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Theory and Methodology

Ranking methods based on valued preference relations: A characterization of the net flow method

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Abstract: This paper deals with the problem of ranking several alternatives on the basis of a valued preference relation. A system of three independent axioms is shown to characterize a ranking method based on 'net flows' which contains as particular cases the rules of Copeland and Borda and is used in one of the PROMETHEE methods

Keywords: Decision theory; valued preferences

1. Introduction

Suppose that you want to compare a number of alternatives taking into account different points of view, e.g. several criteria or the opinion of several voters. A common practice in such situations is to associate with each ordered pair (a, b) of alternatives a number indicating the strength or the credibility of the proposition 'a is at least as good as b', e.g. the sum of the weights of the criteria favouring a or the percentage of voters declaring that a is preferred or indifferent to b. Since Condorcet, we know that, when the different points of view taken into account are conflictual, it may not be easy to compare the alternatives on the basis of these numbers. In this paper we study a particular method allowing to build a ranking, i.e. a complete and transitive binary (crisp) relation [1], on a set of alternatives given such information. In a similar context, Bouyssou and Perny (1990) envisage more general methods building partial rankings, i.e. reflexive and transitive binary relations.

Let A be a finite set of objects called 'alternatives' with at least two elements. We define a valued (binary) relation on A as a function R associating with each ordered pair of alternatives $(a, b) \in A^2$ with $a \neq b$ an element of [0, 1]. From a technical point of view, the condition $a \neq b$ could be omitted from this definition at the cost of a minor modification of our third axiom. However, since it is clear that the values $R(a, a)$ are immaterial in order to rank the alternatives, we will use this definition throughout the paper. A ranking method $\succcurlyeq$ is a function assigning a ranking $\succcurlyeq(R)$ on A to any valued relation R on A.

An obvious way to obtain a ranking method is to associate a score $S(a, R)$ with each alternative

[1] A (crisp) binary relation S on a set A is complete if for all $a, b \in A$ either a S b or b S a. It is transitive if for all a, b, $c \in A$, a S b and b S c imply a S c. It is connected if for all $a, b \in A$ with $a \neq b$, either a S b or b S a. It is asymmetric if for all $a, b \in A$, a S b implies Not b S a. It is reflexive if a S a for all $a \in A$.

$a \in A$ and to rank the alternatives according to their scores, i.e.

$$a \succcurlyeq (R) b \text{ iff } S(a, R) \geq S(b, R). \quad (1)$$

The purpose of this paper is to present an axiomatic characterization of the ranking method based on the following score:

$$S_{\mathrm{NF}}(a, R) = \sum_{c \in A \setminus \{a\}} (R(a, c) - R(c, a)). \quad (2)$$

We will refer to the ranking method defined by (1) and (2) as the Net Flow Method.

When R is crisp, i.e. when $R(a, b)$ can only take the value 0 or 1, this ranking method amounts to the well-known Copeland ranking method (see Goodman, 1954; or Fishburn, 1973). It has been characterized by Rubinstein (1980) when R is a tournament (i.e. a connected and asymmetric crisp binary relation). This result has been extended by Henriet (1985) to the case of crisp and connected relations.

When $R(a, b)$ is interpreted as a percentage of voters considering that a is preferred or indifferent to b, this ranking method is the well-known method of Borda (see Fishburn, 1973). It has been characterized by several authors (Young, 1974; Hansson and Sahlquist, 1976; Nitzan and Rubinstein, 1981) in contexts involving a 'variable electorate'.

The Net Flow Method is also used in the Multiple Criteria Decision Making method PROMETHEE II (Brans and Vincke, 1985).

Our results can be viewed as an extension of the work of Rubinstein and Henriet to the case of valued relations or as an alternative partial characterization of the Borda rule.

When crossing the line between crisp and valued relations, it is necessary to take a position on the nature and the significance of the valuations $R(a, b)$. Contrary to methods using only the Min and/or Max operators, it should be strongly emphasized that the Net Flow Method makes use of the 'cardinal' properties of the valuations. In fact, it is obvious from (1) and (2) that we may well have $\succcurlyeq (R) \neq \succcurlyeq (R_\phi)$, where R_ϕ is defined by $R_\phi(a, b) = \phi(R(a, b))$ for all $a, b \in A$ and ϕ is a strictly increasing transformation on the real line such that $\phi(0) = 0$ and $\phi(1) = 1$. Thus this method does not seem to be appropriate when the comparisons of the valuations only have an ordinal meaning in term of credibility.

2. The main result

Throughout the paper, we note $\approx (R)$ and $\succ (R)$ the symmetric and asymmetric parts of $\succcurlyeq (R)$, i.e. for all $a, b \in A$, [$a \approx (R) b$ iff ($a \succcurlyeq (R) b$ and $b \succcurlyeq (R) a$)] and [$a \succ (R) b$ iff ($a \succcurlyeq (R) b$ and not $b \succcurlyeq (R) a$)].

A ranking method is said to be neutral if and only if, for all permutation σ on A, for all valued relation R on A and all $a, b \in A$,

$$a \succcurlyeq (R) b \Leftrightarrow \sigma(a) \succcurlyeq (R^\sigma) \sigma(b)$$

where R^σ is defined by $R^\sigma(\sigma(a), \sigma(b)) = R(a, b)$ for all $a, b \in A$.

Neutrality expresses the fact that a ranking method does not discriminate between alternatives just because of their labels. It is a classical property in this context (see, e.g., Rubinstein, 1980; or Henriet, 1985). The Net Flow Method is obviously neutral. It is easily checked that neutrality implies that if $R(a, b) = R(b, a)$ and for all $c \in A \setminus \{a, b\}$, $R(a, c) = R(b, c)$ and $R(c, a) = R(c, b)$, then $a \approx (R) b$.

A ranking method is said to be strongly monotonic if the ranking responds 'in the right direction' to a modification of R. More formally, $\succcurlyeq$ is strongly monotonic if and only if for all $a, b \in A$ and for all valued relation R on A,

$$a \succcurlyeq (R) b \Rightarrow a \succ (R') b$$

where R' is identical to R except that [$R(a, c) < R'(a, c)$ or $R(c, a) > R'(c, a)$ for some $c \in A \setminus \{a\}$].

Suppose that R'' is identical to R except that $R(b, d) > R''(b, d)$ or $R(d, b) < R''(d, b)$ for some $d \in A \setminus \{b\}$. It is easy to prove that strong monotonicity implies that [$a \succcurlyeq (R) b \Rightarrow a \succ (R'') b$]. As defined here, strong monotonicity is a very strong property excluding, in particular, the use of any threshold in the treatment of the valuations. However, it is obvious that the Net Flow Method is strongly monotonic.

An important characteristic of a ranking method lies in the way it deals with 'intransitivities' of R. In order to formalize this point, let us recall some well-known definitions used in Graph Theory.

A digraph consists in a set of nodes X and a set of arcs $U \subseteq X^2$. We say that x is the initial extremity and y is the final extremity of the arc $u = (x, y) \in U$.

A circuit (a cycle) of length q in a digraph is an ordered collection of arcs $(u_1, u_2, \ldots, u_q)$ such that for $i = 1, 2, \ldots, q$, the initial extremity of u_i is the final extremity of u_{i-1} and the final extremity of u_i is the initial extremity of u_{i+1} ($u_i \neq u_{i-1}$, one of the extremities of u_i is an extremity of u_{i-1} and the other an extremity of u_{i+1}), where u_0 is interpreted as u_q and u_{q+1} as u_1. A circuit (a cycle) is elementary if and only if each node being the extremity of one arc in the circuit (the cycle) is the extremity of exactly two arcs in the circuit (the cycle).

Let us consider a digraph which set of nodes is A and which set of arc U is $\{(a, b): a, b \in A$ and $a \neq b\}$. It is obvious that there is a one-to-one correspondence between valued relations on A and valuations between 0 and 1 of the arcs of this graph. In the sequel, we identify a valued relation with its associated valued digraph in which the valuation $v_R(u)$ of the arc $u = (a, b)$ is $R(a, b)$.

A transformation on an elementary circuit consists in adding a same positive or negative quantity to the valuations of the arcs in the circuit. A transformation is admissible if the transformed valuations are still between 0 and 1. When we apply an admissible transformation to the graph associated with a valued a valued relation R, we obtain another valued relation R' and we say that R' has been obtained from R through an admissible transformation on an elementary circuit.

A ranking method is independent of circuits if and only if for all valued relation R and R', [R' can be obtained from R through an admissible transformation on an elementary circuit of length 2 or 3] $\Rightarrow\ \succcurlyeq(R) = \succcurlyeq(R')$.

It is obvious that an admissible transformation on an elementray circuit does not alter the score of any of the alternatives, when the scores are defined by (2), so that the Net Flow Method is independent of circuits.

This axiom has a straightforward interpretation. Independence of 2-circuits, i.e. of circuits of length 2, implies that the ranking is only influenced by the differences $R(a, b) - R(b, a)$. Independence of 3-circuits implies that intransitivities of the kind $R(a, b) > 0$, $R(b, c) > 0$ and $R(c, a) > 0$ can be 'wiped out' subtracting $\mathrm{Min}(R(a, b); R(b, c); R(c, a))$ from the 3-circuit $((a, b); (b, c); (c, a))$. Contrary to neutrality and monotonicity, this axiom makes explicit use of the cardinal properties of the valuations. It is obvious that adding to this axiom a condition on 1-circuits would allow to consider valued relations for which $R(a, a)$ is defined.

We are now in position to state our main result.

Theorem. *The Net Flow Method is the only ranking method that is neutral, strongly monotonic and independent of circuits.*

We already noticed that the Net Flow Method is neutral, strongly monotonic and independent of circuits. The proof that it is the only one appears in the next section. Let us first notice that the three axioms characterizing the Net Flow Method are independent as shown by the following examples:

(i) Let $\Phi : A \to \{1, 2, \ldots, |A|\}$ be a one-to-one function. Define $\succcurlyeq$ as

$a \succcurlyeq (R) b$ iff $S_1(a, R) \geq S_1(b, R)$

where for all $c \in A$, $S_1(c, R) = S_{NF}(c, R) \times \Phi(c)$. This ranking method is strongly monotonic and independent of circuits but not neutral.

(ii) Define $\succcurlyeq$ as

$a \succcurlyeq (R) b$ iff $S_2(a, R) S_2(b, R)$

where for all $c \in A$, $S_2(c, R) = -S_{NF}(c, R)$. This ranking method is neutral and independent of circuits but not strongly monotonic.

(iii) Define $\succcurlyeq$ as

$a \succcurlyeq (R) b$ iff $S_3(a, R) \geq S_3(b, R)$

where for all $c \in A$,

$$S_3(c, R) = \sum_{d \in A \setminus \{c\}} R(c, d)^2 - \sum_{d \in A \setminus \{c\}} R(d, c)^2.$$

This ranking method is neutral and strongly monotonic but not independent of circuits.

3. Proofs

Before proving the main theorem we shall go through a number of lemmas. Some of them are interesting on their own given the proximity of this problem with some aspects of the theory of flows on networks (see Ford and Fulkerson (1962)).

Lemma 1. *For all valued relations R and R′ on A, if [R′ can be obtained from R through an admissible transformation on an elementary circuit], then*

[*R' can be obtained from R through a finite number of admissible transformations on elementary circuits of length 2 or 3*].

Proof. The proof is by induction on the length q of the elementary circuit. If $q = 2$ or 3, then the lemma is proved. Suppose now that the lemma is true for all $q \leq k$ with $k \geq 3$ and let us show that it is true for $q = k + 1$. Consider an elementary circuit of length $k + 1$, $u_1 = (a_1, a_2)$, $u_2 = (a_2, a_3), \dots, u_k = (a_k, a_{k+1})$, $u_{k+1} = (a_{k+1}, a_1)$, and suppose that R' has been obtained from R adding δ on the arcs of that circuit. If $\delta = 0$, there is nothing to prove. Suppose that $\delta > 0$ (the proof is similar for $\delta < 0$). We define $r = (a_1, a_k)$ and $s = (a_k, a_1)$. If $v_R(r) \leq 1 - \delta$ and $v_R(s) \leq 1 - \delta$, then we have two circuits $(u_1, u_2, \dots, u_{k-1}, s)$ and (u_k, u_{k+1}, r) of respective length k and 3 on which adding δ is an admissible transformation. Now, subtracting δ from the 2-circuit (r, s) is an admissible transformation which leads to R'. If $v_R(r) > 1 - \delta$ and $v_R(s) \leq 1 - \delta$ (the case $v_R(r) \leq 1 - \delta$ and $v_R(s) > 1 - \delta$ is symmetric), then adding δ on $(u_1, u_2, \dots, u_{k-1}, s)$ is an admissible transformation. Since now the valuations of s and r are strictly positive, we can find a sufficiently large integer n so that subtracting δ/n from the 2-circuit (r, s) is an admissible transformation. Adding δ/n on (u_k, u_{k+1}, r) is now an admissible transformation. Repeating n times these operations leads to R' (see Figure 1). If $v_R(r) > 1 - \delta$ and $v_R(s) > 1 - \delta$, both of $v_R(s)$ and $v_R(r)$ are strictly positive and we can find a sufficiently large integer n so that subtracting δ/n from the 2-circuit (r, s) is an admissible transformation. Adding δ/n on (u_k, u_{k+1}, r) and on $(u_1, u_2, \dots, u_{k-1}, s)$ are now admissible transformations. Repeating this n times leads to R'.

This completes the proof of lemma 1, because since A is finite, the maximum length of an elementary circuit is finite as well. □

Consider an elementary cycle in the graph associated with a valued relation. An arc u_t in a cycle is forward if its common extremity with u_{t-1} is its initial extremity and backward otherwise. A transformation on a elementary cycle consists in adding a positive or negative quantity to the valuation of the forward arcs in the cycle and subtracting it from the valuation of the backward arcs. A transformation on a cycle is admissible if all the transformed valuations are still between 0 and 1. It is obvious that an admissible transformation on an elementary cycle does not alter the score of any of the alternatives when the scores are defined by (2).

Lemma 2. *For all valued relations R and R' on A, if* [*R' can be obtained from R through an admissible transformation on an elementary cycle*] *then*

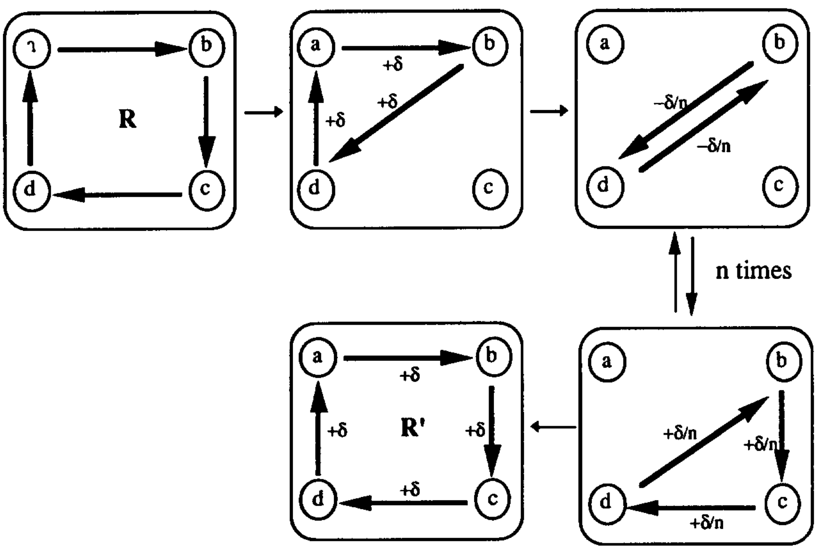

Figure 1. Transformation on a 4-circuit via a number of transformations on 2 or 3-circuits. When $R(b, d) \leq 1 - \delta$ and $R(d, b) > 1 - \delta$, a transformation of δ on the 4-circuit $[(a, b), (b, c), (c, d), (d, a)]$ is obtained after adding δ on the 3-circuit $[(a, b), (b, d), (d, a)]$ and performing n times a transformtion of $-\delta/n$ on the 2-circuit $[(b, d), (d, b)]$ and a transformation of δ/n on the 3-circuit $[(b, c), (c, d), (d, b)]$

[*R' can be obtained from R through a finite number of admissible transformations on elementary circuits*].

Proof. Consider an elementary cycle in the graph associated with R and suppose that R' has been obtained from R by adding δ to the forward arcs of the cycle and subtracting δ from the backward arcs. We respectively note U_F and U_B the set of forward and backward arcs in the cycle. If $\delta = 0$, there is nothing to prove. Suppose that $\delta > 0$ (the proof is similar for $\delta < 0$). Define

$$\alpha_{\max} = \max_{(a,b) \in U_B} v_R(b, a).$$

If $\alpha_{\max} \leq 1 - \delta$, then adding δ on the elementary circuit obtained by considering the arcs in U_F and the set $\{(b, a) \in U: (a, b) \in U_B\}$, is an admissible transformation. Now, subtracting δ from all the 2-circuits of the type $((a, b), (b, a))$ with $(a, b) \in U_B$ are admissible transformations which lead to R'. If $\alpha_{\max} > 1 - \delta$, define

$$U_P = \{(a, b) \in U_B: v_R(b, a) > 1 - \delta\}.$$

For all $(a, b) \in U_P$, we have $v_R(a, b) \geq \delta$ and $v_R(b, a) > 0$. Since $\delta > 0$, we can find a sufficiently large integer n such that subtracting δ/n from all the 2-circuits $((a, b), (b, a))$ with $(a, b) \in U_P$ are admissible transformations. Then adding δ/n to the circuit obtained by considering the arcs in U_F and the arcs (b, a), if (a, b) is in U_B, is an admissible transformation. It is easily seen that it is possible to repeat these operations n times. We thus obtain R' after subtracting δ from the 2-circuits $((a, b), (b, a))$ with $(a, b) \in U_B \setminus U_P$, all these transformations being admissible by construction (see Figure 2).

This completes the proof of Lemma 2, because since A is finite, the maximum length of an elementary cycle, is finite as well. □

Lemma 3. *For all valued relations R and R' on A, $[S_{NF}(c, R) = S_{NF}(c, R')$ for all $c \in A] \Leftrightarrow [R'$ can be obtained from R through a finite number of admissible transformations on elementary cycles*].

Proof. The $\Leftarrow$ part is obvious. In order to prove the $\Rightarrow$ part, suppose that for some R and R' and for all $c \in A$ we have $S_{NF}(c, R) = S_{NF}(c, R')$. If $R = R'$ the lemma is proved. If $R \neq R'$, then $R(a, b) \neq R'(a, b)$ for some a, $b \in A$ with $a \neq b$ and we suppose further for definiteness that $R(a, b) > R'(a, b)$, since the other case is symmetric. We claim that either $R(c, a) > R'(c, a)$ or $R(a, d) < R'(a, d)$ for some c, $d \in A \setminus \{a\}$; since $[R(c, a) \leq R'(c, a)$, $R(a, d) \geq R'(a, d)$ for all c, $d \in A \setminus \{a, b\}$, $R(a, b) > R'(a, b)$ and $R(b, a) \leq R'(b, a)]$ would contradict $S_{NF}(a, R) = S_{NF}(a, R')$. In either case, we can repeat the same argument and therefore, since the number of alternatives is finite, this process will lead to an elementary cycle in the graph associated with R. Let Δ be the minimum over the arcs (e, f) in the cycle of $|R(e, f) -$

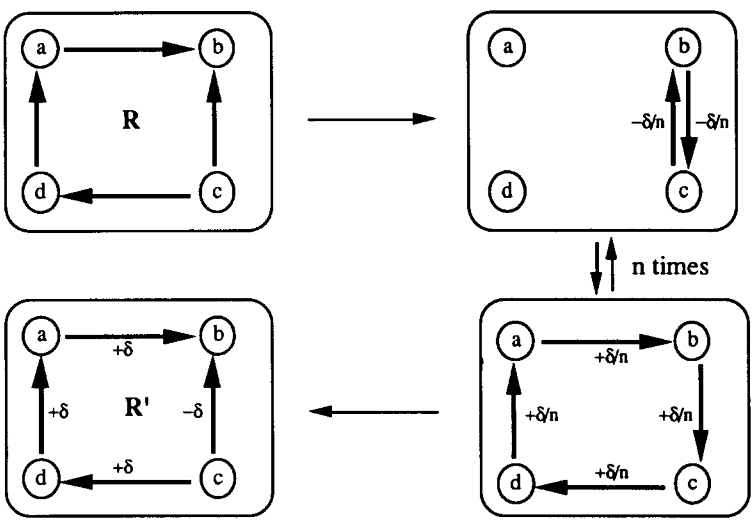

Figure 2. Transformation on a cycle via a number of transformations on circuits. When $R(b, c) > 1 + \delta$, a transformation of δ on the cycle $[(a, b), (c, b), (c, d), (d, a)]$ is obtained by performing n times a transformation of $-\delta/n$ on the 2-circuit $[(b, c), (c, b)]$ and a transformation of δ/n on the 4-circuit $[a, b), (b, c), (c, d), (d, a)]$

$R'(e, f)|$. It is easily checked that adding Δ to the arcs in the cycle such that $R(x, y) < R'(x, y)$ and subtracting it from the arcs in the cycle such that $R(x, y) > R'(x, y)$ is an admissible transformation on the cycle. We thus obtain a valued relation R_1. If $R_1 = R'$ the lemma is proved. If not, we can repeat the same argument starting with R_1 instead of R. Because A is finite, there is only a finite number of arcs such that $R(x, y) \neq R'(x, y)$. Since, at each step the number of arcs on which the current relation and R' are different, is decreased by at least one unit, this process will thus terminate after a finite number of steps, which completes the proof of Lemma 3. □

Proof of the Theorem. All we have to prove is that if $\succcurlyeq$ is neutral, strongly monotonic and independent of circuits, then

$$[a \succcurlyeq (R) b \Leftrightarrow S_{\mathrm{NF}}(a, R) \geq S_{\mathrm{NF}}(b, R)],$$

i.e.

$$S_{\mathrm{NF}}(a, R) = S_{\mathrm{NF}}(b, R) \Rightarrow a \approx (R)\ b \text{ and} \quad (3)$$

$$S_{\mathrm{NF}}(a, R) > S_{\mathrm{NF}}(b, R) \Rightarrow a \succ (R)\ b. \quad (4)$$

First, suppose that $S_{\mathrm{NF}}(a, R) = S_{\mathrm{NF}}(b, R)$ for some $a, b \in A$. We have either $a \succcurlyeq (R)\ b$ or $b \succcurlyeq (R)\ a$. If $a \succcurlyeq (R)\ b$, define θ as the permutation on A transposing a and b. We have $S_{\mathrm{NF}}(c, R) = S_{\mathrm{NF}}(c, R^{\theta})$ for all $c \in A$. Given Lemma 3, we know that R^{θ} can be obtained from R through a finite number of admissible transformations on elementary cycles. Combining Lemmas 1 and 2 we conclude that R^{θ} can be obtained from R through a finite number of admissible transformations on elementary circuits of length 2 or 3. Thus, using independence of circuits we obtain $\succcurlyeq (R) = \succcurlyeq (R^{\theta})$ so that $a \succcurlyeq (R^{\theta})\ b$. Thus, neutrality implies that $b \succcurlyeq (R)\ a$, which establishes (3).

Suppose now that $S_{\mathrm{NF}}(a, R) > S_{\mathrm{NF}}(b, R)$ for some $a, b \in A$ and let $\delta = S_{\mathrm{NF}}(a, R) - S_{\mathrm{NF}}(b, R)$. We define the following sets of alternatives:

$$A_1 = \{c \in A \setminus \{a, b\} \colon R(a, c) > 0\},$$

$$A_2 = \{d \in A \setminus \{a, b\} \colon R(b, d) < 1\},$$

$$A_3 = \{e \in A \setminus \{a, b\} \colon R(e, a) < 1\},$$

$$A_4 = \{f \in A \setminus \{a, b\} \colon R(f, b) > 0\}.$$

We denote by B_i the complement of A_i in $A \setminus \{a, b\}$.

If

$$\delta \leq \sum_{c \in A_1} R(a, c) + \sum_{d \in A_2} (1 - R(b, d)) + \sum_{e \in A_3} (1 - R(e, a)) + \sum_{f \in A_4} R(f, b), \quad (5)$$

it is easy to see that it is possible to obtain a valued relation $\underline{R}$ identical to R except on the ordered pairs of alternatives (a, c) with $c \in A_1$, (e, a) with $e \in A_3$, (b, d) with $d \in A_2$ and (f, b) with $f \in A_4$, such that $S_{\mathrm{NF}}(a, \underline{R}) = S_{\mathrm{NF}}(b, \underline{R})$. Thus (3) implies $a \approx (\underline{R})\ b$ and repeated applications of strong monotonicity lead to $a \succ (R)\ b$.

Let us show that (5) holds. We have

$$\begin{aligned} S_{\mathrm{NF}}(a, R) &= \sum_{c \in A \setminus \{a\}} (R(a, c) - R(c, a)) \\ &= \sum_{c \in A_1} R(a, c) - \sum_{e \in A_3} R(e, a) - |B_3| \\ &\quad + R(a, b) - R(b, a), \\ S_{\mathrm{NF}}(b, R) &= \sum_{c \in A \setminus \{b\}} (R(b, c) - R(c, b)) \\ &= \sum_{d \in A_2} R(b, d) + |B_2| \\ &\quad - \sum_{f \in A_4} R(f, b) + R(b, a) \\ &\quad - R(a, b). \end{aligned}$$

Thus

$$\begin{aligned} \delta = {} & 2(R(a, b) - R(b, a)) - |B_2| - |B_3| \\ & + \sum_{c \in A_1} R(a, c) - \sum_{e \in A_3} R(e, a) \\ & - \sum_{d \in A_2} R(b, d) + \sum_{f \in A_4} R(f, b). \end{aligned}$$

Noticing that $|A_i| + |B_i| = |A| - 2$, it is easy to see that (5) holds as soon as $|A| \geq 3$. If not, then $A = \{a, b\}$, and define $\underline{\underline{R}}$ as $\underline{\underline{R}}(a, b) = \underline{\underline{R}}(b, a) = R(b, a)$. Thus $a \approx (\underline{\underline{R}})\ b$ by (3) and strong monotonicity leads to $a \succ (R)\ b$ which completes the proof of the Theorem. □

Let us finally notice that a similar method of proof can be used to characterize other ranking methods based on scores. For instance, a characterization of the 'leaving flow' method defined by

$$a \succcurlyeq (R) b \text{ iff } \sum_{c \in A \setminus \{a\}} R(a, c) \geq \sum_{c \in A \setminus \{b\}} R(b, c)$$

is at hand keeping neutrality unchanged and modifying monotonicity and independence of circuit in an obvious way. A similar remark holds for the method based on (the opposite) of 'entering flows'. Other extensions of this method of proof may be found in Bouyssou and Perny (1990).

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