European Journal of Operational Research 26 (1986) 150–160
North-Holland

Some remarks on the notion of compensation in MCDM

Denis BOUYSSOU *

Groupe de Recherche sur l'Aide à la Décision et l'Evaluation (Grade), Faculté des sciences de l'administration, Université Laval, Cité universitaire, Quebec G1K 7P4, Canada

Abstract: This paper presents general definitions of compensation and noncompensation in MCDM within the framework of Multiattribute Preference Structures. The interest of using a more or less compensatory aggregation procedure is discussed. General aggregation procedures, that allow to mix compensatory and noncompensatory features in a consistent way, are introduced. They receive a complete axiomatic treatment for the two-attribute case, and it is shown that they contain most currently used aggregation procedures as particular cases.

Keywords: Multiple criteria, decision theory, measurement

Introduction

Aggregating several dimensions, as this is done in MCDM, implies taking a position on the problem of 'compensation'. Surprisingly enough, this topic is absent from the subject index of most textbooks on MCDM (see Zeleny (1982), Goicochea et al. (1982), Chankong and Haimes (1983), Keeney and Raiffa (1976)). When it is dealt with explicitely, compensation seems a controversial topic since, for instance, Hwang and Yoon (1981, p. 25) classify ELECTRE I and II as 'compensatory', whereas Bouyssou and Vansnick (1985) use them to illustrate 'noncompensatory' aggregation procedures. Nevertheless, the literature on MCDM very often appeals to notions such as 'weights', 'tradeoffs', 'lexicographic order' ... which, intuitively are closely related to the problem of compensation.

This paper intends to clarify this notion. Its first section is devoted to the study of possible formal definitions of compensation. In a second section, we shall analyse some desirable properties of MCDM aggregation procedures (MCDM a.p. in the sequel) as regards to compensation. The last two sections will analyse a number of MCDM a.p. in the light of these properties.

* Now at LAMSADE, Université de Paris Dauphine, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16. I thank Philippe Vincke, Jean-Marc Martel and Raymond Nadeau for their helpful comments on earlier drafts of this paper.

Received September 1985

1. On possible definitions of compensation

The aim of this section is to propose fairly general definitions of the notion of compensation. We first argue that meaningful definitions of compensation can only be obtained within the framework of the preference structures encountered in MCDM. The rest of the section is devoted to new definitions of noncompensation and compensation.

1.1. Multiattribute preference structures

Intuitively, compensation refers to the existence of 'tradeoffs', i.e. the possibility of offsetting a 'disadvantage' on some attribute by a sufficiently large 'advantage' on another attribute—whereas smaller 'advantages' would not do the same. Previous works on the notion of compensation (e.g.

0377-2217/86/$3.50

Plott et al. (1975), Fishburn (1976 and 1978), Bouyssou and Vansnick (1985)) have concentrated on the problem of the 'compensatoriness' of a preference relation on multiattributed alternatives. The basic idea used in these papers is simple: A preference relation is noncompensatory if no tradeoffs occur and is compensatory otherwise. The definition of compensation therefore boils down to that of a tradeoff.

In order to arrive at such a definition, it is essential to know what is to be considered as an 'advantage' or as a 'disadvantage' on an attribute or on a group of attributes. In the general case (e.g. when no independence hypotheses are involved) this is obviously very difficult and of little practical interest. In this section, we shall restrict our attention to a particular case that seems to us representative of the type of situations encountered in MCDM.

Let X, a set of alternatives, be the cartesian product of n nonempty sets $X_1, X_2, \ldots, X_n$. An MCDM a.p. can be seen as a way of building a global preference relation $\succcurlyeq$ on X on the basis of a preference relation $\succcurlyeq_i$ on each X_i and 'some other information'. This is mostly done supposing some kind of numerical translation of the $\succcurlyeq_i$ and the 'other information'. Given $\succcurlyeq_1, \succcurlyeq_2, \ldots, \succcurlyeq_n$, we expect $\succcurlyeq$ to satisfy a number of properties, if it has been obtained using an MCDM a.p. We shall say that $(X, \succcurlyeq, \succcurlyeq_1, \succcurlyeq_2, \ldots, \succcurlyeq_n)$ is a multiattribute preference structure (MPS) if:

(1) $\succcurlyeq$ is reflexive and independent (see Krantz et al. (1971, p. 301) for a definition);

(2) for all $i \in \{1, 2, \ldots, n\}$, $\succcurlyeq_i$ is complete and $x_i \succcurlyeq_i y_i$ iff $x_i \succcurlyeq_i^0 y_i$;

(3) for all $x, y \in X$, for all $i \in \{1, 2, \ldots, n\}$ and $z_i, w_i \in X_i$, $x \succ y$ and $z_i \succ_i x_i$ imply $(z_i, (x_j)_{j \neq i}) \succ y$; $x \succ y$ and $y_i \succ_i w_i$ imply $x \succ (w_i, (y_i)_{j \neq i})$; where $\succcurlyeq_I^0$ is the binary relation on $\prod_{i \in I} X_i$ deduced from $\succcurlyeq$ by independence (throughout the paper we shall use $\succ$, $\sim$, $\succ_I$, $\sim_I$ in the usual way, i.e. $x \succ y$ iff $x \succcurlyeq y$ and not $y \succcurlyeq x$, $x \sim y$ iff $x \succcurlyeq y$ and $y \succcurlyeq x$).

The reflexivity of $\succcurlyeq$ is hardly a limitation. The independence hypothesis may seem much more restrictive. Nevertheless most MCDM a.p., often implicitly, use independence in order to arrive at $\succcurlyeq$. Part two of the definition requires each $\succcurlyeq_i$ to be complete, which seems unrestrictive at least in the deterministic case.

It also requires that each $\succcurlyeq_i$ is 'preserved' in the global preference relation. This seems plausible if $\succcurlyeq_i$ is interpreted as a preference relation between 'real' evaluations (as opposed to 'ideal' one —see Roy and Bouyssou (1985, 1986) or Roy (1985) on this point). Therefore, we shall not distinguish $\succcurlyeq_i$ from $\succcurlyeq_i^0$ in the sequel. The last condition is the most important part of this definition. It states a monotonicity condition that, in our opinion, allows to speak of 'advantages' and 'disadvantages' in a consistent way. It is easily seen that in a conjunction with (1) and (2), it entails the transitivity of each $\succ_i$ and that $x_i \succ_i y_i$ for all $i \in I \subset \{1, 2, \ldots, n\}$ implies $(x_i)_{i \in I} \succ_I^0 (y_i)_{i \in I}$. As will become apparent later, a much more demanding condition is obtained if we replace $\succ_i$ by $\succcurlyeq_i$ in part three of the definition. Although these conditions may seem too restrictive from a purely theoretical point of view, we are not aware of any MCDM a.p. that does not produce MPS.

1.2. Noncompensatory MPS

Within the framework of a MPS, the definition of an 'advantage' and of a 'disadvantage' is rather obvious. When comparing x to y, attributes for which $x_i \succ_i y_i$ favor x and attributes for which $y_i \succ_i x_i$ favor y. Given part (3) of our definition, it makes sense to partition $\{1, 2, \ldots, n\}$ into three sets:

$$P(x, y) = \{i \in \{1, 2, \ldots, n\}: \quad x_i \succ_i y_i\},$$

$$P(y, x) = \{i \in \{1, 2, \ldots, n\}: \quad y_i \succ_i x_i\},$$

and

$$I(x, y) = I(y, x) = \{i \in \{1, 2, \ldots, n\}: \quad x_i \sim_i y_i\}.$$

In this context, it seem legitimate to say that $P(x, y)$ represents an 'advantage' when comparing x to y and $P(y, x)$ a 'disadvantage'. However, the status of attributes in $I(x, y)$ is ambiguous. In previous definitions of noncompensation, it was implicitely assumed that they were neutral relatively to the comparison of x and y. When all $\succcurlyeq_i$ are transitive this seems, in general, reasonable. However if $\sim_i$ are not supposed to be transitive this is much more open to criticism. In fact our definition of a MPS does not exclude cases like: $x_i \sim_i y_i$ for all $i \in I$ and $((x_i)_{i \in I}, (z_j)_{j \notin I}) \succ ((y_i)_{i \in I}, (z_j)_{j \notin I})$, in which the conjunction of 'non-noticeable' advantages on some attributes may create an overall effect. If the non-transitivity of $\sim_i$ is due to perception thresholds we might want to exclude this possibility ($x_i \sim_i y_i$ would mean in this context that it is impossible to distinguish x_i from y_i). However as argued by Roy and Bouyssou (1985), the non transitivity of $\sim_i$ is much more often due to the fact that it is essential to build a convincing preference structure on each attribute before aggregating them, a case in which attributes in $I(x, y)$ might not be neutral. (An intermediate situation arises when there is a unique weak order underlying each $\succcurlyeq_i$, e.g. when $\succcurlyeq_i$ is a semi-order. In this case, it would be possible, using the underlying weak orders, to partition further $I(x, y)$ between, neutral, slightly favorable and slightly unfavorable attributes.)

Within this framework we propose the following

Definition. A MPS (X, $\succcurlyeq$, $\succcurlyeq_1$, $\succcurlyeq_2$,..., $\succcurlyeq_n$) is
(a) totally noncompensatory iff for all x, y, z, $w \in X$,

$$(x, y)M(z, w) \Rightarrow [x \succcurlyeq y \text{ iff } z \succcurlyeq w],$$

(b) noncompensatory iff for all x, y, z, $w \in X$,

$$\begin{aligned}&(x, y)M(z, w)\\ &\quad\Rightarrow [(x \succ y \Rightarrow \text{Not } w \succcurlyeq z) \text{ and}\\ &\qquad (x \sim y \Rightarrow \text{Not } w \succ z \text{ and Not } z \succ w)],\end{aligned}$$

where M is a binary relation on X^2 that reads 'have the same preferential profile than' and is defined by

– either $(x, y)M(z, w)$
iff $P(x, y) = P(z, w)$
and $P(y, x) = P(w, z) \cdot (M_1)$

– or $(x, y)M(z, w)$
iff $P(x, y) = P(z, w)$
$P(y, x) = P(w, z)$
and $x_i = y_i$, $z_i = w_i$
for all $i \in I(x, y) \cdot (M_2)$.

Using M_1, our definition of total noncompensation amounts to the 'regular noncompensatory preference structures' in Fishburn (1976). From the preceding discussion it is clear that this definition of M should only be used either when all $\succcurlyeq_i$ are transitive or when we have good reasons to consider that the conjunction of small differences remains a small difference. The implications of this type of noncompensation (in fact a slightly more restrictive one since $\succcurlyeq$ is not supposed to be complete here) have been thoroughly studied by Fishburn (1976) and Bouyssou and Vansnick (1985). It will suffice to say that it allows the definition of a 'more important than' relation between disjoint subsets of attributes ($I \gg J$ iff $x \succ y$ for some x, $y \in X$ such that $P(x, y) = I$, $P(y, x) = J$, $I \approx J$ iff $x \sim y$ for some x, $y \in X$ such that $P(x, y) = I$, $P(y, x) = J$) that can be, under certain conditions, represented by means of additive weights. When this is the case, the model obtained is very close to the concordance part of the ELECTRE I and II methods (Roy, 1968; Roy and Bertier, 1973). It can be shown that the lexicographic order is a particular case of our definition. As noted in Fishburn (1978) the conjunctive and disjunctive screening models do not fit too well into this definition. This is due to the fact that they do not aim at constructing a global preference but rather at separing acceptable from unacceptable actions.

Still using M_1, the notion of noncompensation introduced here is very close to the idea of 'generalized noncompensation' in Bouyssou and Vansnick (1985). As total noncompensation, it forbids reversals of preference when actions have the same preferential profile but introduces the possibility of incomparability. (Not $w \succcurlyeq z$ implies either that $z \succ w$ or that z and w are incomparable). It allows to account for possible discordance effects, as introduced in the ELECTRE methods. It is easily seen that within this case, it is also possible to define an importance relation on disjoint subsets of attributes. It has been shown in the aforementioned paper that when the discordance effects are sufficiently well-behaved, it is possible to define some kind of veto thresholds, avoiding to have $x \succcurlyeq y$ when there is an attribute in $P(y, x)$ for which y is judged 'far better' than x. The implications of noncompensation underlie the ELECTRE I and II methods and have been fully exploited in the TACTIC method (Vansnick, 1986).

Obviously, much more general definitions of noncompensation are obtained using M_2 instead of M_1. These definitions have not been studied in literature for they do not guarantee any more the existence of an unambiguous correspondence between $\succcurlyeq$ on X and an importance relation on the set of subsets of attributes.

1.3. Compensatory MPS

Considering the fact that noncompensation amounts to forbidding tradeoffs, it seems reasonable to say that a MPS is minimally compensatory when it is not noncompensatory. Using M_2, we thus obtain a definition of minimal compensation that generalizes that of Fishburn (1978):

Definition. A MPS $(X, \succcurlyeq, \succcurlyeq_1, \succcurlyeq_2, \ldots, \succcurlyeq_n)$ is minimally compensatory iff $P(x, y) = P(z, w)$, $P(y, x) = P(w, z)$, $x_i = y_i$ and $z_i = w_i$ for all $i \in I(x, y)$, $x \succcurlyeq y$ and $w \succ z$ for some $x, y, z, w \in X$.

In this case we say that the attributes in $P(x, y)$ minimally compensate those in $P(y, x)$.

Though compensation has traditionally been associated, often implicitly, with the possibility of 'matching' exactly some positive difference on I by some negative difference on J (this is the idea underlying the use of indifference curves), this definition appears much too restrictive when X is supposed to be finite. This notion of minimal compensation can be strengthened in several directions. A notion of 'total minimal compensation' can be obtained if we require that given any nonempty disjoint subsets of attributes I and J, I minimally compensates J. Furthermore, it is possible to say that I 'strongly compensates' J requiring that for all $x, y \in X$ such that $x \succ y$, $P(x, y) = J$, $P(y, x) = I$, there is a $z \in X$ such that $z \succcurlyeq x$ and $z_i = y_i$ for all $i \notin I$. Therefore a notion of perfect compensation is at hand if we ask for strong compensation to hold both ways between any two disjoint (nonempty) subsets of attributes. Strong compensation imposes severe structural restrictions on X. The reader may find interesting to compare this notion with the solvability assumptions used in the additive conjoint measurement (Krantz et al., 1971, Chapter 6).

1.4. Compensatory and noncompensatory MCDM a.p.

Though, in our opinion, no MCDM a.p. can pretend to be able to deal, in a reasonable way, with any type of set X and of preferences $\succcurlyeq_1, \succcurlyeq_2, \ldots, \succcurlyeq_n$ (the case in which n is large but X contains a small number of actions is typically not covered by most MCDM a.p.), they generally have a domain of application including many types of X and of $\succcurlyeq_i$.

The way each a.p. transforms information in order to arrive at $\succcurlyeq$ can be called its 'aggregation convention', which is generally well illustrated by the numerical transformation used. In order to avoid useless definition, we just propose at this point to say that the aggregation convention of an a.p. is minimally compensatory if for some set X, some $\succcurlyeq_1, \succcurlyeq_2, \ldots, \succcurlyeq_n$ (and some other information), it can produce a relation $\succcurlyeq$ in which I minimally compensates J, for some I, J and noncompensatory otherwise. Clearly, the convention underlying the additive utility model ($x \succcurlyeq y$ iff $\sum_{i=1}^{n} u_i(x_i) \geqslant \sum_{i=1}^{n} u_i(y_i)$) is minimally compensatory whereas a lexicographic or a concordance-discordance convention is noncompensatory.

From a practical point of view, these definitions are far from being completely satisfactory since they do not allow to rank a.p. from the most to the least compensatory (but from the preceding discussion we feel that such an objective will probably be very difficult to reach in the general case). They nevertheless give a basis to discuss the desirable properties that an a.p. should exhibit as regards to compensation.

2. The 'compensatoriness' of aggregation procedures

The idea that MCDM a.p. should be minimally compensatory underlies most of the work that has been done in this area, notable exceptions being the methods using outranking relations based on a concordance-discordance principle. In fact, the additive utility model is certainly the most popular a.p. in the field of MCDM. However, noncompensatory a.p. do have a number of very interesting features. First, by definition, they only require 'inter-attribute' information in terms of an importance relation and discordance set. Within the context of highly complex and conflictual decision processes, this may prove fruitful since such a.p. do not force the decision makers to express tradeoffs—a highly sensitive information indeed. Secondly, noncompensatory a.p., when they appeal to the idea of a veto effect, tend to 'rank' actions with 'well-balanced' evaluations before actions that may be well evaluated on a number of attributes but are very bad on others (in some situations, compensatory a.p. may produce a reverse ranking). Such a tendency appears to be very desirable since it may facilitate negociations between actors having strongly conflictual value systems (Bouyssou, 1984). It follows from there that one may wish to use an a.p. having some noncompensatory features, without ignoring the fact that people do make tradeoffs, but simply because such a.p. can prove very efficient to construct a reasonable global preference relation (for arguments favoring the use of some noncompensation in other contexts, we refer to Einhorn (1970)).

As local tradeoffs are generally easily expressed, it may thus be interesting in many situations to use an a.p. that is sufficiently flexible to admit compensation for small 'preference differences' and noncompensation elsewhere (see also Luce (1978) who emphasizes the interest of such models from a descriptive point of view). This idea underlies the next two sections.

It should be emphasized that standard compensatory a.p. (e.g. the additive utility model or the additive difference model) can be used to generate preference relations exhibiting only local tradeoffs in certain cases (in a 'paramorphic' sense, see Einhorn (1970) on this point). However the noncompensatory component (this could be formally defined saying that the MPS $(X, \succcurlyeq, \succcurlyeq_1, \succcurlyeq_2, \ldots, \succcurlyeq_n)$ is minimally compensatory but that for some $\succcurlyeq_1', \succcurlyeq_2', \ldots, \succcurlyeq_n')$ such that $\succ_i' \subset \succ_i$ for all $i \in \{1, 2, \ldots, n\}$, $(X, \succcurlyeq, \succcurlyeq_1', \succcurlyeq_2', \ldots, \succcurlyeq_n')$ is noncompensatory) of preference relations exhibited by such compensatory a.p. is not truly noncompensatory in that it is only due to the particular evaluations of the alternatives and not the way they are aggregated. From a practical point of view (in a constructive perspective) this is essential since, in order to implement such a.p., the analyst has to gather inter-attribute information with a compensatory scheme in mind. Here we are interested in a.p. that are flexible enough to allow to gather information using a 'tradeoff reasoning' for small preference differences and a noncompensatory one elsewhere.

At this point, it may be worth mentioning that a common argument against the use of noncompensatory a.p. is that they may produce non-transitive global preferences $\succcurlyeq$. As we shall see, this is not specific to noncompensatory a.p. As regards to the transitivity of $\succ$ (that of $\sim$ is liable to the classical criticisms of the transitivity of indifference), let us only mention that from the descriptive (May, 1954; Tversky, 1969), normative (Burros, 1979) and prescriptive (see the numerous applications of the ELECTRE methods as reported in Siskos et al. (1983)) points of view, it does not seem to be a compulsory requirement. When intransitivities do exist then, depending on the nature of the decision problem, one may implement a number of methods to use $\succcurlyeq$ in order to arrive at a decision prescription. Since this is out of the scope of the a.p. we shall not deal with this problem explicitly. Let us only mention here that these methods may sometimes obscure the more or less compensatory nature of the preference relation produced by the a.p.

3. A review of some aggregation procedures

This section will review a number of a.p. being rather flexible as regards to compensation. We will identify an a.p. with the numerical translation of preferential information it uses. The problem of the axiomatic foundations of MCDM a.p. is differed to the next section.

In a recent and illuminating paper Jacquet-Lagrèze (1982) has shown how most MCDM a.p. derive from the same general and, in fact, very intuitive principles. In order to compare x to y, a very general procedure consists in weighting the pros and cons of the assertion 'x is at least as good as y' and in declaring that 'x is at least as good as y' if the pros clearly outweight the cons. Obviously, one may be more or less confident in the assertion depending on the difference in 'weights'. It should be noticed that, when more than two actions are to be compared, this 'weighting' technique does not guarantee that comparisons will be transitive, since transitivity is essentially a ternary property.

Jacquet-Lagrèze (1982) has shown that most MCDM methods evaluate the pros (resp. the cons) as the sum of pros (resp. cons) on each attribute, and that on each attribute pros and cons are evaluated on the basis of a binary relation $\succcurlyeq_i$ and 'some other information' mainly concerning the importance of the attribute and evaluation of 'preference differences'. This general framework proves very fruitful for analysing MCDM a.p. from the point of view of compensation for it is sufficiently general to include compensatory and noncompensatory a.p. as particular cases. We shall restrict our attention in this paper to a.p. exhibiting only one type of global preference relation.

The additive difference model was proposed by Tversky (1969) in order to account for intransitive $\succcurlyeq$, using a very natural intra-dimensional information processing strategy perfectly in line with the idea of additivity of pros and cons. Stated formally a preference relation $\succcurlyeq$ satisfies the additive difference model if there exist real-valued functions $u_1, u_2, \ldots, u_n$ and increasing functions $\Phi_1, \Phi_2, \ldots, \Phi_n$ defined on some real intervals such that

$$x \succcurlyeq y \quad \text{iff} \quad \sum_{i=1}^{n} \Phi_i(u_i(x_i) - u_i(y_i)) \geqslant 0 \tag{1}$$

and

$$\Phi_i(-\delta) = -\Phi_i(\delta) \quad \text{for all } i \in \{1, 2, \ldots, n\} \tag{1'a}$$

and for all $\delta \in \mathbb{R}$ such that

$$u_i(x_i) - u_i(y_i) = \delta \quad \text{for some } x_i,\ y_i \in X_i.$$

Keeping in line with the original work of Tversky, it has always been understood that the difference functions Φ_i should be strictly increasing.

Apart from the fact that (1) and (1′a) imply that $\succcurlyeq$ is complete, which may not always be realistic for decision-aid purposes (see Roy (1985)), the additive difference model has two major drawbacks that were already noted by Fishburn (1980). Eqs. (1) and (1′a), together with the hypothese of strictly increasing Φ_i, obviously imply that $\succcurlyeq$ has no truly noncompensatory component and that the preference relations on each attribute (which are unambiguously defined since (1) and (1′a) imply that $\succcurlyeq$ is independent) are complete and transitive.

These two severe limitations are absent if we suppose that the difference functions are only increasing (i.e. $\delta \geqslant \delta' \Rightarrow \Phi_i(\delta) \geqslant \Phi_i(\delta')$). This gives rise to what we could call a 'weak additive difference model'. Though flat portions of Φ_i may seem strange, they allow to drop the assumption of the transitivity of $\sim_i$ retaining only that of $\succ_i$, which seems realistic in many contexts. Furthermore the weak additive difference model allows to mix compensatory and noncompensatory aspects in the same model, keeping in line with a growing literature on this topic (see Luce (1978) and Fishburn (1980)). It is easy to see that a flat Φ_i around 0 entails a nontransitive $\sim_i$ whereas a flat Φ_i for large differences indicates that only local tradeoffs occur.

Keeping in line with the idea of additivity of pros and cons, it is possible to envisage a much more general MCDM a.p. requiring the existence of real valued functions p_i or X_i^2 such that

$$x \succcurlyeq y \quad \text{iff} \quad \sum_{i=1}^{n} p_i(x_i, y_i) \geqslant 0 \tag{2}$$

and

$$p_i(x_i, y_i) = -p_i(y_i, x_i) \quad \text{for all } i = 1, 2, \ldots, n \text{ and for all } x_i,\ y_i \in X_i. \tag{2'a}$$

Conditions (1′a) and (2′a) impose a strong rationality requirement on the weights of pros and cons implying that $\succcurlyeq$ is necessarily complete. Much more general models can be obtained respectively replacing these conditions by:

$$\Phi_i(\delta) \cdot \Phi_i(-\delta) \leqslant 0 \quad \text{for all } i \in \{1, 2, \ldots, n\}$$

and for all $\delta \in \mathbb{R}$ such that

$$u_i(x_i) - u_i(y_i) = \delta \text{ for some } x_i,\ y_i \in X_i \tag{1'b}$$

and

$$p_i(x_i, y_i) \cdot p_i(y_i, x_i) \leqslant 0 \quad \text{for all } i \in \{1, 2, \ldots, n\} \text{ and for all } x_i,\ y_i \in X_i. \tag{2'b}$$

Using these conditions, (1) and (2) allow incomparability and take into account possible discordance effects that imply that some cons are 'intolerable'. The price to pay for the generality of these models is that discordance is introduced additively, which may be open to criticism.

All these models imply that $\succcurlyeq$ is independent and, denoting by $\succcurlyeq_i$ the relation induced on X_i by independence, we have the following

Proposition. (1) *If* $\succcurlyeq$ *on X satisfies* (1) *with either* (1′a) *or* (1′b), *then* $(X, \succcurlyeq, \succcurlyeq_2, \ldots, \succcurlyeq_n)$ *is a MPS.*

(2) *If* $\succcurlyeq$ *on x satisfies* (2) *with either* (2′a) *or* (2′b) *then* $(X, \succcurlyeq, \succcurlyeq_1, \succcurlyeq_2, \ldots, \succcurlyeq_n)$ *is a MPS if*

$$x_i \succ_i y_i \rightarrow p_i(x_i, z_i) \geqslant p_i(y_i, z_i) \text{ and}$$

$$p(z_i, y_i) \geqslant p_i(z_i, x_i) \quad \text{for all } i \in \{1, 2, \ldots, n\} \text{ and for all } x_i,\ y_i,\ z_i \in X_i. \tag{3}$$

Proof. Left to the reader.

The additive utility model or the TACTIC method (as presented in Bouyssou and Vansnick (1985)) obviously are particular cases of these models. This is not the case for ELECTRE I and II due to their treatment of attributes in $I(x, y)$. Taking them into account would require a reformulation of (1′b) and (2′b).

The link between these models and methods building a valued global preference relation (such as ELECTRE III—see Roy (1978) or PROMETHEE—see Brans and Vincke (1982), and also the pioneering work of Goodman (1951)) is more subtle since, in general, these methods directly use the valued preference relation to arrive at a prescription without building, as an intermediate step, a non-necessarily transitive $\succcurlyeq$, which could be analyzed in our framework. However, if we interpret the valued preference relation by declaring $x \succcurlyeq y$ iff the value attached to the arc (x, y) is greater or equal than the value attached to (y, x)—and we feel that this interpretation is in line with the 'flow' technique used in PROMETHEE—the link becomes obvious. On the contrary, if—as the distillation algorithm of ELECTRE III suggests—we declare that $x \succcurlyeq y$ iff the value attached to the arc (x, y) exceeds some threshold, our model would require some more sophistication to encompass this case.

A rather unpleasant feature of these a.p. is that they imply the neutrality of attributes in $I(x, y)$. As discussed earlier, this is probably too restrictive. The addition of a threshold in the formulation of (1) and (2) would overcome this difficulty but, since this was not critical for our purposes, we did not analyse this point further.

The a.p. presented in this section may seem exceedingly general and are compatible with many different interpretations, some of which being obviously out of the scope of MCDM. Their interest nevertheless lies in the fact that they contain many methods as particular cases and remain completely flexible from the point of view of both transitivity and compensation.

4. On the axiomatization of MCDM aggregation procedures

In the preceding section we introduced a variety of rather flexible a.p. and it may be interesting to know whether they can be axiomatized from a measurement theoretic point of view. Though it would be illusory to think that such an axiomatic analysis can give a justification to those a.p. (see Roy and Bouyssou (1986)), it surely allows a deeper understanding of the methods using them.

All the structures we introduced fall into what Krantz et al. (1971) called nondecomposable conjoint structures. Until recently this kind of structures received little attention, most of the axiomatic work dealing with multiattribute preferences having been done within the framework of classical utility theory. However, beginning with the work of Tversky (1969), there seems to be a growing interest in this topic as shown by the works of Fishburn (1978, 1980, 1985), Luce (1978), Huber (1979), Roy (1985), Croon (1984), Bouyssou and Vansnick (1986).

With the emphasis on compensation, an important problem is the choice of appropriate structural assumptions in order to obtain the desired representation, since those structural assumptions may render void some interpretations of the a.p.. For instance if we need to use unrestricted solvability (see Krantz et al. (1971, p. 256) for a precise definition) in our axioms then any kind of noncompensation is obviously excluded. In order to maintain the flexibility of the interpretation of these models, one is bound to use restricted solvability—see Krantz et al. (1971)—or a density condition. Thus the choice between two sets of structural assumptions is much more critical here than it is for standard additive conjoint measurement. This problem is not purely technical since it is well-known that unicity results vitaly depend on structural assumptions. One possible way to avoid this problem has been taken by Luce (1978). In order to combine a two-component additive utility model for small differences and a lexicographic ordering elsewhere, he explicitly states in his axioms where compensation is supposed to take place (the notion of 'small' differences is captured through the definition of 'indifference intervals'), i.e. where structural assumptions can be safely imposed. A similar step has been taken by Fishburn (1980), though his use of topological concepts renders difficult the interpretation of his structural assumptions (that Croon (1984) attempted to recast into an algebraic format using extremely strong solvability conditions), in order to axiomatize a two-component additive difference model for 'small' differences together with a lexicographic ordering. It should also be mentioned that Beals et al. (1968) and Tversky and Krantz (1970) have proposed in the context of similarity judgements models resembling (1) and (2). However their axioms are not easily transposable into a preferential context.

Throughout the rest of the paper we shall restrict our attention to the $n = 2$ case. As will become apparent, this case is fundamentally different from the $n \geqslant 3$ case for it is strongly related to ordinal rather than conjoint measurement (and this explains why we will not state unicity results here). We have the following Theorem.

Theorem 1. *Let $\succcurlyeq$ be a binary relation on a finite or denumerable set $X_1 \times X_2$. There exist two real-valued functions satisfying* (2) *and* (2′a) *iff*

A_1: *$\succcurlyeq$ is complete i.e. $x \succcurlyeq y$ or $y \succcurlyeq x$ for all x, $y \in X$ and*

A_2: *$\succcurlyeq$ verifies triple cancellation i.e. for all x_1, y_1, z_1, $w_1 \in X_1$, x_2, y_2, z_2, $w_2 \in X_2$, $x_1x_2 \succcurlyeq y_1y_2$, $y_1z_2 \succcurlyeq x_1w_2$ and $z_1w_2 \succcurlyeq w_1z_2 \to z_1x_2 \succcurlyeq w_1y_2$.*

Proof. Necessity is obvious. Sufficiency is straightforward. First observe that we can always suppose without loss of generality that $X_1 \cap X_2 = \emptyset$, since we can build a disjoint duplication of these sets as this is done in Doignon et al. (1984). Let us consider the binary relation B on $X_1^2 \cup X_2^2$ defined by

$$\alpha\beta B\lambda\delta \quad \text{iff} \quad \begin{aligned} &\alpha, \beta \in X_1, \lambda, \delta \in X_2 \text{ and } \alpha\delta \succ \beta\lambda, \\ &\alpha, \beta \in X_2, \lambda, \delta \in X_1 \text{ and } \delta\alpha \succ \lambda\beta, \\ &\alpha, \beta, \lambda, \delta \in X_1 \text{ and } \alpha\beta \succ_1 \lambda\delta, \\ &\alpha, \beta, \lambda, \delta \in X_2 \text{ and } \alpha\beta \succ_2 \lambda\delta, \end{aligned}$$

where $x_1y_1 \succ_1^* z_1w_1$ iff x_1, y_1, z_1, $w_1 \in X_1$ and $[x_1x_2 \succ y_1y_2$ and $w_1y_2 \succcurlyeq z_1x_2]$ or $[x_1x_2 \succcurlyeq y_1y_2$ and $w_1y_2 \succ z_1x_2]$ for some x_2, $y_2 \in X_2$ and $x_2y_2 \succ_2^* z_2w_2$ iff x_2, y_2, z_2, $w_2 \in X_2$ and $[x_1x_2 \succ y_1y_2$ and $y_1w_2 \succcurlyeq x_1z_2]$ or $[x_1x_2 \succcurlyeq y_1y_2$ and $y_1w_2 \succ x_1z_2]$ for some x_1, $y_1 \in X_1$.

Given the definition of B we claim that the desired representation exists if B is asymmetric and negatively transitive. Indeed, since $X_1^2 \cup X_2^2$ is countable, it admits a numerical representation h and we have

$$x_1x_2 \succ y_1y_2 \quad \text{iff} \quad h(x_1, y_1) > h(y_2, x_2)$$
$$\text{iff } h(x_2, y_2) > h(y_1, x_1).$$

To obtain the desired representation, it suffices to take

$$p_1(x_1, y_1) = h(x_1, y_1) - h(y_1, x_1)$$

and

$$p_2(x_2, y_2) = h(x_2, y_2) - h(y_2, x_2).$$

The proof that A_1 and A_2 imply that B is a weak order is long but straightforward and is left to the reader. □

This very simple result prompts a series of remarks. First, $\succcurlyeq$ being reflexive by A_1, A_2 implies that it is independent which is not surprising. Secondly, it is easily seen that the representation obtained is not regular (i.e. $p_i(x_i, y_i) = p_i(z_i, w_i)$ does not imply that $p_i'(x_i, y_i) = p_i'(z_i, w_i)$ for all other admissible representations p_i'). Thirdly, using an appropriate order density condition, this result can be generalized to the non-countable case. Fourthly, we conjecture that no such results are available for the $n \geqslant 3$ case (note that necessary and sufficient conditions can straightforwardly be obtained using the method of Scott (1964) in the finite case).

We state without proof the following theorem.

Theorem 1′. *Let $\succcurlyeq$ be a binary relation on a finite or denumerable set $X_1 \times X_2$. There exist two real-valued functions satisfying* (2), (2′a) *and* (3) *iff*

– A_1 *and* A_2,

– A_3: *For all x_1, y_1, x_1', $y_1' \in X_1$, x_2, y_2, x_2', $y_2' \in X_2$;*

$x_1x_2 \succcurlyeq y_1y_2$ *and* $x_1'x_2 \succ x_1x_2 \Rightarrow x_1'x_2 \succcurlyeq y_1y_2$;

$x_1x_2 \succcurlyeq y_1y_2$ *and* $x_1x_2' \succ x_1x_2 \Rightarrow x_1x_2' \succcurlyeq y_1y_2$;

$x_1x_2 \succcurlyeq y_1y_2$ *and* $y_1x_2 \succ y_1'x_2 \Rightarrow x_1x_2 \succcurlyeq y_1'y_2$;

$x_1x_2 \succcurlyeq y_1y_2$ *and* $x_1y_2 \succ x_1y_2' \Rightarrow x_1x_2 \succcurlyeq y_1y_2'$.

Given a binary relation $\succcurlyeq$ on $X_1 \times X_2$ we define $\succcurlyeq_1$ and $\succcurlyeq_2$ by: $x_1 \succcurlyeq_1 y_1$ iff x_1, $y_1 \in X_1$ and $x_1x_2 \succcurlyeq y_1x_2$ for all $x_2 \in X_2$ and $x_2 \succcurlyeq y_2$ iff x_2, $y_2 \in X_2$ and $x_1x_2 \succcurlyeq x_1y_2$ for all $x_1 \in X_1$. We use $\succ_1$, $\sim_1$, $\succ_2$ and $\sim_2$ in the usual way. We have the following theorem.

Theorem 2. *Let $\succcurlyeq$ be a binary relation on a countable set $X_1 \times X_2$. There exist two real-valued functions satisfying* (2) *and* (2'b) *iff*

- A_4: *Strong independence.* $\succcurlyeq_1$ *and* $\succcurlyeq_2$ *are complete.*
- A_5: *Monotonicity. For all* $x_1, y_1 \in X_1$ *and* $x_2, y_2 \in X_2$, $x_1 \succcurlyeq_1 y_1$ *and* $x_2 \succcurlyeq_2 y_2 \Rightarrow x_1x_2 \succcurlyeq y_1y_2$ *and if either* $x_1 \succ_1 y_1$ *or* $x_2 \succ_2 y_2$ *then* $x_1x_2 \succ y_1y_2$.
- A_6: *Weak cancellation. for all* $x_1, y_1, z_1, w_1, \in X_1$ *and* $x_2, y_2, z_2, w_2 \in X_2$: $x_1x_2 \succcurlyeq y_1y_2$ *and* $z_1z_2 \succcurlyeq w_1w_2$ *imply either* $x_1z_2 \succcurlyeq y_1w_2$ *or* $z_1x_2 \succcurlyeq w_1y_2$.

Proof. *Necessity.* Since (2'b) implies $p_i(x_i, x_i) = 0$, we have $x_i \succcurlyeq_i y_i \Leftrightarrow p_i(x_i, y_i) \geq 0$. Thus Not $x_i \succcurlyeq_i y_i$ and Note $y_i \succcurlyeq_i x_i$ imply $p_i(x_i, y_i)$. $p_i(y_i, x_i) > 0$ which shows the necessity of A_4. The necessity of the first part of A_5 is obvious. Suppose that $x_1 \succcurlyeq_1 y_1$ and $x_2 \succ_2 y_2$. Thus, $p_1(x_1, y_1) \geqslant 0$, $p_2(x_2, y_2) \geqslant 0$, and $p_2(y_2, x_2) < 0$. From (2'b) we have $p_1(y_1, x_1) \leqslant 0$, so that $x_1x_2 \succcurlyeq y_1y_2$ and Not $y_1y_2 \succcurlyeq x_1x_2$ which show the necessity of A_5. Suppose now that $x_1x_2 \succcurlyeq y_1y_2$, $z_1z_2 \succcurlyeq w_1w_2$, Not $x_1z_2 \succcurlyeq y_1w_2$ and Not $z_1x_2 \succcurlyeq w_1y_2$. Thus $p_1(x_1, y_1) + p_2(x_2, y_2) \geqslant 0$, $p_1(z_1, w_1) + p_2(z_2, w_2) \geqslant 0$, $p_1(x_1, y_1) + p_2(z_2, w_2) < 0$, and $p_1(z_1, w_1) + p_2(x_2, y_2) < 0$ which leads to $p_1(x_1, y_1) < p_1(z_1, w_1)$ and $p_1(z_1, w_1) < p_1(x_1, y_1)$, a contradiction. Hence A_6 is necessary.

Sufficiency. As before we shall unrestrictively suppose that $X_1^2 \cap X_2^2 = \emptyset$. We define a relation D between X_1^2 and X_2^2 by: $x_1y_1Dy_2x_2$ iff $x_1x_2 \succcurlyeq y_1y_2$. A_6 implies that D is a biorder in the sense of Doignon et al. (1984). Thus, as all sets are countable, their proposition 7 implies the existence of a real-valued function h defined up to a strictly increasing monotone transformation such that

$$x_1y_2 \succcurlyeq y_1x_2 \quad \text{iff} \quad h(x_1, y_1) \geqslant h(x_2, y_2),$$

for all $x_2, y_2 \in X_2$, $[z_1x_2 \succcurlyeq w_1y_2 \Rightarrow x_1x_2 \succcurlyeq y_1y_2]$ iff $h(x_1, y_1) \geqslant h(z_1, w_1)$,

for all $x_1, y_1 \in X_1$ $[x_1y_2 \succcurlyeq y_1x_2 \Rightarrow x_1w_2 \succcurlyeq y_1z_2]$ iff $h(x_2, y_2) \geqslant h(z_2, w_2)$,

for all $z_1, w_1 \in X_1$, $z_2, w_2 \in X_2$, $[z_1y_2 \succcurlyeq w_1x_2$ and $x_1w_2 \succcurlyeq y_1z_2 \Rightarrow z_1w_2 \succcurlyeq w_1z_2]$ iff $h(x_2, y_2) \geqslant h(x_1, y_1)$.

We know from A_5 that $x_1 \sim_1 y_1$ and $x_2 \sim_2 y_2$ imply $x_1y_2 \succcurlyeq y_1x_2$ so that $h(x_1, y_1) \geqslant h(x_2, y_2)$. Suppose now that for some $z_1, w_1 \in X_1$ and $z_2, w_2 \in X_2$ we have $z_1y_2 \succcurlyeq w_1x_2$, $x_1w_2 \succcurlyeq y_1z_2$ and Not $z_1w_2 \succcurlyeq w_1z_2$. Form A_4 and A_5 we have either $w_1 \succ_1 z_1$ or $z_2 \succ_2 w_2$ for otherwise $z_1 \succcurlyeq_1 w_1$ and $w_2 \succcurlyeq_2 z_2$ would imply $z_1w_2 \succcurlyeq w_1z_2$. But using A_5 again we see that $w_1 \succ_1 z_1$ and $x_2 \succcurlyeq_2 y_2$ imply $w_1x_2 \succ z_1y_2$, a contradiction. Similarly $z_2 \succ_2 w_2$ is impossible since $y_1 \succcurlyeq_1 x_1$ and A_5 would imply $y_1z_2 \succ x_1w_2$. Thus $h(x_2, y_2) \geqslant h(x_1, y_1)$. Therefore $x_1 \sim_1 y_1$, $x_2 \sim_2 y_2$ imply $h(x_1, y_1) = h(x_2, y_2) = \delta$ and we can always choose h so that $\delta = 0$.

We now claim that taking

$$p_1(x_1, y_1) = h(x_1, y_1) \quad \text{for all } x_1, y_1 \in X_1$$

and

$$p_2(x_2, y_2) = -h(y_2, x_2) \quad \text{for all } x_2, y_2 \in X_2$$

gives the desired representation. Indeed, we have

$$x_1x_2 \succcurlyeq y_1y_2 \quad \text{iff} \quad p_1(x_1, y_1) + p_2(x_2, y_2) \geqslant 0.$$

Furthermore, $x_i \sim_i y_i$ implies $p_i(x_i, y_i) \cdot p_i(y_i, x_i) = 0$ for $i = 1, 2$. But $x_i \succ_i y_i$ implies $p_i(x_i, y_i) \geqslant 0$ and $p_i(y_i, x_i) < 0$, so that $p_i(x_i, y_i) \cdot p_i(y_i, x_i) \leqslant 0$. This completes the proof. □

A_4 and A_5 assert that incomparability only occur, when criteria are conflicting and are rather unrestrictive within the framework of an MPS. A_6 is a rather weak cancellation condition, which is implied by triple cancellation when $\succcurlyeq$ is complete. It amounts to defining a biorder in the sense of Doignon et al. (1984) on $X_1^2 \times X_2^2$ and implies on its own the existence of two functions such that $x_1x_2 \succcurlyeq y_1y_2$ iff $p_1(x_1, y_1) + p_2(x_2, y_2) \geqslant 0$. Again, using an appropriate density condition, this result can be generalized to the non-countable case. As in the case of Theorem 1, we are not presently aware of any satisfactory generalization of this result for the $n \geqslant 3$ case. An immediate corrolary of Theorem 2 is the next theorem.

Theorem 2'. *Let* $\succcurlyeq$ *be a binary relation on a finite or denumerable set of* $X_1 \times X_2$. *There exist real-valued functions satisfying* (2), (2'b) *and* (3) *iff* A_3, A_4, A_5 *and* A_6.

The case of the weak additive difference model is more difficult in the general case. However in the finite case, it is straightforward to give necessary and sufficient conditions for (1). As in the proof of Theorem 1 we define $x_1y_1 \succ_1^* z_1w_1$ iff $[x_1x_2 \succ y_1y_2$ and $w_1y_2 \succcurlyeq z_1x_2]$ or $[x_1x_2 \succcurlyeq y_1y_2$ and $w_1y_2 \succ z_1x_2]$ for some $x_2, y_2 \in X_2$ and $x_2y_2 \succ_2^* z_2w_2$ iff $[x_1x_2 \succ y_1y_2$ and $y_1w_2 \succcurlyeq x_1z_2]$ or $[x_1x_2 \succcurlyeq y_1y_2$ and $y_1w_2 \succ x_1z_2]$ for some $x_1, y_1 \in X_1$. We have the following theorem.

Theorem 3. *Let $\succcurlyeq$ be a binary relation on a finite set $X_1 \times X_2$. There exist real-valued functions u_1, u_2, Φ_1, Φ_2 satisfying* (1) *and* (1′a) *with Φ_1 and Φ_2 increasing iff*

A_1 *and* A_2,

A_7: *For $i = 1, 2$, for all $m = 2, 3, \ldots$ and $[x_i^1, x_i^2, \ldots, x_i^m, y_i^1, y_i^2, \ldots, y_i^m, z_i^1, z_i^2, \ldots, z_i^m, w_i^1, w_i^2, \ldots, w_i^m] \in X_i$, $[x_i^1, \ldots, x_i^m, y_i^1, \ldots, y_i^m$ is a permutation of $z_i^1, \ldots, z_i^m, w_i^1, \ldots, w_i^m$ and $w_i^jy_i^j \succ_i^* x_i^jz_i^j$ for each $j < m]$ $\Rightarrow$ Not $w_i^my_i^m \succ_i^* x_i^mz_i^m$.*

Proof. The necessity of A_1 and A_2 is obvious. The necessity of A_7 is proved observing that $w_i^jy_i^j \succ_i^* x_i^jz_i^j$ implies $\Phi_i(u_i(w_i^j) - u_i(y_i^j)) > \Phi_i(u_i(x_i^j) - u_i(z_i^j))$. Thus since Φ_i is increasing, $u_i(w_i^j) - u_i(y_i^j) > u_i(x_i^j) - u(z_i^j)$ and $w_i^my_i^m \succ_i^* x_i^mz_i^m$ contradicts the permutation hypothesis.

To show sufficiency it suffices to observe that A_1 and A_2 implies the existence of real-valued functions p_1 and p_2 satisfying (2) and (2′a) and $p_i(x_i, y_i) > p_i(z_i, w_i)$ iff $x_iy_i \succ_i^* z_iw_i$ for $i = 1, 2$. A_7 implies, by Theorem 6.1 in Fishburn (1970), the existence of real-valued functions u_1 and u_2 such that

$$x_iy_i \succ_i^* z_iw_i \;\Rightarrow\; u_i(x_i) - u_i(y_i) > u_i(z_i) - u_i(w_i).$$

We define Φ_i on $\{\Delta: \Delta \in \mathbb{R}$ and $u_i(x_i) - u_i(y_i) = \Delta$ for some $x_i, y_i \in X_i\}$ by $\Phi_i(u_i(x_i) - u_i(y_i)) = p_i(x_i, y_i)$. Given the properties of p_i and $\succ_i^*$, Φ_i is obviously well-defined. To show that it is increasing suppose that $u_i(x_i) - u_i(y_i) \geqslant u_i(z_i) - u_i(w_i)$ then Not $z_iw_i \succ_i^* x_iy_i$. Thus $p_i(z_i, w_i) \leqslant p_i(x_i, y_i)$ so that Φ_i is increasing. □

Given the nature of A_7, this result is far from being satisfactory. It can be shown that A_7 does not imply A_2 or A_1 and that A_1, A_2 and A_7 holding for $i = 1$ (resp. 2) does not imply that A_7 holds for $i = 2$ (resp. 1). Though A_7 is rather difficult to interpret, it implies that $\succ_i$ is transitive for $i = 1, 2$. In fact suppose that $x_i \succ_i y_i$, $y_i \succ_i z_i$ and $z_i \succcurlyeq_i x_i$ thus $x_iy_i \succ_i^* x_iz_i$, $y_iz_i \succ_i^* x_iz_i$ and $x_iy_i \succ_i^* z_iy_i$ which is impossible since $(x_i, x_i, z_i, y_i, z_i, y_i)$ is a permutation of $(z_i, z_i, y_i, x_i, y_i, x_i)$.

The reader will check that, on the basis of Theorem 2, it is possible to obtain a counterpart of Theorem 3 using (1′b) instead of (1′a).

Conclusions

We shall briefly indicate in this section some directions that seem to offer good opportunities for future research on the subject of this paper. First, we restricted our attention throughout the paper to MCDM a.p. exhibiting only one type of preference relation and it would be interesting to know if our definitions and results can be extended to the case of valued preference relations. Secondly, we used a rather restrictive interpretation of Jacquet-Lagrèze's ideas in order to obtain simple aggregation models. The validity of this interpretation certainly deserves closer scrutiny. Thirdly, it seems crucial to know whether there exist satisfactory sufficient axiomatizations of models (1) and (2) for the $n \geqslant 3$ case using structural assumptions that do not exclude the presence of noncompensatory components in these models. Lastly, one could envisage the definition of a 'more compensatory than' relation between a.p. on the basis of the numerical representation used in (1) or (2).

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