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Theory and Methodology

Democracy and efficiency: A note on "Arrow's theorem is not a surprising result"

Denis Bouyssou
ESSEC, BP105, F-95021 Cergy, France

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Abstract: It has been shown that Arrow's impossibility result can be avoided when the notion of aggregation procedure is extended to include procedures leading to more than one relation on the set of alternatives. The purpose of this note is to study the structure of these aggregation procedures, generalizing previous results obtained by Phillipe Vincke. Under 'Arrowian' conditions, we prove that such procedures lead to oligarchies. The size of these oligarchies is discussed.

Keywords: Social Choice Theory, Arrow's theorem

1. Introduction

A central theme in Social Choice Theory is to study how the preferences of several individuals for various alternatives can be aggregated in a 'reasonable' way (see, e.g., Sen, 1986). Starting with the work of Arrow (1963), many results show that apparently innocuous conditions relating individual preferences to social preferences are incompatible.

Vincke (1982) proved that the situation is somewhat different if we do not try to completely aggregate individual preferences, i.e. if we consider aggregation procedures that may lead to more than one preference relation at the aggregate level, these several preference relations being interpreted as potential results of a final aggregation. A closely related extension has been studied by Weymark (1983) who considers aggregation procedures leading to a single but not necessarily complete preference relation.

The purpose of this note is to study the structure of the aggregation procedures proposed by Vincke (1982). We introduce our definitions and notations in the next section before presenting our results in Section 3.

2. Definitions and notations

A binary relation S on a set X is a subset of X^2. As usual, we write $x\ S\ y$ instead of $(x, y) \in S$.

A binary relation S on X is

– reflexive if $x\ S\ x$ for all $x \in X$,

– complete if $x\ S\ y$ or $y\ S\ x$, for all $x, y \in X$ and

– transitive if $x\ S\ y$ and $y\ S\ z$ imply $x\ S\ z$, for all $x, y, z \in X$.

A complete and transitive binary relation will be called a *ranking*. We define P_X as the set of all rankings on a set X.

Given a binary relation S on a set X, we denote by $\alpha(S)$ its asymmetric part, i.e. a binary relation on X defined by $x\ \alpha(S)\ y$ if and only if $[x\ S\ y$ and $\text{Not}(y\ S\ x)]$.

Given a (strictly) positive integer k and a set X, $\mathscr{P}_k(X)$ will denote the set of all nonempty subsets of X with at most k elements.

We formalize our problem as follows. Let A be a finite set of objects called 'alternatives' with at least three elements, and N a finite set, the $|N| = n$ elements of N being interpreted as 'individuals' having preferences for the alternatives. In this note, it is supposed that the individuals express their preferences for the alternatives as rankings on A and that any ranking on A can be the preference relation of some individual.

Given a (strictly) positive integer k, we define a *k-aggregation procedure* as a function associating at most k rankings to any n-tuple of rankings, i.e. a function:

$$F : [P_A]^n \to \mathscr{P}_k(P_A)$$

$$(R_1, R_2, \ldots, R_n) \mapsto F(R_1, R_2, \ldots, R_n).$$

Thus, if k and k' are (strictly) positive integers such that $k \leqslant k'$, any k-aggregation procedure is a k'-aggregation procedure.

Vincke (1982) generalizes the classical conditions introduced by Arrow (1963) for 1-aggregation procedures as follows (for notational convenience, we abbreviate $(R_1, R_2, \ldots, R_n)$ as $(\langle R_i \rangle)$ in the rest of this note). For all $(\langle R_i \rangle)$, $(\langle \underline{R}_i \rangle) \in [P_A]^n$ and all $a, b \in A$, a k-aggregation procedure F satisfies:

Condition P. If $[a\ \alpha(R_i)\ b$ for all $i \in N] \Rightarrow [a\ \alpha(R)\ b$ for all $R \in F(\langle R_i \rangle)]$, and

Condition I. If $[\forall i \in N,\ (a\ R_i\ b \Leftrightarrow a\ \underline{R}_i\ b)$ and $(b\ R_i\ a \Leftrightarrow b\ \underline{R}_i\ a)] \Rightarrow [\forall R \in F(\langle R_i \rangle),\ \exists \underline{R} \in F(\langle \underline{R}_i \rangle)$ such that $(a\ R\ b \Leftrightarrow a\ \underline{R}\ b)$ and $(b\ R\ a \Leftrightarrow b\ \underline{R}\ a)]$.

It is easy to see that conditions P and I coincide with the unanimity and independence conditions introduced by Arrow (1963) when applied to 1-aggregation procedures. Thus, following Vincke (1982), a k-aggregation procedure satisfying P and I is called an Arrowian procedure.

Let F be a k-aggregation procedure. An individual $j \in N$ is said to be a *dictator* for F if, for all profiles $(\langle R_i \rangle)$, there is an element of $F(\langle R_i \rangle)$, reflecting all her strict preferences, i.e. $\forall(\langle R_i \rangle) \in [P_A]^n$, $\exists R \in F(\langle R_i \rangle)$ such that, $\forall a, b \in A$, $[a\ \alpha(R_j)\ b \Rightarrow a\ \alpha(R)\ b]$.

An individual $j \in N$ is a *weak dictator* for F if, for all profiles $(\langle R_i \rangle)$, all her strict preferences are reflected in $F(\langle R_i \rangle)$, i.e. $\forall(\langle R_i \rangle) \in [P_A]^n$, $\forall a, b \in A$, $\exists R \in F(\langle R_i \rangle)$ such that $[a\ \alpha(R_j)\ b \Rightarrow a\ \alpha(R)\ b]$.

It is obvious that a dictator is also a weak dictator, whereas the converse is not true.

3. Results

Within the framework described in the preceding section, it is easy to see that Arrow's theorem can be expressed as:

Proposition 1. *There is no Arrowian* 1*-aggregation procedure without dictator.*

Contrasting with this negative result, Vincke (1982) proves:

Proposition 2. *There is a positive integer $\underline{k}$ such that there is an Arrowian $\underline{k}$-aggregation procedure without dictator.*

Thus, by taking a sufficiently large integer k, it is always possible to find an Arrowian k-aggregation procedure without dictator. This apparently positive result would however be disappointing if the minimal $\underline{k}$ for which there is an Arrowian $\underline{k}$-aggregation procedure without dictator were to be large compared to n. This would imply that the only Arrowian and non-dictatorial k-aggregation procedures are very inefficient. It turns out that it is possible to strengthen Proposition 2 as:

Proposition 3. *There is an Arrowian* 2*-aggregation procedure without dictator.*

Proof. We prove Proposition 3 by giving an example of an Arrowian 2-aggregation procedure without dictator. Let $A = \{a_1, a_2, \ldots, a_m\}$ and let R^1,

R^2, R^3 and R^4 be the rankings defined by (when describing a ranking, it is understood that a precedes b in the list if a is strictly preferred to b and that alternatives between brackets are indifferent):
R^1: $a_1, a_2, a_3 \cdots a_{m-1}\, a_m$,
R^2: $a_m\, a_{m-1}\, a_{m-2} \cdots a_2\, a_1$,
R^3: $a_2\, a_1\, a_3\, a_4 \cdots a_{m-1}\, a_m$,
R^4: $a_m\, a_{m-1} \cdots a_4\, a_3\, a_1\, a_2$.
Let f be the 2-aggregation procedure defined by:

$$f(R_1, R_2, \ldots, R_n) = \begin{cases} \{R^3\} \cup \{R^4\} & \text{if } R_1 = R_2 = \cdots = R_{n-1} = R^1 \text{ and } R_n = R^2, \\ \{R_1\} \cup \{R_n\}, & \text{otherwise.} \end{cases}$$

It is easy to prove that f has no dictator and satisfies P and I. □

Thus, as soon as $k \geqslant 2$, there is an Arrowian k-aggregation procedure without dictator. Let us observe however that, in the example used in the proof of Proposition 3, both 1 and n are weak dictators. This is not surprising since Vincke (1982) proves:

Proposition 4. *There is no Arrowian k-aggregation procedure without weak dictator.*

Our next proposition shows that Proposition 4 can be greatly strengthened. Given a k-aggregation procedure F, define an *oligarchy* O as a subset of N such that for all $a, b \in A$ and all $(\langle R_i \rangle) \in [P_A]^n$:

$$a\ \alpha(R_j)\ b \text{ for all } j \in O \Rightarrow a\ \alpha(R)\ b \text{ for all } R \in F(\langle R_i \rangle),$$

$$a\ \alpha(R_j)\ b \text{ for some } j \in O \Rightarrow a\ \alpha(R)\ b \text{ for some } R \in F(\langle R_i \rangle).$$

In the example used in the proof of Proposition 3, $\{1, n\}$ is an oligarchy. By definition, all members of an oligarchy are weak dictators. Conversely, it is easy to see that an oligarchy must include all weak dictators. We have:

Proposition 5. *Every Arrowian k-aggregation procedure has a unique oligarchy.*

Proof. Proposition 5 is a direct consequence of Theorem 1 in Weymark (1983) stating that, for all functions G associating a reflexive and transitive binary relation on A to any n-tuple of rankings on A and satisfying conditions I and P, there is a nonempty subset O of N such that, for all $a, b \in A$ and all $(\langle R_i \rangle) \in [P_A]^n$:

$$a\ \alpha(R_j)\ b \text{ for all } j \in O \Rightarrow a\ \alpha(S)\ b,$$

$$a\ \alpha(R_j)\ b \text{ for some } j \in O \Rightarrow \text{Not}(b\ S\ a)$$

where $S = G(\langle R_i \rangle)$.

Consider the function g defined on $[P_A]^n$ by:

$$g(\langle R_i \rangle) = \bigcap_{R \in F(\langle R_i \rangle)} R$$

where F is an Arrowian k-aggregation procedure. It is easily checked that g satisfies all the conditions of the theorem of Weymark. Given the definition of an oligarchy, this proves that F has an oligarchy. The proof is completed observing that there can be at most one oligarchy since if O and Q are distinct oligarchies, $a\ \alpha(R_j)\ b$ for all $j \in O$ and $b\ \alpha(R_l)\ a$ for some $l \in Q \backslash O$ would imply $a\ \alpha(R)\ b$ and $b\ \alpha(R)\ a$. □

From Proposition 5, we know that a k-aggregation procedure concentrates much power in the hands of the members of the unique oligarchy O. We conclude this note by some remarks about the size $|O|$ of this oligarchy.

Proposition 1 says that when $k = 1$, then there is a dictator so that $|O| = 1$. Apart from this degenerate case it is difficult, in general, to evaluate $|O|$. Based on well-known results about the dimension of a partial order (see, e.g., Dushnick and Miller, 1941, or Doignon et al., 1984), it is possible to obtain a simple result when it is supposed that the set of alternatives is sufficiently rich. We have:

Proposition 6. *As soon as $|A| \geqslant 2(k+1)$, the size of the oligarchy associated to an Arrowian k-aggregation procedure is at most k.*

Proof. Let $A = \{a_1, a_2, \ldots, a_{2k+2}, \ldots, a_m\}$ and F be an Arrowian k-aggregation procedure on A. Suppose that F has an oligarchy $O = \{j_1, j_2, \ldots, j_r\} \subset N$ with $|O| = r > k$. Consider the following preferences for the r members of the oligarchy:

$$j_1\colon a_2\, a_3 \cdots a_{k+1}\, \underline{a}_{k+2}\, \underline{a}_1\, a_{k+3}\, a_{k+4} \cdots a_{2k+2}\, [a_{2k+3} \cdots a_m],$$

$$j_2\colon a_1\, a_3\, a_4 \ldots a_{k+1}\, \underline{a}_{k+3}\, \underline{a}_2\, a_{k+2}\, a_{k+4}\, a_{k+5} \cdots a_{2k+2}\, [a_{2k+3} \ldots a_m],$$

$$j_3\colon a_1\, a_2\, a_4\, a_5 \ldots a_{k+1}\, \underline{a}_{k+4}\, \underline{a}_3\, a_{k+2}\, a_{k+3}\, a_{k+5}\, a_{k+6} \cdots a_{2k+2}\, [a_{2k+3} \ldots a_m],$$

$$\vdots$$

$$j_{k+1}\colon a_1\, a_2\, a_3 \cdots a_k\, \underline{a}_{2k+2}\, \underline{a}_{k+1}\, a_{k+2}\, a_{k+3} \cdots a_{2k+1}\, [a_{2k+3} \ldots a_m],$$

$$j_{k+2}, j_{k+3}, \ldots, j_r\colon [a_1\, a_2 \ldots a_{k+1}][a_{k+2}\, a_{k+3} \ldots a_{2k+2}][a_{2k+3} \ldots a_m].$$

Since j_1 is a member of the oligarchy, we know that $a_{k+2}\ \alpha(R^1)\ a_1$ for some $R^1 \in F(\langle R_i \rangle)$. For all $j \in O$, we have:

$$a_1\ \alpha(R_j)\ a_l \quad \text{for} \quad l = k+3,\ k+4, \ldots, m, \text{ and}$$

$$a_{l'}\ \alpha(R_j)\ a_{k+2} \quad \text{for} \quad l' = 2, 3, \ldots, k+1.$$

Thus, given the definition of an oligarchy, these preferences must be part of all the elements of $F(\langle R_i \rangle)$. Since $a_{k+2}\ \alpha(R^1)\ a_1$, it is easy to see that R^1 cannot contain any of the k other underlined preferences. Applying a similar argument to all the underlined preferences shows that $|F(\langle R_i \rangle)| \geqslant k+1$, contradicting the fact that F is a k-aggregation procedure. □

This last result is easily interpreted. When k is small, a k-aggregation procedure gives much power to a small group of individuals since the oligarchy contains at most k individuals. Larger values of k allow a fairer distribution of power but at the cost of a loss of efficiency. Thus, if the reasonableness of conditions P and I is admitted, Proposition 6 establishes a tradeoff between efficiency and democracy for k-aggregation procedures. In this framework, Arrow's theorem can be seen as depicting an extreme aspect of this tradeoff: when efficiency is at its maximum, democracy is at its minimum.

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