A NOTE ON THE SUM OF DIFFERENCES CHOICE FUNCTION FOR FUZZY PREFERENCE RELATIONS

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Abstract

This note deals with the problem of choice functions based on fuzzy preference relations. We study a choice function based on the sum of the differences and show that it is the only one to satisfy a system of three independent properties.

Key words: Decision Making, Fuzzy Preferences, Choice functions.

I- Introduction

As argued in [1], there are many situations in which a choice is to be made from a set of alternatives on the basis of fuzzy preferences. How should one make such a choice ?

A possible way to analyze this problem is to list a number of rationality requirements and look for a method of choice that would satisfy them. This is the route followed in [1] where nine possible methods are presented and analyzed. However, as noted in this paper, the very definition of rational choice based on fuzzy preferences is far from being obvious.

Instead of asking what are the properties of a method of choice, we may try to find a number of properties that this method would be the only one to satisfy, i.e., to characterize this method. Such a characterization would hopefully allow a better understanding of the method and offer a sound basis to compare it with other methods. The purpose of this note is to illustrate this alternative route by presenting a characterization of a method of choice based on the sum of differences. This method have been presented and analyzed in [1]. The results presented here heavily rests on [2] in which a ranking method based on sum of differences is characterized. We formalize our problem in the next section and present our results in section 3.

II- Notations and Definitions

Let A be a finite set of objects called "alternatives" with at least two elements, P(A) being the set of all nonempty subsets of A. We define a fuzzy (binary) relation on a set A as a function R associating with each ordered pair of alternatives $(a, b) \in A^2$ with $a \neq b$ an element of [0, 1]. From a technical point of view, the condition $a \neq b$ could be omitted from this definition at the cost of minor modifications in the sequel. However, since it is clear that the values R(a, a) are of little help in order to find "good" alternatives, we shall use this definition throughout this note. We define

R(A) as the set of all fuzzy relations on A. It should be noted that, contrary to what is done in [1], we do not restrict our attention here to connected fuzzy relations. Fuzzy preferences often arise from poor and conflicting information concerning the alternatives and it seems reasonable not to exclude fuzzy relations for which $R(a, b) + R(b, a) < 1$ for some $a, b \in A$.

A (preference-based) Choice Function (CF) on A is a function $C : P(A) \times R(A) \to P(A)$, such that $C(B, R) \subseteq B$, for all $R \in R(A)$ and all $B \in P(A)$,

i.e., a function associating with each nonempty subset B of A and each fuzzy relation R on A a nonempty choice set C(B, R) included in B which we may interpret as the set of the "best" alternatives in B given the relation R.

A simple way to obtain a CF is to associate a score W(a, B, R) with each alternative $a \in B$ based on the behavior of R on B and to include in the choice set C(B, R) the alternatives with the highest score, i.e.

$$C(B, R) = \{b \in B : W(b, B, R) \geq W(a, B, R) \text{ for all } a \in B\} \quad (1)$$

All the CF analyzed in [1] are of this type. In the next section, we characterize the CF based on the following "sum of differences" score:

$$SD(a, B, R) = \sum_{c \in B \setminus \{a\}} (R(a,c) - R(c,a)) \quad (2)$$

We shall refer to the CF based on this score as the SD CF. Apart from its simplicity and its intuitive appeal, a reason to study the SD CF lies in its long history in the literature on choice functions. When R is crisp, i.e. when R(a, b) can only take the value 0 or 1, the SD CF amounts to the well-known Copeland CF (see [3] or [4]). It has been characterized in [6], extending previous results in [7], when R is a crisp and connected relation. When R is fuzzy the SD CF has been analyzed at length in [1]. Furthermore, if R(a, b) is interpreted as a percentage of voters considering that a is preferred or indifferent to b, this CF amounts to the well-known Borda rule (see [3], [5], [8]).

III- Results

A CF is said to be neutral if for all $\sigma \in G(A)$, all $R \in R(A)$, all $B \in P(A)$:

$[a \in C(B, R) \Rightarrow \sigma(a) \in C(\sigma(B), R^{\sigma})]$

where $\Gamma(A)$ is the set of all permutations on A, R^{σ} is such that $R^{\sigma}(\sigma(a), \sigma(b)) = R(a, b)$ for all $a, b \in A$ and $\sigma(B) = \{\sigma(a) : a \in B\}$.

Neutrality expresses the fact that a CF does not discriminate between alternatives just because of their labels. It is a classical property in this context (see, e.g., [6] or [7]). It is obvious that the SD CF is neutral.

A CF is said to be monotonic if an alternative in the choice set remains in the choice set after its position has been improved vis-à-vis some other alternative. More formally, a CF is monotonic if for all $R \in R(A)$ and all $B \in P(A)$,

$[a \in C(B, R) \Rightarrow a \in C(B, R')]$

where R′ is identical to R except that $R(a, c) < R'(a, c)$ or $R(c, a) > R'(c, a)$ for some $c \in B\backslash\{a\}$.

A CF is said to be strongly monotonic if for all $R \in R(A)$ and all $B \in P(A)$,

$[a \in C(B, R) \Rightarrow \{a\} = C(B, R')]$

where R′ is defined as previously. Though strong monotonicty is a very strong property, it is obvious that the SD CF is strongly monotonic and, thus, monotonic.

In order to introduce our last property, let us recall some well-known definitions used in Graph Theory. A digraph consists of a set of nodes X and a set of arcs $U \subseteq X^2$. We say that x is the initial extremity and y is the final extremity of the arc $u = (x, y) \in U$.

A cycle of length q in a digraph is an ordered collection of arcs $(u_1, u_2, ..., u_q)$ such that for i = 1, 2, ..., q, one of the extremities of u_i is an extremity of u_{i-1} and the other is an extremity of u_{i+1} (u_0 being interpreted as u_q and u_{q+1} as u_1). A cycle is elementary if and only if each node being the extremity of one arc in the cycle is the extremity of exactly two arcs in the cycle. A cycle is said to be in $Y \subseteq X$ if all the extremities of the arcs in the cycle are in Y. An arc u_i in a cycle is forward if its common extremity with u_{i-1} is its initial extremity and backward otherwise. A cycle containing only forward arcs is called a circuit.

Let us consider a digraph, its set of nodes being A and its set of arc being $U = \{(a, b) : a, b \in A$ and $a \neq b\}$. It is not difficult to see that there is a one-to-one correspondence between fuzzy relations on A and valuations between 0 and 1 of the arcs of this graph. In the sequel, we identify a fuzzy relation R with its associated valued digraph in which the valuation $v_R(u)$ of the arc u = (a, b) is R(a, b).

Consider an elementary cycle in the graph associated with a fuzzy relation. A transformation on an elementary cycle consists in adding the same positive or negative quantity to the valuation of the forward arcs in the cycle and subtracting it from the valuation of the backward arcs. A transformation on an elementary cycle is *admissible* if all the transformed valuations are still between 0 and 1. When we apply an admissible transformation on an elementary cycle in $B \subseteq A$ to the graph associated with a fuzzy relation R, we obtain another fuzzy relation R′. Consider a fuzzy relation R″ such that $R''(a, b) = R'(a, b)$ for all $a, b \in B$ with $a \neq b$. In such a situation, we say that R″ on B can be obtained from R on B through an admissible transformation on an elementary cycle in B. Admissible transformations on elementary cycles are closely related to the SD scores of the alternatives and we have:

Lemma 1. For all R, R′ ∈ R(A) and all B ∈ P(A),

[SD(a, B, R) = SD(a, B, R′) for all a ∈ B] ⇔ [R′ on B can be obtained from R on B through a finite number of admissible transformations on elementary cycles in B].

Proof. The ⇐ part is obvious. In order to prove the ⇒ part, suppose that for some R, R′ ∈ R(A) and some B ∈ P(A), SD(c, B, R) = SD(c, B, R′) for all c ∈ B.

If R = R′ on B, the lemma is proved.

If R ≠ R′ on B then R(a, b) ≠ R(a, b) for some a, b ∈ B with a ≠ b and we suppose for definiteness that R(a, b) > R′(a, b) (the other case being symmetric).

We claim that either R(c, a) > R′(c, a) or R(a, d) < R′(a, d) for some c, d ∈ B\{a}, for otherwise R(c, a) ≤ R′(c, a), R(a, d) ≥ R′(a, d) for all c, d ∈ B\{a, b}, R(a, b) > R′(a, b) and R(b, a) ≤ R′(b, a) would contradict SD(a, B, R) = SD(a, B, R′). In either case, we can repeat the same argument and therefore, since the number of alternatives in B is finite, this process will lead to an elementary cycle in B in the graph associated with R. Let Δ be the minimum over the arcs in the cycle of |R(x, y) – R′(x, y)|. It is easily checked that adding Δ to the arcs in the cycle such that R(x, y) < R′(x, y) and subtracting it from the arcs in the cycle such that R(x, y) > R′(x, y) is an admissible transformation on the cycle. We thus obtain a fuzzy relation R_1. If R_1 = R on B, the lemma is proved. If not, we can repeat the same argument starting with R_1 instead of R.

Because B is finite, there is only a finite number of arcs in B such that R(x, y) ≠ R′(x, y). Since, at each step the number of arcs in B on which the current relation and R are different is decreased by at least one unit, this process will thus terminate after a finite number of steps, which completes the proof of lemma 1. ■

The following two lemmas show that any admissible transformation on an elementary cycle can be performed through a finite number of admissible transformations on elementary circuits of length 2 or 3.

Lemma 2. For all R, R ∈ R(A) and all B ∈ P(A), if [R on B can be obtained from R on B through an admissible transformation on an elementary cycle in B] then [R on B can be obtained from R on B through a finite number of admissible transformations on elementary circuits in B].

Proof. Consider an elementary cycle in B in the graph associated with R and suppose that R′ on B can be obtained from R on B by adding δ to the forward arcs of the cycle and subtracting δ from the backward arcs. We respectively note U_F and U_B the set of forward and backward arcs in the cycle. If δ = 0, there is nothing to prove. Suppose that δ > 0 (the other case being symmetric).

Define $\alpha_{max} = \max_{(a,b) \in U_B} v_R(b,a)$.

If $\alpha_{max} \leq 1 - \delta$ then adding δ on the elementary circuit (all the circuits that we shall consider in the proof are obviously in B) obtained by considering the arcs in U_F and the set {(b, a) ∈ U : (a, b) ∈ U_B}, is an admissible transformation. Now, subtracting δ from all the 2-circuits of the type ((a, b), (b, a)) with (a, b) ∈ U_B are admissible transformations which lead to R′ on B.

If $\alpha_{max} > 1 - \delta$, define $U_P = \{(a, b) \in U_B : v_R(b, a) > 1 - \delta\}$. For all (a, b) ∈ U_P, we have $v_R(a, b) \geq \delta$ and $v_R(b, a) > 0$. Since δ > 0, we can find a sufficiently large integer n such that subtracting δ/n from all the 2-circuits ((a, b), (b, a)) with (a, b) ∈ U_P are admissible transformations. Then adding δ/n on the elementary circuit obtained by considering the arcs in U_F and the arcs (b, a) if (a, b) is in U_B, is an admissible transformation. It is easily seen that it is possible to repeat these operations n times. We thus obtain R′ on B after subtracting δ from the 2-circuits B ((a, b), (b, a)) with (a, b) ∈ $U_B \backslash U_P$, all these transformations being admissible by construction. This completes the proof of lemma 2. ■

Lemma 3. For all R, R′ ∈ R(A) and all B ∈ P(A), if [R′ on B can be obtained from R on B through an admissible transformation on an elementary circuit in B] then [R′ on B can be obtained from R on B through a finite number of admissible transformations on elementary circuits in B of length 2 or 3].

Proof. The proof is by induction on the length q of the elementary circuit in B (which is necessarily finite). If q = 2 or 3, then the lemma is proved. Suppose now that the lemma is true for all q ≤ k with k ≥ 3 and let us show that it is true for q = k+1. Consider an elementary circuit in B of length k+1:

$u_1 = (a_1, a_2)$, $u_2 = (a_2, a_3)$, ..., $u_k = (a_k, a_{k+1})$, $u_{k+1} = (a_{k+1}, a_1)$, and suppose that R′ on B can been obtained from R on B adding δ on the arcs of that circuit.

If δ = 0, there is nothing to prove.

Suppose that δ > 0 (the proof being similar for δ < 0). We define $r = (a_1, a_k)$ and $s = (a_k, a_1)$.

If $v_R(r) \leq 1 - \delta$ and $v_R(s) \leq 1 - \delta$, then we have two elementary circuits (all the circuits that we shall consider in the proof are obviously in B) $(u_1, u_2, ..., u_{k-1}, s)$ and (u_k, u_{k+1}, r) of respective length k and 3 on which adding δ is an admissible transformation. Now, subtracting δ from the 2-circuit (r, s) is an admissible transformation which leads to R′ on B.

If $v_R(r) > 1 - \delta$ and $v_R(s) \leq 1 - \delta$ (the case $v_R(r) \leq 1 - \delta$ and $v_R(s) > 1 - \delta$ being symmetric), then adding δ on $(u_1, u_2, ..., u_{k-1}, s)$ is an admissible transformation. Since now the valuations of r and s are strictly positive, we can find a sufficiently large integer n so that subtracting δ/n from the 2-circuit (r, s) is an admissible transformation. Adding δ/n on (u_k, u_{k+1}, r) is now an admissible transformation. Repeating n times these operations leads to R′ on B.

If $v_R(r) > 1 - \delta$ and $v_R(s) > 1 - \delta$, both $v_R(s)$ and $v_R(r)$ are strictly positive and we can find a sufficiently large integer n so that subtracting δ/n from the 2-circuit (r, s) is an admissible transformation. Adding δ/n on (u_k, u_{k+1}, r) and on $(u_1, u_2, ..., u_{k-1}, s)$ are now admissible transformations. Repeating n times these operations leads to R′ on B. This completes the proof of lemma 3. ■

Combining the preceding three lemmas suggests the following property:

A CF is independent of circuits if for all R, R′ ∈ R(A) and all B ∈ P(A),

[R′ on B can be obtained from R on B through an admissible transformation on an elementary circuit in B of length 2 or 3] ⇒ C(B, R) = C(B, R′).

This property has a straightforward interpretation. Independence of 2-circuits implies that the choice is only influenced by the differences R(a, b) – R(b, a). Independence of 3-circuits implies that intransitivities of the kind R(a, b) > 0, R(b, c) > 0 and R(c, a) > 0 can be "wiped out" subtracting Min(R(a, b) ; R(b, c) ; R(c, a)) from the 3-circuit ((a, b) ; (b, c) ; (c, a)). Contrary to neutrality and monotonicity, this property makes an explicit use of the cardinal properties of the numbers R(a, b).

Given two Choice functions C and C′ on A, we say that C is smaller than C′ if, for all R ∈ R(A) and all B ∈ P(A), C(B, R) ⊆ C′(B, R).

We are now in position to state our main result:

Theorem

- The SD CF is the smallest CF that is neutral, monotonic and independent of circuits;
- The SD CF is the only CF that is neutral, strongly monotonic and independent of circuits.

The following lemma will be helpful in the proof of the theorem:

Lemma 4. If a Choice Function C is monotonic and independent of circuits then, for all R ∈ R(A) and all B ∈ P(A),

[SD(a, B, R) ≥ SD(b, B, R) and b ∈ C(B, R), for some a, b ∈ B] ⇒

[b ∈ C(B, R′) for some fuzzy relation R′ on A such that

SD(b, B, R′) = SD(a, B, R)

$SD(a, B, R') = SD(b, B, R)$ and

$SD(c, B, R') = SD(c, B, R)$ for all $c \in B\backslash\{a, b\}$].

Proof of lemma 4.

If $SD(a, B, R) = SD(b, B, R)$, taking $R' = R$ establishes the thesis.

Suppose now that $\delta = SD(a, B, R) - SD(b, B, R) > 0$.

If $\delta \leq R(a, b) + (1 - R(b, a))$, taking R' identical to R except that $R'(a, b) < R(a, b)$ and/or $R'(b, a) > R(b, a)$ and applying monotonicity, establishes the thesis.

If $\delta > R(a, b) + (1 - R(b, a))$, consider the fuzzy relation R_1 identical to R except that $R_1(a, b) = 0$ and $R_1(b, a) = 1$. Since $b \in C(B, R)$, monotonicity implies that $b \in C(B, R_1)$. We obviously have $SD(c, B, R) = SD(c, B, R_1)$ for all $c \in B\backslash\{a, b\}$, whereas $SD(a, B, R_1) = SD(a, B, R) - R(a, b)$ and $SD(b, B, R_1) = SD(b, B, R) + 1 - R(a, b)$.

Let $\delta_1 = SD(a, B, R_1) - SD(b, B, R_1)$.

We define the following sets of alternatives:

$B_1 = \{c \in B \setminus \{a,b\} : R_1(a, c) > R_1(b, c)\}$,

$B_2 = \{d \in B \setminus \{a,b\} : R_1(d, b) < R_1(d, a)\}$.

Since $\delta_1 > 0$, it is obvious that at least one of B_1 and B_2 is nonempty. Suppose that B_1 is nonempty and let $c \in B_1$. Consider the elementary cycle in B [(a, b), (b, c), (a, c)] in the graph associated with R_1 in which (a, b) is a forward arc. It is easy to see that performing a transformation of ε such that $0 < \varepsilon \leq R(a, c) - R(b, c)$, is an admissible transformation on that cycle. After such an admissible transformation, we obtain a relation R'_1. Given lemmas 1, 2 and 3, we know that $C(B, R_1) = C(B, R'_1)$ by independence of circuits so that $b \in C(B, R'_1)$.

Consider now a relation R_2 identical to R'_1 except that $R_2(a, b) = 0$. Monotonicity implies that $b \in C(B, R_2)$ whereas we obviously have $SD(c, B, R_1) = SD(c, B, R_2)$ for all $c \in B\backslash\{a, b\}$, $SD(a, B, R_2) = SD(a, B, R_1) - \varepsilon$ and $SD(b, B, R_2) = SD(b, B, R_1) + \varepsilon$. It is easy to see that a similar process can be conducted by taking an alternative d in B_2 and performing a transformation of ε with $0 < \varepsilon \leq R(d, b) - R(d, a)$ on the elementary cycle [(a, b), (d, b), (d, a)].

Since it is obvious that:

$$\sum_{c \in B_1} (R_1(a,c) - R_1(b,c)) + \sum_{d \in B_2} (R_1(d,b) - R_1(d,a)) > \delta_1,$$

repeating these operations, starting either with an element of B_1 or of B_2, with suitably chosen ε will lead to the desired relation in a finite number of steps.

This completes the proof of lemma 4. ■

Proof of the Theorem.

Part 1.

In order to prove the first part of the theorem, all we have to show is that if a Choice Function C is neutral, monotonic and independent of circuits then, for all R ∈ R(A), all B ∈ P(A) and all a, b ∈ B:

$SD(a, B, R) \geq SD(b, B, R)$ and $b \in C(B, R) \Rightarrow a \in C(B, R)$ (3)

Suppose that $SD(a, B, R) \geq SD(b, B, R)$ and $b \in C(B, R)$. Define θ as the permutation on A transposing a and b so that θ(B) = B. Consider now the relation R′ as defined in lemma 4. We have $b \in C(B, R')$. We also have $SD(c, B, R^{\theta}) = SD(c, B, R')$ for all c ∈ B. Given lemmas 1, 2 and 3, repeated applications of independence of circuits lead to $C(B, R^{\theta}) = C(B, R')$. Thus, $b \in C(B, R^{\theta})$ and neutrality implies a ∈ C(B, R), which completes the proof of the first part of the theorem.

Part 2.

In order to prove the second part of the theorem, it suffices to observe that, if monotonicity is replaced by strong monotonicity the previous demonstration and that of lemma 4 shows that:

$SD(a, B, R) > SD(b, B, R)$ and $b \in C(B, R)$,

is impossible, which completes the proof. ■

It is worth observing that the SD CF is not the only CF being neutral, monotonic and independent of circuit. For instance, this is also the case for the CF defined by:

$C(B, R) = \{b \in B : SD(b, B, R) + \varepsilon \geq SD(a, B, R) \text{ for all } a \in B\}$, with $\varepsilon > 0$.

Let us also notice that the three properties characterizing the SD CF are independent as shown by the following examples:

i- Let $F : A \to \{1, 2, ..., |A|\}$ be a one-to-one function.

Define a CF by (1) with $W(a, B, R) = SD(a, B, R)/F(a)$.

This CF is strongly monotonic and independent of circuits but not neutral.

ii- Define a CF by (1) with $W(a, B, R) = -SD(a, B, R)$.

This CF is neutral and independent of circuits but not strongly monotonic.

iii- Define a CF by (1) with:

$$W(a, B, R) = \sum_{c \in B \setminus \{a\}} (R(a,c)^2 - R(c,a)^2),$$

This CF is neutral and strongly monotonic but not independent of circuits.

Let us finally observe that obvious modifications of the three properties used in this note allow to characterize the SF CF and the SA CF introduced in [1] that are respectively based on the following two scores $SF(a,B,R) = \sum_{c \in B \setminus \{a\}} R(a,c)$ and $SA(a,B,R) = -\sum_{c \in B \setminus \{a\}} R(c,a)$

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