiński, 1998, Mousseau et al., 2000, 2001, 2003, Ngo The and Mousseau, 2002).

In a regrettably unpublished paper, Roy (2002) was the first to observe that the two versions of ELECTRE TRI-B are not linked via the transposition operation. He nevertheless observed that when applying the transposition operation to ETRI-B-pc one obtains a method that is "close" to ETRI-B-pd and vice versa. The purpose of this section is to analyze further the behavior of ELECTRE TRI-B with respect to the transposition operation.

Roy and Bouyssou (1993, p. 315) have proposed to analyze ELECTRE TRI-B considering six requirements: *uniqueness* (each alternative is assigned to a unique category), *independence* (the assignment of an alternative does not depend on the assignment of other alternatives), *conformity* (if $\pi^{k+1}\ P_\lambda\ a$ and $a\ P_\lambda\ \pi^k$ the alternative a is assigned to C^k. Moreover, if $a\ I_\lambda\ \pi^k$, then a is assigned to C^k), *monotonicity* (if $a\ \Delta\ b$, then a is assigned to a category that is at least as good as the category to which b is assigned), *homogeneity* (alternatives comparing similarly to all profiles must be assigned to the same category), *stability* (the assignment of the alternatives should be consistent with the merging or the splitting of categories via the suppression of limiting profiles or the addition of new ones). Roy and Bouyssou (1993, p. 398–399) have shown that both ETRI-B-pc, and ETRI-B-pd, satisfy these six requirements. The only difficulty lies in the last part of the conformity requirement asking that an alternative that is indifferent to the limiting profile π^k should be assigned to C^k. Indeed, it can happen that an alternative is indifferent to *several* consecutive profiles, in which case it is clearly impossible to satisfy the conformity requirement. Roy and Bouyssou (1993, p. 395) have introduced an additional condition, called "compatibility" requiring that if an alternative is indifferent to a profile it cannot be indifferent to other profiles. With this additional condition, the conformity requirement is satisfied by both versions of ELECTRE TRI-B. Roy and Bouyssou (1993, p. 396) nevertheless state that ELECTRE TRI-B can still be used in cases in which this additional condition fails.

Most of our discussion will be centered around the second part of the conformity requirement. At first sight, it seems perfectly sensible. Roy and Bouyssou (1993) have motivated this condition considering the case of an assignment method that is based on the aggregation of the n criteria into a single one via a function $V(g_1, g_2, \dots, g_n)$. In such a case, in order to sort alternatives, one chooses $r + 2$ thresholds $\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_r, \lambda_{r+1}$ such that $\lambda_1 < \lambda_2 < \dots < \lambda_r$ and with λ_0 (resp. λ_{r+1}) arbitrarily low (resp. high). We can then use the following sorting rule:

$$a \in C^k \Leftrightarrow \lambda_k \leq V(g_1(a), g_2(a), \dots, g_n(a)) < \lambda_{k+1}.$$

Clearly, with such a rule, the position of the non-strict inequality is purely conventional. The chosen convention is here to have all categories "closed below". This is the same convention that is at work in the second part of the conformity requirement. Once this convention has been settled, one can give a precise meaning to the profiles. They act as the lower limit of the categories, with the understanding that, at the lower limit, an alternative still belongs the category.

The analogy with a method that is based on the aggregation of all criteria into one can be misleading however. In this situation, the case of an alternative $a \in A$ such that $V(g_1(a), g_2(a), \dots, g_n(a))$ is equal to one of the thresholds λ_k is clearly exceptional (at least when the function V is taken to be strictly increasing in all its arguments as, e.g., in the UTADIS method, see Greco, Mousseau, and Słowiński, 2010, Jacquet-Lagrèze, 1995, Zopounidis and Doumpos, 2000). Therefore, in this case, the convention of taking the categories to be closed below (or closed above) has hardly any practical consequences. This is not the case when using a preference model such as the one used in ELECTRE TRI-B. Indeed, with the preference model used in ELECTRE TRI-B, we cannot suppose anymore that indifference to a profile is exceptional and that the convention of taking categories to be closed below has no practical consequences. Indifference to a profile is not an exceptional case with ELECTRE TRI-B.

Moreover, the last part of the conformity requirement conflicts with the transposition operation. If a method satisfies the last part of the conformity requirement, applying the transposition operation leads to a method in which categories are closed above instead of being closed below. This conflict was first observed in Roy (2002). Roy (2002) concludes in favor of the satisfaction of the last part of conformity requirement rather than with in favor of the consistency with respect to transposition operation. Although the text is brief on this point, we suspect that this has mainly to do with the definition of the profiles. In order to define the profiles, it is clearly useful to know to what category an alternative that is exactly identical to the profile should belong to. Although we recognize the strength of this argument, we consider the conflict between the conformity requirement and the transposition operation to be troublesome. The strict adherence to the conformity requirement together with the use of a preference model in which being indifferent to a profile is not exceptional have consequences that are not easy to justify.

Consider for instance a situation in which alternatives are evaluated on 11 criteria (on a 0 to 100 scale) and are to be sorted into two ordered categories. Suppose that, on each criterion, the preference and indifference thresholds are both equal to 11 and that the veto thresholds have been chosen to be large enough so as to play no rôle. Suppose furthermore that all criteria have an equal weight and that λ is taken to be 0.52. If the limiting profile separating the two categories is at 50 on all criteria, we will conclude with both versions of ELECTRE TRI-B that an alternative that is evaluated as $(40, 40, 40, 40, 40, 40, 40, 30, 30, 30, 30)$ will belong to the higher category since it is indifferent to the profile. This shows that indifference in the preference model used in ELECTRE TRI-B can be quite "thick". We say that *a strongly dominates b* if $g_i(a) > g_i(b)$, for all $i \in N$, which we denote by $a \; \Delta^* \; b$. The above remark about the "thickness" of indifference implies that both versions of ETRI-B violate a requirement stating that

$$\pi^{k+1} \; \Delta^* \; a \; \Delta^* \; \pi^k \Rightarrow a \in C^k.$$

In our view, this is the sign that the price to pay for a strict adherence to the last part of the conformity requirement is quite high: the choice between categories closed below or closed above is not as conventional with the preference model used in ELECTRE TRI-B as it is with other preference models.

In order to clarify this point further, it will be useful to consider two new versions of ELECTRE TRI-B called dual of pseudo-conjunctive and dual of pseudo-disjunctive (often abbreviated as ETRI-B-dpc and ETRI-B-dpd).

Definition 12 (ELECTRE TRI-B, dual of pseudo-conjunctive)
Increase k from 1 until the first value k such that $\pi^k \; S_\lambda \; a$. Assign alternative a to C^{k-1}.

Definition 13 (ELECTRE TRI-B, dual of pseudo-disjunctive)
Decrease k from $r+1$ until the first value k such that $a \; P_\lambda \; \pi^k$. Assign alternative a to C^{k-1}.

It is easy to check that ETRI-B-pc, and ETRI-B-dpc, corresponds via the transposition operation. The same is true for ETRI-B-pd, and ETRI-B-dpd. The two variants ETRI-B-dpc, and ETRI-B-dpd just introduced share the same properties as the original versions of ETRI-B, except that for the last part of the conformity principle. With these two variants, categories are now closed above.

We have seen above that the choice of categories closed above or closed below was not purely conventional in ELECTRE TRI-B. To deal with this problem, we suggest to view a limiting profile as a *frontier*: at the frontier, we have good reasons to be in the upper category but we also have equally good reasons to be in the lower category. This conception of profiles is more complex than the original one. It nevertheless has the advantage of explicitly recognizing the difficulty of combining a strict convention (closed above or closed below) together with a preference model in which indifference situations are not exceptional. In particular, in our view, in the case evoked above in which $\pi^{k+1} \; \Delta^* \; a \; \Delta^* \; \pi^k$, the possibility of assigning a to C^k should clearly not be discarded.

This is why we suggest the following new variant of ELECTRE TRI-B. It consists in using ETRI-B-pc and ETRI-B-dpc conjointly[2].

Our proposition has advantages and disadvantages. Before analyzing them, let us stress that our proposal amounts to abandon the last part of the conformity requirement in favor of consistency with the transposition operation. Let us also point out that we only modify ELECTRE TRI-B in the case in which an alternative is indifferent to one or several profiles. Whereas the two versions of ELECTRE TRI-B assigned such an alternative to the same category (i.e., the highest category k such that $a \; S_\lambda \; \pi^k$), it is assigned by ETRI-B-pc to the highest category k such that $a \; S_\lambda \; \pi^k$ and by ETRI-B-dpc to the lowest category ℓ such that $\pi^\ell \; S_\lambda \; a$.

Let us first begin by pointing out why this proposal may not be fully satisfactory.

1. Clearly our proposal rests on a conception of profiles that is more complex than the original ones. Profiles are here at the frontier between two categories and may belong to each of them. This is a clear drawback. Let us nevertheless stress that in many real-world applications, the definition of the profiles is done via an elicitation technique, on the basis of assignment examples. In such a case, the complex nature of the profiles is not a real issue. Moreover, this conception allows us to ensure consistency with respect to the transposition operation and to avoid situations in which an alternative

[2]We could have instead chosen to use ETRI-B-pd and ETRI-B-dpd conjointly. Our choice is motivated here by the fact that the construction of the outranking relation focuses on an "at least as good as" relation instead of a "strictly better than" one. This is also in line with the analysis in Bouyssou and Marchant (2007a,b).

is strongly dominated by a limiting profile while being, by both versions, assigned to the category that is limited below by this profile.

2. In ELECTRE TRI-B, an alternative is assigned by ETRI-B-pc, to a category that is never higher than the category to which it is assigned by ETRI-B-pd. This is no more the case with our proposal. While, when an alternative is incomparable to a set of consecutive profiles, the assignment with ETRI-B-pc is always lower than the assignment with ETRI-B-dpc, the situation is now reversed when an alternative is indifferent to several consecutive profiles. This explains why we consider that the two versions should be used conjointly.

Let us now try to explain why our proposal may be of some interest.

1. First, observe that the proposed variant is only marginally different from the original one. They only differ in the case of indifference with one or several consecutive profiles. In this case, instead of sticking to a convention that has far reaching consequences, we exploit the fact that the alternative may be assigned in different ways. Observe that, when applying ETRI-B-pc and ETRI-B-pd conjointly, each alternative is assigned to an interval of categories. When we do the same with ETRI-B-pc, and ETRI-B-dpc, we obtain an interval that is never smaller than the preceding one. This interval is identical to the first one, except when an alternative is indifferent to one or several profiles. In this case, the interval obtained with our variant is wider than the interval obtained with ELECTRE TRI-B (in which case, the interval is reduced to a single category). Therefore, our variant appears to be more "cautious" than the original one by explicitly pointing out the ambiguity resulting from being indifferent to a profile (or several consecutive profiles).

2. Second, observe that, using our variant, when $\pi^{k+1} \; \Delta^* \; a \; \Delta^* \; \pi^k$, it is always true, contrary to what happens with ELECTRE TRI-B, that alternative a can be assigned to C^k by one version of our variant.

3. Third, the new variant is fully compatible with the transposition operation. If the consistency with respect to this operation is crucial for ELECTRE TRI-C, it is hard to understand why it should be abandoned with ELECTRE TRI-B.

4. Fourth, our variant only makes use of S_λ and does not appeal to P_λ. On a theoretical level, we view this as an advantage. Indeed, Bouyssou and Pirlot (2009, 2013) have shown that the properties of P_λ are significantly different from the ones of S_λ. They are also far more complex and difficult to analyze.

This should not be a surprise since the preference model used in ELECTRE methods concentrates on "at least as good as" and not on "strictly better than" relations (for an outranking method concentrating on strict preference, see Vansnick, 1986).

5. Fifth, since ETRI-B-dpc is nothing but ETRI-B-pc applied to a transposed problem, all properties of ETRI-B-pc are shared by ETRI-B-dpc. Indeed, ETRI-B-dpc can be viewed as ETRI-B-pc with a different choice of parameters. This shows that the axiomatic analysis of Bouyssou and Marchant (2007a,b) applies *without any change* to ETRI-B-dpc. Moreover the elicitation techniques developed for ETRI-B-pc also apply *without any change* to ETRI-B-dpc. This last point is major advantage since the elicitation techniques that were developed for ETRI-B-pd, are rare and far more complex than the ones developed for ETRI-B-pc (they use integer linear programming instead of linear programming).

5 Conclusion

This paper has analyzed several aspects of ELECTRE TRI-B, motivated by the recent proposal of ELECTRE TRI-C.

We have first shown that the relations between ELECTRE TRI-B and ELECTRE TRI-C are complex. Indeed, we have shown that there are ordered partitions that can be obtained with ELECTRE TRI-B and that cannot be obtained with ELECTRE TRI-C and vice versa. This is not intended to be a criticism of either ELECTRE TRI-B or ELECTRE TRI-C. Instead, we view this fact as the sign that the situation with central profiles is less similar to the situation with limiting profiles as one might think (Figueira, 2013 and Roy, 2013 insist on this point). This clearly calls for further research. In particular, three points would deserve further work. First, it would be nice to know whether it is possible to propose a method working with central profiles that would have simpler relations with ELECTRE TRI-B, in the sense explored in Section 3. Second, it would be interesting to investigate the theoretical properties of ELECTRE TRI-C, mimicking what Bouyssou and Marchant (2007a,b) have done with ELECTRE TRI-B. Finally, ELECTRE TRI-C has been introduced without the proposal of specific elicitation techniques. This is clearly an important direction for future research (see Almeida-Dias, 2011, Ch. 7, for preliminary results on the subject).

We have also used the transposition operation introduced by Almeida-Dias et al. (2010) to suggest an original variant of ELECTRE TRI-B. The interest and usefulness of this variant should be carefully analyzed. We have already shown that it has, compared to the original ELECTRE TRI-B, advantages with respect to the existence of elicitation techniques and with respect to its theoretical properties.

Appendix

A Construction of the outranking relation

This section details the construction of the outranking relation in ELECTRE TRI. It is supposed that all criteria are pseudo-criteria, in the sense of Roy (1996). This means that we associate with each criterion $i \in N$ two thresholds: a preference threshold p_i and an indifference threshold q_i. Formally, p_i and q_i are functions from $\mathbb{R}$ into $\mathbb{R}^+$ such that, for all $a, b \in A$, $q_i(g_i(a)) \leq p_i(g_i(a))$ and $g_i(a) \geq g_i(b)$ implies that $g_i(a)+q_i(g_i(a)) \geq g_i(b)+q_i(g_i(b))$ and $g_i(a)+p_i(g_i(a)) \geq g_i(b)+p_i(g_i(b))$. The interpretation of these functions is as follows. If the value $g_i(a) - g_i(b)$ is positive but less than $q_i(g_i(b))$, it is supposed that this difference is insignificant given the way g_i has been built and that, on this criterion, the two alternatives should be considered indifferent. If $g_i(a) - g_i(b)$ is positive and larger than $p_i(g_i(b))$, it is supposed that the difference is significant and that, on this criterion, a is strictly preferred to b. The intermediate situation is supposed to model an hesitation between indifference and strict preference.

Each criterion $i \in N$ is assigned a weight w_i. We suppose, w.l.o.g., that weights have been normalized so that $\sum_{i=1}^n w_i = 1$.

The above information is used to define, on each criterion $i \in N$, a valued relation on A, i.e., a function from $A \times A$ into $[0, 1]$, called the partial concordance relation such that:

$$c_i(a,b) = \begin{cases} 1 & \text{if } g_i(b) - g_i(a) \leq q_i(g_i(a)), \\ 0 & \text{if } g_i(b) - g_i(a) > p_i(g_i(a)), \\ \frac{p_i(g_i(a)) - (g_i(b) - g_i(a))}{p_i(g_i(a)) - q_i(g_i(a))} & \text{otherwise.} \end{cases}$$

The definition of the valued relation c_i is illustrated in Figure 3.

The valued relations c_i are then aggregated into a single valued concordance relation c letting, for all $a, b \in A$,

$$c(a,b) = \sum_{i=1}^{n} w_i c_i(a,b),$$

The definition of the valued concordance relation c mainly rests on "ordinal considerations" and does not make central use of the idea of "preference difference". In order to prevent an alternative a to be preferred to an alternative b when there

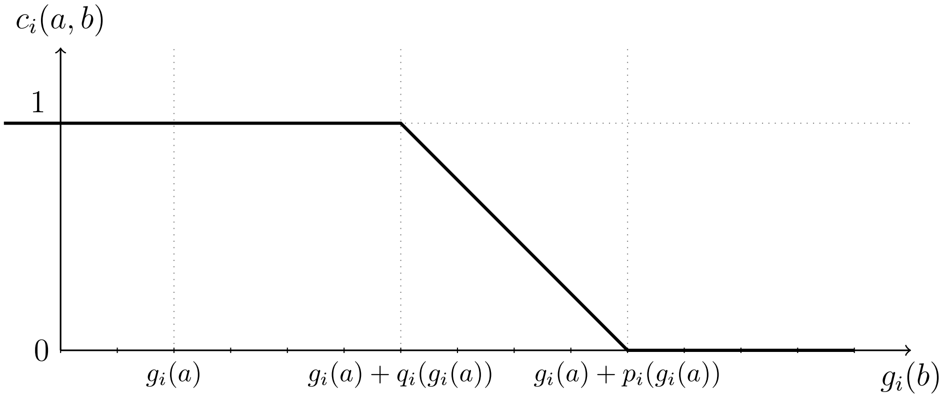

Figure 3: Computing $c_i(a,b)$

is a criterion on which there is a "very large" preference difference in favor of b, one makes use of the idea of discordance.

A veto threshold v_i is defined on each criterion $i \in N$. Formally, v_i is a function from $\mathbb{R}$ into $\mathbb{R}^+$ such that, for all $a, b \in A$, $p_i(g_i(a)) \leq v_i(g_i(a))$ and $g_i(a) \geq g_i(b)$ implies that $g_i(a) + v_i(g_i(a)) \geq g_i(b) + v_i(g_i(b))$. The interpretation is that if the value $g_i(b) - g_i(a)$ is larger than $v_i(g_i(a))$, this large preference difference in favor of b cannot be compensated and forbids to conclude that a is at least as good as b.

Formally the veto thresholds are used to define on each criterion $i \in N$ a valued relation on A, i.e., a function from $A \times A$ into $[0,1]$, called the partial discordance relation such that:

$$d_i(a,b) = \begin{cases} 1 & \text{if } g_i(b) - g_i(a) > v_i(g_i(a)), \\ 0 & \text{if } g_i(b) - g_i(a) \leq p_i(g_i(a)), \\ \frac{(g_i(b) - g_i(a) - p_i(g_i(a))}{v_i(g_i(a)) - p_i(g_i(a))} & \text{otherwise.} \end{cases}$$

The definition of the relation d_i is illustrated in Figure 4 The n relations d_i are aggregated into one discordance relation letting, for all $a, b \in A$

$$d(a,b) = \begin{cases} 0 & \text{if } D_{ab} = \{j \in N : d_j(a,b) > c(a,b)\} = \varnothing, \\ 1 - \prod_{i \in D_{ab}} \frac{1 - d_i(a,b)}{1 - c(a,b)} & \text{otherwise,} \end{cases}$$

On the basis of the relations c and d a valued outranking relation s is built letting, for all $a, b \in A$,

$$s(a,b) = c(a,b) \times (1 - d(a,b)).$$

It is not difficult to check that if the veto thresholds are chosen to be large enough, we have, for all $a, b \in A$, $s(a,b) = c(a,b)$.

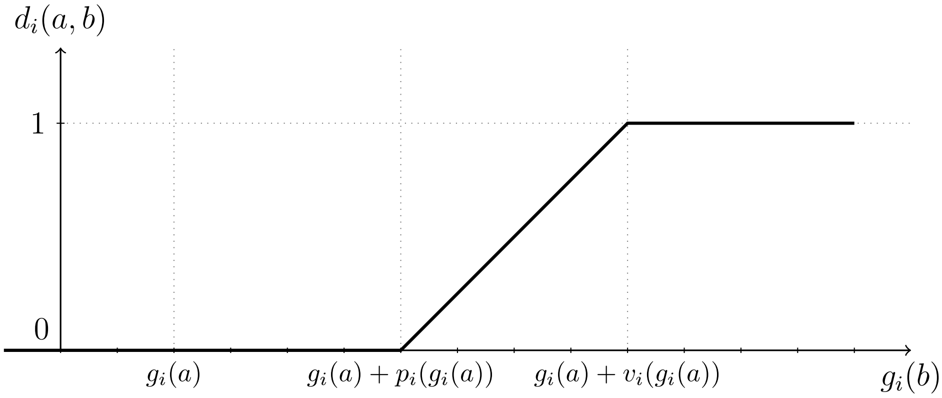

Figure 4: Computing $d_i(a, b)$

On the basis of the valued relation s a binary relation S on A is defined letting:

$$a \; S_\lambda \; b \Leftrightarrow s(a,b) \geq \lambda,$$

where $\lambda \in [0,1]$ is a cutting level (usually taken to be above 1/2).

References

J. Almeida-Dias. *Multiple criteria decision aiding for sorting problems: Concepts, methodologies, and applications.* PhD dissertation, Universidade Téchnica de Lisboa and Université Paris Dauphine, Lisbon and Paris, 2011.

J. Almeida-Dias, J. R. Figueira, and B. Roy. ELECTRE TRI-C: A multiple criteria sorting method based on characteristic reference actions. *European Journal of Operational Research*, 204(3):565–580, 2010.

J. Almeida-Dias, J. R. Figueira, and B. Roy. A multiple criteria sorting method where each category is characterized by several reference actions: The ELECTRE TRI-nC method. *European Journal of Operational Research*, 217(3):567–579, 2012.

D. Bouyssou. Outranking relations: Do they have special properties? *Journal of Multi-Criteria Decision Analysis*, 5(2):99–111, 1996.

D. Bouyssou and T. Marchant. An axiomatic approach to noncompensatory sorting methods in MCDM, I: The case of two categories. *European Journal of Operational Research*, 178(1):217–245, 2007a.

D. Bouyssou and T. Marchant. An axiomatic approach to noncompensatory sorting methods in MCDM, II: More than two categories. *European Journal of Operational Research*, 178(1):246–276, 2007b.

D. Bouyssou and M. Pirlot. Analyzing the correspondence between strict and non-strict outranking relations. Communication to EURO XXIII, Bonn, July 2009, EURO, 2009.

D. Bouyssou and M. Pirlot. On the relationship between strict and non-strict outranking relations. Working paper, LAMSADE, Université Paris Dauphine, 2013.

O. Cailloux, P. Meyer, and V. Mousseau. Eliciting ELECTRE TRI category limits for a group of decision makers. *European Journal of Operational Research*, 223(1):133–140, 2012.

S. Damart, L. C. Dias, and V. Mousseau. Supporting groups in sorting decisions: Methodology and use of a multi-criteria aggregation / disaggregation DSS. *Decision Support Systems*, 43(4):1464–1475, 2007.

L. C. Dias and J. Clímaco. ELECTRE TRI for groups with imprecise information on parameter values. *Group Decision and Negotiation*, 9(5):355–377, 2000.

L. C. Dias and V. Mousseau. IRIS: A DSS for multiple criteria sorting problems. *Journal of Multi-Criteria Decision Analysis*, 12:285–298, 2003.

L. C. Dias and V. Mousseau. Inferring ELECTRE's veto-related parameters from outranking examples. *European Journal of Operational Research*, 170(1):172–191, 2006.

L. C. Dias, V. Mousseau, J. Figueira, and J. Clímaco. An aggregation / disaggregation approach to obtain robust conclusions with ELECTRE TRI. *European Journal of Operational Research*, 138(2):332–48, 2002.

J. R. Figueira. Personal communication to the authors, 16 September 2013.

J. R. Figueira, V. Mousseau, and B. Roy. ELECTRE methods. In J. R. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple Criteria Decision Analysis: State of the Art Surveys*, number 28 in International Series in Operations Research & Management Science, pages 133–162. Springer Verlag, Boston, Dordrecht, London, 2005. ISBN 0-387-23067-X.

J. R. Figueira, S. Greco, B. Roy, and R. Słowiński. An overview of ELECTRE methods and their recent extensions. *Journal of Multi-Criteria Decision Analysis*, 20(1–2):61–85, 2013.

S. Greco, V. Mousseau, and R. Słowiński. Multiple criteria sorting with a set of additive value functionss. *European Journal of Operational Research*, 207(3):1455–1470, 2010.

É. Jacquet-Lagrèze. An application of the UTA discriminant model for the evaluation of R&D projects. In P. Pardalos, Y. Siskos, and C. Zopounidis, editors, *Advances in Multicriteria Analysis*, pages 203–211. Kluwer, Dordrecht, 1995.

A. Leroy, V. Mousseau, and M. Pirlot. Learning the parameters of a multiple criteria sorting method. In R. I. Brafman, F. S. Roberts, and A. Tsoukiàs, editors, *ADT 2011*, number 6992 in LNAI, pages 219–233. Springer, 2011.

V. Mousseau and L. C. Dias. Valued outranking relations in ELECTRE providing manageable disaggregation procedures. *European Journal of Operational Research*, 156(2):467–482, 2004.

V. Mousseau and R. Słowiński. Inferring an ELECTRE TRI model from assignment examples. *Journal of Global Optimization*, 12(2):157–174, 1998.

V. Mousseau, R. Słowiński, and P. Zielniewicz. A user-oriented implementation of the ELECTRE TRI method integrating preference elicitation support. *Computers & Operations Research*, 27(7–8):757–777, 2000.

V. Mousseau, J. Figueira, and J.-P. Naux. Using assignment examples to infer weights for ELECTRE TRI method: Some experimental results. *European Journal of Operational Research*, 130(2):263–275, 2001.

V. Mousseau, J. R. Figueira, L. C. Dias, C. Gomes da Silva, and J. Clímaco. Resolving inconsistencies among constraints on the parameters of an MCDA model. *European Journal of Operational Research*, 147(1):72–93, 2003.

V. Mousseau, L. C. Dias, and J. R. Figueira. Dealing with inconsistent judgments in multiple criteria sorting models. *4OR, A Quarterly Journal of Operations Research*, 4 (2):147–158, 2006.

A. Ngo The and V. Mousseau. Using assignment examples to infer category limits for the ELECTRE TRI method. *Journal of Multi-Criteria Decision Analysis*, 11(1):29–43, 2002.

B. Roy. *Multicriteria methodology for decision aiding.* Kluwer, Dordrecht, 1996. Original version in French: "*Méthodologie multicritère d'aide à la décision*", Economica, Paris, 1985.

B. Roy. Présentation et interprétation de la méthode ELECTRE TRI pour affecter des zones dans des catégories de risque. Document du LAMSADE 124, LAMSADE, Université Paris Dauphine, Paris, France, March 2002. 25 pages.

B. Roy. Personal communication to the authors, 1 October 2013.

B. Roy and D. Bouyssou. *Aide multicritère à la décision : méthodes et cas.* Economica, Paris, 1993.

B. Roy and P. Vincke. Pseudo-orders: Definition, properties and numerical representation. *Mathematical Social Sciences*, 14(2):263–274, 1987.

J.-C. Vansnick. On the problems of weights in MCDM (the noncompensatory approach). *European Journal of Operational Research*, 24(2):288–294, 1986.

Y. Wei. *Aide multicritère à la décision dans le cadre de la problématique du tri : concepts, méthodes et applications.* Thèse de doctorat, Université Paris Dauphine, Paris, France, 1992. (in French).

J. Zheng. *Preference Elicitation for reference based aggregation models: Algorithms and Procedures.* PhD thesis, École Centrale, Paris, 2012.

J. Zheng, S. A. Metchebon, V. Mousseau, and M. Pirlot. Learning an optimistic ELECTRE TRI sorting rule. Research Report CER 12–06, École Centrale, Paris, Laboratoire de Génie Industriel, 2012.

C. Zopounidis and M. Doumpos. PREFDIS: A multicriteria decision support system for sorting decision problems. *Computers & Operations Research*, 27(7–8):779–797, 2000.

Contents