Outranking relations: Do they have special properties ? [1]

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Outranking relations are most often built using a concordance-discordance principle. Such relations are, in general, neither transitive nor complete. This is not to say that the concordance-discordance principle does not impose some "structural" restrictions on these relations. We show why this question may be of some importance for analyzing the various techniques designed to build a recommendation on the basis of such relations. These restrictions are studied for the ELECTRE and PROMETHEE methods.

Keywords : MCDM, Outranking Relations, Social Choice Theory, Binary Choice Probabilities.

I. Introduction.

This paper is concerned with the outranking approach to Multiple Criteria Decision Aid (useful references concerning these methods include Schärlig (1985), Vincke (1992a), Roy (1991) and Roy and Bouyssou (1993)). Methods related to this approach, including the well-known family of ELECTRE methods, are often presented as the combination of two steps:

- a "construction step" in which one or several outranking relations are built and
- an "exploitation step" in which outranking relations are used to derive a recommendation.

The construction step consists in comparing alternatives taking all criteria into account. This leads to a preference model taking the form of one or several binary relations – the so-called "outranking relations" – that may be crisp or valued (our definitions and notations concerning binary relations are introduced at the end of this section). Outranking relations, in most methods, are built using a concordance-discordance principle. This principle leads to declaring that an alternative is "at least as good as" another when:

- a "sufficient" majority of criteria supports this proposition (concordance principle) and
- the opposition of the minority is not "too strong" (non-discordance principle).

It is well-known that this principle does not, in general, lead to binary relations possessing "remarkable properties" such as transitivity and completeness (this being true for valued relations independently on how we interpret these properties in the valued case, see, *e.g.* Perny (1992) or Perny and Roy (1992)).

The exploitation step aims at building a recommendation on the basis of such preference models. Depending on the problem, this recommendation may take the form of the selection

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of a subset of alternatives, the ranking of all alternatives or the sorting of alternatives into different categories. Since the preference models that are used do not, in general, possess "remarkable properties" this is not an obvious task. This calls for the application of specific techniques that depend on the type of recommendation that is looked for (see Vanderpooten (1990) or Roy and Bouyssou (1993)).

Outranking methods have often been criticized for their lack of axiomatic foundations. Recently some papers have attempted a theoretical analysis of the construction step (see, *e.g.*, Bouyssou (1986 and 1992a), Bouyssou and Vansnick (1986), Perny (1992), Vansnick (1986)) and of the exploitation step (see, *e.g.*, Bouyssou (1992b and 1995), Bouyssou and Perny (1992), Pirlot (1995) or Vincke (1992b)) of various outranking methods. In these papers outranking methods are not considered as a whole. They analyze their two steps in a separate way. The absence of "remarkable properties" of outranking relations has often been seen as the rationale for separating the analysis of the exploitation step from that of the construction step. In particular, papers dealing with the exploitation step have attempted to analyze the properties of several exploitation techniques assuming that these techniques are to be applied to *any* preference structure. This is a weak line of reasoning however. Even if we know that outranking relations do not possess "remarkable properties" they may well possess some "structural properties" when they are built using a concordance-discordance principle, *i.e.* it may well be impossible to obtain *any* preference structure using a construction technique based on concordance-discordance. The existence of such "structural properties" would render difficult the interpretation of the above-mentioned analyses of exploitation techniques. Exploitation techniques could be envisaged that would not be appealing when applied to any relation but that would behave more nicely when applied to relations possessing some "structural properties".

The aim of this paper is to investigate the existence of such "structural properties". We summarize our main conclusions below:

– with the construction techniques used in ELECTRE I and ELECTRE III outranking relations do not possess "structural properties": with ELECTRE I it is possible to obtain any crisp reflexive binary relation as an outranking relation. The same is true for ELECTRE III in the valued case. In these methods, this allows, to some extent, to separate the analysis of the exploitation step from that of the construction step ;

– the situation is different when turning to methods, such as PROMETHEE, that do not make use of the discordance concept: they lead to outranking relations having "structural properties" ;

– the characterization of these "structural properties" turns out to be a difficult task. This problem is deeply linked with classic problems in the theory of choice ;

– in spite of the absence of a complete characterization of the "structural properties" of outranking relations in PROMETHEE-like methods, we outline how an axiomatic analysis of exploitation techniques can be conducted taking into account these properties. The paper is organized as follows. The case of ELECTRE I and III is examined in section 2. In section 3 we turn to methods, such as PROMETHEE, that do not make use of the discordance concept. A final section presents some directions for future research. In the rest of this section we introduce our basic notations and definitions.

A *valued* (binary) *relation* T on a set X is a function from X×X into [0 ; 1]. It is said to be *reflexive* (resp. *irreflexive*) if T(x, x) = 1 (resp. 0), for all x ∈ X. A valued relation T on X such that T(x, y) ∈ {0, 1}, for all x, y ∈ X, is said to be *crisp*. As is usual, we note x T y instead of T(x, y) = 1 when T is a crisp relation. Concerning crisp relations, we will make use of the following classic definitions (see Fishburn (1970) or Roubens and Vincke (1985)). Let T be a crisp relation on X. This relation is said to be:

–*reflexive* if [x T x],

–*complete* if [x T y or y T x],

–*weakly complete* if [x ≠ y ⇒ x T y or y T x],

–*transitive* if [x T y and y T z ⇒ x T z],

–*antisymmetric* if [x T y and y T x ⇒ x = y],

–*asymmetric* if [x T y ⇒ Not(y T x)],

–*Ferrers* if [(x T y and z T w) ⇒ (x T w or z T y)],

–*semi-transitive* if [(x T y and y T z) ⇒ (x T w or w T z)],

for all x, y, z, w ∈ X.

We say that a crisp relation is:

– a *linear order* if it is complete, antisymmetric and transitive,

– a *strict linear order* if it is weakly complete, asymmetric and negatively transitive,

– a *weak order* (sometimes called a pre-order) if it is complete and transitive,

– a *strict weak order* if it is asymmetric and negatively transitive,

– a *semi-order* if it is complete, Ferrers and semi-transitive,

– a *strict semi-order* if it is irreflexive, Ferrers and semi-transitive,

– an *interval order* if it is complete and Ferrers

– a *strict interval order* if it is irreflexive and Ferrers,

– a *strict partial order* if it is irreflexive and transitive,

We note $\mathcal{LO}_X$ (resp. $\mathcal{SLO}_X$, $\mathcal{WO}_X$, $\mathcal{SWO}_X$, $\mathcal{SO}_X$, $\mathcal{SSO}_X$, $\mathcal{IO}_X$, $\mathcal{SIO}_X$, $\mathcal{SPO}_X$) the set of all linear orders (resp. strict linear orders, weak orders, strict weak orders, semi-orders, strict semi-orders, interval orders, strict interval orders, strict partial orders) on a set X, dropping the subscript when there is no risk of confusion about the underlying set. It is well-known (see Fishburn (1970) or Roubens and Vincke (1985)) that $\mathcal{LO}_X \subseteq \mathcal{WO}_X \subseteq \mathcal{SO}_X \subseteq \mathcal{IO}_X$ and $\mathcal{SLO}_X \subseteq \mathcal{SWO}_X \subseteq \mathcal{SSO}_X \subseteq \mathcal{SIO}_X \subseteq \mathcal{SPO}_X$, all inclusions being strict as soon as X is large enough.

We note $\iota(T)$ (resp. $\alpha(T)$) the symmetric (resp. asymmetric) part of the crisp relation T, *i.e.*

the crisp relations respectively defined by:

$x\ \iota(T)\ y \Leftrightarrow [x\ T\ y \text{ and } y\ T\ x]$ and

$x\ \alpha(T)\ y \Leftrightarrow [x\ T\ y \text{ and Not}(y\ T\ x)]$.

It is easy to see (see, *e.g.*, Roubens and Vincke (1985)) that for any crisp relation T on a set X,

$T \in \mathcal{LO}_X \Leftrightarrow \alpha(T) \in \mathcal{SLO}_X$, $T \in \mathcal{WO}_X \Leftrightarrow \alpha(T) \in \mathcal{SWO}_X$, $T \in \mathcal{SO}_X \Leftrightarrow \alpha(T) \in \mathcal{SSO}_X$ and $T \in \mathcal{IO}_X \Leftrightarrow \alpha(T) \in \mathcal{SIO}_X$.

Throughout this paper A = {a, b, c, …} will denote a *finite* set with $|A| = m \geq 2$ elements. We interpret the elements of A as "alternatives" to be compared using an outranking method.

II. ELECTRE I and ELECTRE III.

II-1. A basic result.

ELECTRE I (Roy (1968)) and ELECTRE III (Roy (1978)) are among the most popular outranking methods. ELECTRE I aims at building a crisp outranking relation starting with a set of alternatives evaluated on several "true-criteria". ELECTRE III builds a valued outranking relation starting with a set of alternatives evaluated on several "pseudo-criteria" (on the notions of true and pseudo criterion, see Roy (1985)). We briefly recall here, from a purely algorithmic point of view, the principles of the construction techniques that are used in these two methods (for a thorough discussion of these methods, see Roy and Bouyssou (1993)). As indicated in section 1, we denote by A a finite set (of "alternatives") with m elements.

Consider an "ELECTRE I situation on A" consisting in:

– a strictly positive integer n (the "number of criteria"),
– a real number (the "concordance threshold") $s \in [0,5 ; 1]$,
– n functions (the "criteria") $g_1, g_2, \ldots, g_n$ from A into $\mathbb{R}$,
– n functions (the "veto thresholds") $v_1, v_2, \ldots, v_n$ from $\mathbb{R}$ into $\mathbb{R}_+$ such that, $\forall i \in \{1, 2, \ldots n\}$ and $\forall a, b \in A$, $g_i(a) \geq g_i(b) \Rightarrow g_i(a) + v_i(g_i(a)) \geq g_i(b) + v_i(g_i(b))$,
– n strictly positive real numbers (the "weights") $k_1, k_2, \ldots, k_n$.

Starting with an "ELECTRE I situation on A", the construction technique of ELECTRE I builds a crisp relation S on A (*i.e.*, a subset of AxA) letting, for all $a, b \in A$:

$a\ S\ b \Leftrightarrow [a\ C\ b \text{ and Not } (a\ V\ b)]$

where

$$a\ C\ b \iff \frac{\sum_{j\ :\ g_j(a) \geq g_j(b)} k_j}{\sum_{i=1}^{n} k_i} \geq s$$

and

$$a\ V\ b \iff [\exists\, i \in \{1, 2, \ldots, n\} \text{ such that } g_i(b) > g_i(a) + v_i(g_i(a))].$$

The crisp relation C (resp. V) is called the concordance (resp. discordance) relation of

ELECTRE I.

Consider an "ELECTRE III situation on A" consisting in:

– a strictly positive integer n (the "number of criteria"),

– n functions (the "criteria") $g_1, g_2, \ldots, g_n$ from A into 1,

– 3n functions (the "indifference, preference and veto thresholds") $q_1, p_1, v_1, q_2, p_2, v_2, \ldots, q_n, p_n, v_n$ from 1 into 1_+ such that, $\forall$ i $\in$ {1, 2, …, n}, $\forall$ a, b $\in$ A:
$q_i(g_i(a)) \leq p_i(g_i(a)) \leq v_i(g_i(a))$ and
$g_i(a) \geq g_i(b) \Rightarrow [g_i(a) + q_i(g_i(a)) \geq g_i(b) + q_i(g_i(b)),\ g_i(a) + p_i(g_i(a)) \geq g_i(b) + p_i(g_i(b))$ and $g_i(a) + v_i(g_i(a)) \geq g_i(b) + v_i(g_i(b))]$

– n strictly positive real numbers (the "weights") $k_1, k_2, \ldots, k_n$.

Starting with an "ELECTRE III situation on A", the construction technique of ELECTRE III builds a valued relation S on A (*i.e.* a function from AxA into [0 ; 1]) letting, for all a, b $\in$ A:

$$S(a, b) = C(a, b)\cdot(1 - D(a, b))$$

where

$$C(a, b) = \frac{\sum_{i=1}^{n} k_i \cdot C_i(a, b)}{\sum_{i=1}^{n} k_i} \text{ with } C_i(a, b) = \begin{cases} 1 \text{ if } g_i(b) - g_i(a) \leq q_i(g_i(a)) \\ 0 \text{ if } g_i(b) - g_i(a) > p_i(g_i(a)) \\ \dfrac{p_i(g_i(a)) - (g_i(b) - g_i(a))}{p_i(g_i(a)) - q_i(g_i(a)} \text{ otherwise.} \end{cases}$$

and

$$D(a, b) = \begin{cases} 0 \text{ if } D_{ab} = \{j \in \{1, 2, \ldots, n\} : D_j(a, b) > C(a, b)\} = \emptyset \\ 1 - \prod_{i \in D_{ab}} \dfrac{1 - D_i(a, b)}{1 - C(a, b)} \text{ otherwise} \end{cases}$$

with

$$D_i(a, b) = \begin{cases} 1 \text{ if } g_i(b) - g_i(a) > v_i(g_i(a)) \\ 0 \text{ if } g_i(b) - g_i(a) \leq p_i(g_i(a)) \\ \dfrac{g_i(b) - g_i(a) - p_i(g_i(a))}{v_i(g_i(a)) - p_i(g_i(a))} \text{ otherwise.} \end{cases}$$

The valued relation C is called the concordance relation of ELECTRE III.

It is easily seen that an outranking relation S built with either method is necessarily reflexive (*i.e.*, $\forall$ a $\in$ A, a S a in the crisp case and S(a, a) = 1 in the valued case). Apart from reflexivity, do these relations possess any "structural property" ? The following simple proposition shows that this is not the case.

Proposition 1.

(a) Let T be any reflexive crisp relation on a finite set A. There is an "ELECTRE I situation on A" such that applying the construction technique of ELECTRE I to this situation leads to an outranking relation identical to T.

(b) Let T be any reflexive valued relation on a finite set A. There is an "ELECTRE III situation on A" such that applying the construction technique of ELECTRE III to this situation leads to an outranking relation identical to T.

Proof. It consists in exhibiting the appropriate "situation". In both cases the "situation" consists in:

one criterion having "much weight" on which all alternatives have identical evaluations;

several criteria having "little weight". On each of them we introduce a "discordance effect" for a selected pair of alternatives.

(a) Let T be a reflexive crisp relation on a finite set A. Let $u = m^2 - |T|$, where $m = |A|$. If $u = 0$, ELECTRE I will lead to T using a "situation" consisting in a unique criterion on which all alternatives have an identical evaluation. When $u \neq 0$, consider a one-to-one correspondence between $\{2, 3, \dots, u+1\}$ and the ordered pairs $(a, b) \in A^2$ such that Not(a T b). Let us build a "situation" such that:

- $n = 1 + u$,
- $s = 1/2$,
- $g_1(c) = 0, \forall c \in A$, and $v_1(x) = 1, \forall x \in \mathbf{1}$,

  – for $i = 2, 3, \dots, u+1$, suppose that i corresponds to the ordered pair (a, b) such that Not(a T b) and let us choose the functions g_i and v_i so that:

  $g_i(a) = 0$, $g_i(b) = 1$, $g_i(c) = 0.5, \forall c \in A \setminus \{a, b\}$,

  $v_i(x) = 0.6, \forall x \in \mathbf{1}$.

  – $k_1 = 1/2$, $k_2 = k_3 = \dots = k_{1+u} = 1/2u$.

It is easily seen that applying ELECTRE I to this "situation" will lead to an outranking relation identical to T. Since all alternatives have an identical evaluation on g_1 and $k_1 = s = 1/2$, the concordance relation C is complete. For each ordered pair (a, b) such that Not(a T b) we have introduced a criterion for which a V b which allows to recover T.

(b) Let T be a reflexive valued relation on a finite set A. If $T(a, b) = 1, \forall a, b \in A$, ELECTRE III will lead to T using a "situation" consisting in a single criterion on which all alternatives have an identical evaluation. When this is not the case, let:

$U = \{(a, b) \in A^2 : T(a, b) \neq 1\}$, $u = |U|$ and

$$t^{Max} = \max_{(a, b) \in U} T(a, b).$$

By hypothesis, we have, $t^{Max} \in [0 ; 1[$ and $u \in \{1, 2, \dots, m(m-1)\}$.

Let t be such that $t^{Max} < t < 1$. We note:

$$\gamma = t + (u-1)\frac{1-t}{u}.$$

It is clear that $t^{Max} < \gamma < 1$. Consider a one-to-one correspondence between U and $\{2, 3, \dots, u+1\}$.

Let us build a "situation" such that:

– $n = 1 + u$,

– $g_1(c) = 0$, $\forall$ c $\in$ A, and $q_1(x) = p_1(x) = 0$, $v_1(x) = 1$, $\forall x \in \mathbb{1}$,

– for i = 2, 3, …, u+1, suppose that i corresponds to the ordered pair (a, b) $\in$ U and let us choose the functions g_i, q_i, p_i and v_i so that:

$g_i(b) = 1$, $g_i(a) = 0$, $g_i(c) = 0.5$, $\forall$ c $\in$ A\{a, b},

$q_i(x) = p_i(x) = 0.5$, $\forall x \in \mathbb{1}$,

$v_i(x) = \zeta_i$, $\forall x \in \mathbb{1}$, where:

$$\zeta_i = 0.5 + \frac{0.5\gamma}{\gamma - (1-\gamma)T(a,b)},$$

– $k_1 = t$, $k_i = (1-t)/u$ for i = 2, 3, …, u+1.

Observe that $\zeta_i \geq 1$, i = 2, 3, ..., u+1, so that the functions v_i are admissible for ELECTRE III.

The application of ELECTRE III to this "situation" leads to an outranking relation S.

Consider an ordered pair (a, b) such that T(a, b) = 1. It is easy to see that we have $C_i(a, b) = 1$ and $D_i(a, b) = 0$, $\forall$ i $\in$ {1, 2, …, u+1} so that S(a, b) = 1.

Consider now an ordered pair (a, b) $\in$ U, *i.e.* such that T(a, b) < 1. Let us denote by i the unique element in {2, 3, …, u+1} corresponding to this ordered pair. We have for the concordance part:

$C_j(a, b) = 1$, $\forall$ j $\neq$ i, and $C_i(a, b) = 0$ so that

C(a, b) = γ and T(a, b) < C(a, b) < 1.

For the discordance part, we have:

$D_j(a, b) = 0$, $\forall$ j $\neq$ i, and

$$D_i(a,\ b) = \text{Min}[1\ ;\ \text{Max}[0\ ;\ \frac{g_i(b) - g_i(a) - p_i(g_i(a))}{v_i(g_i(a)) - p_i(g_i(a))}]] = \text{Min}[1\ ;\ \text{Max}[0\ ;\ \frac{1-0.5}{\zeta_i - 0.5}]]$$

$$= \text{Min}[1\ ;\ \frac{0.5}{[0.5 + \frac{0.5\gamma}{\gamma-(1-\gamma)T(a,b)}] - 0.5}] = \frac{\gamma - (1-\gamma)T(a,b)}{\gamma}.$$

Since $\frac{\gamma - (1-\gamma)T(a,b)}{\gamma} > \gamma$, we have :

$$S(a,\ b) = C(a,\ b)\frac{1\ -\ D_i(a,\ b)}{1\ -\ C(a,\ b)} = \gamma \frac{1 - \frac{\gamma-(1-\gamma)T(a,b)}{\gamma}}{1-\gamma} = T(a,b).$$

This completes the proof. □

Proposition 1 implies that for the axiomatic investigation of exploitation techniques to be coupled with ELECTRE I and ELECTRE III, it makes sense to suppose that they will be confronted to any (reflexive) preference structure. Such investigations were conducted in the crisp case by Vincke (1992b) and in the valued case by Pirlot (1995), Bouyssou (1992b and 1995) and Bouyssou and Perny (1992). However, one should not conclude from proposition 1 that it is legitimate for ELECTRE I and ELECTRE III to completely separate the construction and the exploitation step. In particular in the valued case, this proposition says nothing about the nature and the interpretation of the "valuations" used to model preferences and consequently, about the operations that we can legitimately use on them so as to stay consistent with the way they have been built. This difficult problem concerning valued relations is still widely open (see Perny (1992)).

II-2. Remarks and extensions.

In this subsection, we add some comments to the very simple proposition 1.

i) The construction step of ELECTRE I being a particular case of that of ELECTRE IS (see Roy and Skalka (1984) or Roy and Bouyssou (1993)), part (a) of proposition 1 directly applies to ELECTRE IS. A simple modification of this proof shows that any two nested crisp reflexive relations on a finite set can be obtained as the result of the ELECTRE II construction technique (see Roy and Bertier (1973)).

ii) The proof proposition 1 uses "situations" involving up to m(m-1) + 1 criteria (the next remark shows that m(m+1) criteria are always sufficient for ELECTRE I). In most cases, it is possible to use more "realistic" "situations", *i.e.* "situations" using a more reasonable number of criteria. For each method it would be interesting to know what is the minimum number of criteria that has to be used in order to be able to recover any reflexive relation. This raises interesting combinatorial problems that will not be dealt with here (similar problems arise with the method of majority decisions; a basic reference on the subject is Stearns (1957)).

iii) The proof of part (a) of proposition 1 makes great use of the possibility to introduce discordance effects "at will" with ELECTRE I. Building upon a famous result of McGarvey (1953) concerning the method of majority decision, let us show that part (a) of the proposition remains true even if when there are no discordance effects, *i.e.* when the veto thresholds are large enough to imply V = Ø. Let T be a reflexive crisp relation on A. For any pair of distinct alternatives {a, b} we have one and only one of the following situations:

i- [a T b and b T a]

ii- [a T b and Not(b T a)]

iii- [Not(a T b) and Not(b T a)].

Consider a one-to-one correspondence between {1, 3, …, m(m-1) - 1} and the m(m-1)/2 pairs of distinct alternatives in A. Consider a "situation" such that:

– n = m(m - 1)

– 1/2 < s < 1/2 + 1/m(m-1)

– for i = 1, 3, …, m(m-1) - 1. Suppose that i corresponds to the pair {a, b} of distinct alternatives. Consider a one-to-one correspondence f between A\{a, b} and {1, 2, …, m - 2} and define g_i and g_{i+1} letting:

$g_i(a) = g_i(b) = m - 1$, $g_i(c) = f(c)$, $\forall\ c \in A\backslash\{a, b\}$ and

$g_{i+1}(a) = g_{i+1}(b) = 1$, $g_{i+1}(c) = m - f(c)$, $\forall\ c \in A\backslash\{a, b\}$ if [a T b and b T a],

$g_i(a) = m$, $g_i(b) = m - 1$, $g_i(c) = f(c)$, $\forall\ c \in A\backslash\{a, b\}$ and

$g_{i+1}(a) = 1$, $g_{i+1}(b) = 0$, $g_{i+1}(c) = m - f(c)$, $\forall\ c \in A\backslash\{a, b\}$ if [a T b and Not(b T a)],

$g_i(a) = m$, $g_i(b) = m - 1$, $g_i(c) = f(c)$, $\forall\ c \in A\backslash\{a, b\}$ and

$g_{i+1}(a) = 0$, $g_{i+1}(b) = 1$, $g_{i+1}(c) = m - f(c)$, $\forall\ c \in A\backslash\{a, b\}$ if [Not(a T b) and Not(b T a)],

– $k_1 = k_2 = \ldots = k_n = 1/n$.

Let us show that C = T, C being the concordance relation obtained by applying ELECTRE I to this "situation". Let {a, b} be a pair of distinct alternatives in A.

We note $r_{ab} = \sum_{j\ :\ g_j(a) \geq g_j(b)} k_j$ and $r_{ba} = \sum_{j\ :\ g_j(b) \geq g_j(a)} k_j$.

It is easy to see that:

$r_{ab} = r_{ba} = m(m-1)/2 + 1$ if a T b and b T a,

$r_{ab} = m(m-1)/2 + 1$ and $r_{ba} = m(m-1)/2 - 1$ if a T b and Not(b T a),

$r_{ab} = r_{ba} = m(m-1)/2$ if Not(a T b) and Not(b T a).

Thus, for all $s \in\ ]1/2\ ;\ 1/2 + 1/m(m-1)[$, we have T = C.

Extending in a similar way a previous result by Deb (1976), it is not difficult to show that the following stronger result holds:

Let T be a reflexive crisp relation on a finite set A and $\lambda \in\ ]0.5\ ;\ 1[$. There is an "ELECTRE I situation on A" with $s = \lambda$ such that applying the construction technique of ELECTRE I to this situation leads to a concordance relation C identical to T.

This result does not hold for $\lambda = 1$ (resp. 0.5) since, in that case, C is necessarily transitive (resp. complete).

Let us finally mention that these techniques can easily be transposed to other ways of building crisp concordance relations. For instance, it can be shown that every asymmetric crisp relation T can be obtained as a concordance relation in the TACTIC method (see Vansnick (1986); let us recall that concordance relations in TACTIC are necessarily asymmetric).

iv) Contrary to the situation with ELECTRE I, it is not possible to obtain any reflexive valued relation as the result of ELECTRE III if the discordance part of the method is not used, *i.e.* when S(a, b) = C(a, b), $\forall$ a, b $\in$ A, which amounts to choosing "very large" veto thresholds v_i. Consider the following valued relation T (valued relations in matrix form are always read from row to column):

T	a	b	c
a	1	1	0
b	0	1	1
c	1	0	1

Suppose that there is an "ELECTRE III situation on A" such that C = T (from part (b) of proposition 1, we know that there is a "situation" such that S = T). In such a "situation" we have, $\forall$ i $\in$ {1, 2, …, n}, $g_i(a) > g_i(b) + p_i(g_i(b))$, $g_i(b) > g_i(c) + p_i(g_i(c))$, $g_i(c) > g_i(a) + p_i(g_i(a))$, which is contradictory since p_i is always non-negative. Thus, concordance relations in ELECTRE III possess "structural properties". We study them in the next section.

III. Valued concordance relations and "binary choice probabilities".

III-1. Valued concordance relations, stepped relations and semi-orders.

In order to analyze the "structural properties" of valued concordance relations, we use the following general framework. Consider a "Generalized Strict Concordance situation on A" consisting in:

– a strictly positive integer n,
– n functions $g_1, g_2, \ldots, g_n$ from A into 1,
– n functions $t_1, t_2, \ldots, t_n$ from 1^2 into [0 ; 1] such that, $\forall$ i $\in$ {1, 2, …, n}, t_i is non-decreasing (resp. non-increasing) in its first (resp. second) argument and $t_i(x, x) = 0$, $\forall \supseteq x \in$ 1,
– n strictly positive real numbers $k_1, k_2, \ldots, k_n$.

On the basis of such a "situation", the "Generalized Strict Concordance" method or more briefly the GSC method (inspired by Perny (1992) and Perny and Roy (1992)) leads to a valued relation P on A letting, $\forall$ a, b $\in$ A:

$$P(a,\ b)\ =\ \frac{\sum_{i=1}^{n} k_i \cdot\ t_i(g_i(a),\ g_i(b))}{\sum_{i=1}^{n} k_i}.$$

We denote by $\mathcal{GSC}_A$ (or simply $\mathcal{GSC}$ when there is no risk of confusion) the set of all valued relations on A that can be obtained with the GSC method on the basis of a "Generalized Strict Concordance situation on A". Since it was supposed that $t_i(x, x) = 0$, all relations in $\mathcal{GSC}$ are obviously irreflexive (*i.e.*, T(a, a) = 0, $\forall$ a $\in$ A).

The interest of the GSC method lies in its links with the PROMETHEE method (see Brans and Vincke (1985) or Brans *et al.* (1984)) and the concordance part of ELECTRE III. First, it is easy to show that the PROMETHEE method is a particular case of the GSC method with $t_i(g_i(a), g_i(b)) = \Delta_i(g_i(a) - g_i(b))$, the functions Δ_i used in PROMETHEE being non-decreasing and such that $\Delta_i(0) = 0$. Second consider an "ELECTRE III situation" and let:

$t_i(g_i(a), g_i(b)) = 1 - C_i(b, a)$.

Such functions t_i are clearly admissible in the GSC method. Thus to each concordance relation C obtained with ELECTRE III corresponds a valued relation P obtained with the GSC method such that P(a, b) = 1 - C(b, a), $\forall$ a, b $\in$ A.

In what follows we study the structural properties of relations in $\mathcal{GSC}$. The following definitions will prove useful for this purpose.

Definition 1. Let T be a valued relation on a finite set A. The relation T is said to be a *t-g* *relation* if there are:

– a real-valued function g on A and

– a function t from g[A]×g[A] into [0 ; 1] being non-decreasing (resp. non-increasing) in its first (resp. second) argument and such that t(x, x) = 0, ∀⊇x ∈ g[A],

such that, for all a, b ∈ A, T(a, b) = t(g(a), g(b)).

The notion of t-g relation is very closely related to that of "monotone scalability" used in Monjardet (1984) (after Fishburn (1973)), the only difference being the addition here of a restriction on t(x, x). By construction, relations in $\mathcal{GSC}$ are "convex mixtures" of t-g relations.

Definition 2. Let T be a valued relation on a finite set A. We say that T is *upper diagonal stepped* if there is a linear order (*i.e.*, a complete, antisymmetric and transitive crisp relation) V on A such that, for all a, b ∈ A:

a V b ⇒ T(b, a) = 0 and

a V b ⇒ T(a, c) ≥ T(b, c) and T(c, a) ≤ T(c, b), ∀ c ∈ A.

Apart from the restriction that a V b ⇒ T(b, a) = 0, an upper diagonal stepped relation is identical to a relation having a "monotone board" as defined in Monjardet (1984).

Definition 3. Let T be a valued relation on a finite set A. We say that T is *linear* if, for all a, b, c, d ∈ A, [T(a, c) > T(b, c) or T(c, a) < T(c, b)] ⇒ [T(a, d) ≥ T(b, d) and T(d, a) ≤ T(d, b)].

The following lemma is a direct consequence of Theorem 13 in Monjardet (1984). For the sake of completeness we outline its proof.

Lemma 1. Let T be a valued relation on a finite set A. The following statements are equivalent:

(i) T is a t-g relation,

(ii) T is irreflexive and linear,

(iii) T is upper diagonal stepped.

Proof.

(i) ⇒ (ii). Obvious.

(ii) ⇒ (iii). Define the crisp relation S_T, the strict trace of T, letting, for all a, b ∈ A,

a S_T b ⇔ [T(a, c) > T(b, c) or T(c, a) < T(c, b) for some c ∈ A]. The relation S_T is clearly negatively transitive. It is easy to see that it is also asymmetric when T is linear so that $S_T \in \mathcal{SWO}_A$. Consider now any linear order V such that $S_T \subseteq \alpha(V)$ (such a linear order exists by Szpilrajn's lemma). Using the irreflexivity and the linearity of T, it is easy to prove that T is upper diagonal stepped using such a linear order.

(iii) ⇒ (i). Since A is a finite set and V is a linear order, there is a real-valued function g on A such that, for all a, b ∈ A, g(a) ≥ g(b) ⇔ a V b (see, *e.g.* Roubens and Vincke (1985)). Given such a function g define t(g(a), g(b)) = T(a, b). The valued relation T being upper diagonal stepped, it is easy to prove that t is a well-defined real-valued function on g[A]xg[A], has the required monotonicity property and is such that t(g(a), g(a)) = 0, for all a ∈ A (for details see theorems 1 and A in Fishburn (1973)). □

Following the terminology used in Doignon *et al.* (1986), a valued relation satisfying the conditions of lemma 1 will be called a *strict linear semiordered valued relation*, or, more briefly, a *valued strict semi-order*. More general forms of linearity and closely related definitions and results can be found in Doignon *et al.* (1986). This paper also gives many references and a thorough historical background on the subject.

Definition 4. Let $\mathcal{K}$ be a set of crisp relations on a finite set A. The valued relation T on A is said to be *representable* in $\mathcal{K}$ if there is a function φ from $\mathcal{K}$ into [0 ; 1] such that:

$$\sum_{K \in \mathcal{K}} \varphi(K) = 1,$$

for which:

$$T(a, b) = \sum_{K \in \mathcal{K}} \varphi(K) \cdot K(a, b), \ \forall\, a, b \in A.$$

From lemma 1, we know that relation in $\mathcal{GSC}$ are "convex mixtures" of upper diagonal stepped relations, *i.e.*, valued strict semi-orders. We proceed by showing that valued strict semi-orders are particular convex mixtures of elements of $\mathcal{SSO}$, *i.e.* crisp strict semi-orders.

Lemma 2. Let T be a crisp relation on a finite set A. The relation T is upper diagonal stepped if and only if T ∈ $\mathcal{SSO}_A$.

Proof. Results immediately from the classical properties of strict semi-orders, see, *e.g.*, Fishburn (1970) or Roubens and Vincke (1985). □

Lemma 3. Let T be a valued relation on a finite set A. If T is upper diagonal stepped then it is representable in $\mathcal{SSO}_A$.

Proof. Since A is finite and T is upper diagonal stepped, it takes at most m(m-1)/2 strictly positive values. These m(m-1)/2, non-necessarily distinct, strictly positive are such that:

$0 < q_1 \leq q_2 \leq \ldots \leq q_{m(m-1)/2} \leq 1$. Let $q^* = 1 - q_{m(m-1)/2}$.

For any i = 1, 2, …, m(m-1)/2, define the crisp relation T_i on A letting, for all a, b ∈ A:

$a\, T_i\, b \Leftrightarrow T(a, b) \geq q_i$. Since T is upper diagonal stepped, it is easy to see that T_i is upper diagonal stepped, for i = 1, 2, …, m(m-1)/2. Thus, we know from lemma 2, that $T_i \in \mathcal{SSO}_A$. Let T* be the (crisp) empty relation on A, *i.e.* the relation such that Not(a T* b), for all a, b ∈ A. It is clear that T* ∈ $\mathcal{SSO}_A$.

Define a function φ from $\mathcal{SSO}_A$ into [0 ; 1] letting, for all W ∈ $\mathcal{SSO}_A$:

$$\varphi(W) = \begin{cases} q^* \text{ if } W = T^* \\ q_1 \text{ if } W = T_1, \\ q_i - q_{i-1} \text{ if } W = T_i, \text{ for } i = 2, 3, \ldots, m(m-1)/2 \\ 0 \text{ otherwise.} \end{cases}$$

It is easy to see that:

$$\sum_{W \in \mathcal{SSO}_A} \varphi(W) = 1,$$

and that, with this function, T is representable in $\mathcal{SSO}_A$. □

Simple examples show that the converse of lemma 3 not true. Combining lemmas 1, 2 and 3 allows for a first characterization of the elements of $\mathcal{GSC}$, showing that a valued relation is a convex mixture of valued strict semi-orders if and only if it is a convex mixture of crisp strict semi-orders.

Proposition 2.

Let P be a valued relation on a finite set A. Then $P \in \mathcal{GSC}_A$ if and only if it representable in the set $\mathcal{SSO}_A$ of all strict semi-orders on A.

Proof.

a) [P is representable in $\mathcal{SSO}_A \Rightarrow P \in \mathcal{GSC}_A$]. Let $\mathcal{K} = \{T_1, T_2, ..., T_\ell\}$ be the set of all strict semi-orders T in $\mathcal{SSO}_A$ such that $\varphi(T) > 0$ (since A is finite so is $\mathcal{SSO}$ and, hence, $\mathcal{K}$). The set A being finite, for any T_i, i = 1, 2, …, ℓ, there is (see, *e.g.*, Roubens and Vincke (1985)) a function u_i from A into $\mathbb{R}$ such that, $\forall$ a, b $\in$ A, $[a\ T_i\ b] \Leftrightarrow [u_i(a) > u_i(b) + 1]$.

Consider a "situation" involving ℓ criteria and let:

$k_i = \varphi(T_i)$, $g_i = u_i$, $t_i(x, y) = 1$ if $x > y + 1$ and 0 otherwise. It is obvious that applying the GSC method to this "situation" leads to P.

b) [$P \in \mathcal{GSC}_A \Rightarrow$ P is representable in $\mathcal{SSO}_A$]. To prove that P is representable in $\mathcal{SSO}_A$, it is sufficient to prove that the relations defined by $P_i(a, b) = t_i(g_i(a), g_i(b))$, $\forall$ a, b $\in$ A, are representable in $\mathcal{SSO}_A$, since P is a convex mixture of the relations P_i. From lemma 1, we know that the relations P_i are upper diagonal stepped and the use of lemma 3 completes the proof. □

We already know that PROMETHEE is a particular case of the GSC method. The above proof shows that if a valued relation is representable $\mathcal{SSO}$, it can be obtained as the result of PROMETHEE since the functions $t_i(x, y)$ used in the proof of proposition 2 only depend on x - y. Thus, a valued relation can be obtained with the GSC method if and only if it can be obtained with PROMETHEE[2].

Consider now the "Generalized Large Concordance" method (GLC) which is identical to GSC except that it uses functions t_i such that $t_i(x, x) = 1$. The concordance part of ELECTRE III is obviously a particular case of the GLC method. The immediate transposition of proposition 2 and the preceding remark to this case shows the equivalence of the following three propositions:

– R can be obtained with the GLC method,
– R can be obtained as a concordance relation in ELECTRE III,
– R is representable in $\mathcal{SO}_A$, the set of all semi-orders on A.

III-2. "Generalized Concordance" and binary choice probabilities.

2 This does not imply that any t-g relation can be obtained with PROMETHEE with a *single* criterion, *i.e.* with a function t depending only on the difference g(a) - g(b). On this point see Marchant (1995).

Proposition 2 leads us to study the conditions for a valued relation to be representable in the set $\mathcal{SSO}$ of strict semi-orders. Observe that irreflexivity is clearly a necessary condition for the representation in any set of irreflexive crisp relations and, thus, in $\mathcal{SSO}$. In the rest of this subsection, we therefore concentrate on conditions between "off-diagonal" elements.
Before turning to the representation problem in $\mathcal{SSO}$, let us recall that the characterization of valued relations representable in the set $\mathcal{SLO}$ of strict linear orders (notice that, apart from reflexivity conditions, this problem is equivalent to that of the representation in $\mathcal{LO}$) has received much attention in literature in which it is known as the "binary choice probabilities" problem (among the numerous papers dealing with that problem, dating back to the early fifties, let us mention the recent contributions of Cohen and Falmagne (1990), Dridi (1980), Fishburn (1987 and 1990), Fishburn and Falmagne (1989), Gilboa (1990), Gilboa and Monderer (1992), Koppen (1995) and Suck (1992); Fishburn (1992) offers an excellent survey of the available results). This is a difficult problem. Apart from the trivial irreflexivity requirement, it amounts to characterizing the set of all facets of a polyhedron in 1m(m-1) having $|\mathcal{SLO}|$ = m! vertices. We will not try here to give an exhaustive survey of the important and complex literature on the subject. We will just mention a few points that are important for our purposes.
Let P be a valued relation on A. Consider the following conditions, for all distinct a, b, c ∈ A:

P(a, b) + P(b, a) = 1 and (1)

P(a, b) + P(b, c) ≤ 1 + P(a, c). (2)

Since a strict linear order is weakly complete, asymmetric and transitive, the necessity of (1) and (2) for the representability of P in $\mathcal{SLO}$ follows. Let us notice that (2) is a necessary condition for the representation in any set of transitive crisp relations (*e.g.*, $\mathcal{WO}$, $\mathcal{SWO}$, $\mathcal{SIO}$ or $\mathcal{SPO}$). Condition (2) is more often presented under the form of the "triangle inequality":

P(a, b) + P(b, c) ≥ P(a, c). (2′)

which, together with (1), is equivalent to (2).
Together with irreflexivity, conditions (1) and (2) are known to be sufficient for the representation in $\mathcal{SLO}$ when m ≤ 5 (see Dridi (1980) and Fishburn (1987)). They are no more sufficient as soon as m ≥ 6 as shown by the following well-known example (see, *e.g.*, Dridi (1980) or Gilboa (1990); Dridi (1980) offers a whole family of such examples):

P	a	b	c	d	e	f
a	0	1/2	1/2	1	1	1/2
b	1/2	0	1/2	1	1/2	1
c	1/2	1/2	0	1/2	1	1
d	0	0	1/2	0	1/2	1/2
e	0	1/2	0	1/2	0	1/2
f	1/2	0	0	1/2	1/2	0

Routine verification shows that P is irreflexive and satisfies (1) and (2). It is not representable in $\mathcal{SLO}$ however. This is shown observing that strict linear orders T can be used to represent P only if they are such that:

a T d, a T e, b T d, b T f, c T e and c T f, (3)

Any strict linear order T satisfying (3) can include at most one of the following relations: f T a, e T b and d T c. Thus, it is impossible to have at the same time P(f, a) = 1/2, P(e, b) = 1/2 and P(d, c) = 1/2.

It is known that no *finite* set of necessary and sufficient conditions can guarantee the representation of a relation in $\mathcal{SLO}$ for all m (see Fishburn (1990) or Fishburn and Falmagne (1989)). Such conditions exist for each value of m however. The discovery of such conditions for m = 6, 7, … is an open – and difficult – problem (Fishburn (1992) mentions a recent result, obtained by enumeration, of G. Reinelt giving such conditions for m = 6).

The preceding remarks lead us for the representation problem in $\mathcal{SSO}$ to consider the following conditions, for all distinct a, b, c $\in$ A:

P(a, b) + P(b, a) ≤ 1 and (1′)

P(a, b) + P(b, c) ≤ 1 + P(a, c). (2)

Together with irreflexivity, their necessity for the representation in $\mathcal{SSO}$ is obvious (notice that in presence of (1′), (2′) is no more equivalent to (2)). Since they are also necessary for the representation in any set of asymmetric and transitive relations (*e.g.* $\mathcal{SIO}$ or $\mathcal{SPO}$), we have good reasons to believe that they are not sufficient as soon as m is large enough (*i.e.* when m ≥ 4, since in that case there are asymmetric and transitive relations which are not strict semi-orders). The following example shows that this is indeed the case:

P	a	b	c	d
a	0	0	1/2	1/2
b	0	0	1/2	0
c	0	1/2	0	1/2
d	1/2	0	0	0

It is obvious to see that P is irreflexive and satisfies (1′) and (2). Suppose that P is representable in $\mathcal{SSO}$. The sum of the "weights" φ(T) of the strict semi-orders T for which b T c has to be 1/2. For these strict semi-orders it is impossible to have c T d since transitivity would imply b T d. The sum of the weights of the strict semi-orders for which c T b has to be 1/2. For these strict semi-orders it is impossible to have a T c since transitivity would imply a T b. If P is representable in $\mathcal{SSO}$, it has to be representable using only two families of strict semi-orders that are respectively such that:

b T c, a T c and

c T b, c T d.

No strict semi-order of the first family can have d T a because transitivity would imply d T c. But d T a is also impossible in the second family since transitivity would lead to c T a. We have thus shown that P cannot be represented in $\mathcal{SSO}$.

Let us notice that the preceding argument only makes use of the transitivity of the elements of $\mathcal{SSO}$. Thus this example shows that (1′) and (2) together with irreflexivity are not sufficient for the representability of P in any set of asymmetric and transitive crisp relations as soon as m ≥ 4, *e.g.* $\mathcal{SIO}$ or $\mathcal{SPO}$). A tedious but simple proof shows that (1′) and (2) together with irreflexivity are sufficient for the representation in $\mathcal{SSO}$ when m = 3. We do not reproduce it here.

The preceding remarks lead us to believe that the problem of the characterization of the elements in $\mathcal{GSC}$ is not easier than the "binary choice probabilities" problem as soon as m is moderately large. In view of the preceding example with m = 4, we will not, in this paper, pursue any further in that direction. We conclude this section with two remarks.

i) Not having a simple characterization of $\mathcal{GSC}$ for all values of m is an incentive to try to directly infer from proposition 2 a number of useful properties of this set. In particular, if we choose the functions t_i and the weights k_i so as to be rational, every relation P in $\mathcal{GSC}$ can be interpreted as the summary of an "election" in which each voter would indicate her preference for the elements of A through a strict semi-order. The values P(a, b) in this context represent the percentage of voters having declared that "a is strictly preferred to b". For instance, in such a context, the "net flow" exploitation technique used in PROMETHEE II is equivalent to ranking alternatives according to Borda's rule. This simple analogy can easily be exploited to transfer in our context many useful results in Social Choice Theory concerning the characterization of choice or ranking procedures (concerning Borda's rule see Young (1974), Hansson and Sahlquist (1976) and Debord (1987, 1993)). Such transpositions are often self-evident since most results of this type in Social Choice Theory are valid whenever the "voters" have (strict) preferences included in any set of relations containing the set of strict linear orders $\mathcal{SLO}$. Bouyssou (1993) offers examples of such transpositions.

ii) The absence of simple characterization of the elements of $\mathcal{GSC}$ does not facilitate the analysis of exploitation techniques to be coupled with the GSC method when the above-mentioned analogy cannot be used. The following very simple proposition might be helpful to quickly eliminate "bad" exploitation techniques.

Proposition 3. Let P be an irreflexive valued relation on A. We have:

$$\sum_{a,b \in A} P(a,b) \leq 1 \Rightarrow P \in \mathsf{GSC}_A.$$

Proof. Given proposition 2, we only have to show that P is representable in $\mathcal{SSO}_A$. For all distinct a, b ∈ A, consider the crisp relation P_{ab} on A such that, for all c, d ∈ A, c P_{ab} d ⇔ [c = a and d = b]. It is clear from lemma 2 that, for all distinct a, b ∈ A, $P_{ab} \in \mathcal{SSO}_A$. Since

$$\sum_{a,b \in A} P(a,b) \leq 1$$

and P is irreflexive, it is obvious that P is representable in $\mathcal{SSO}_A$ using the relations P_{ab} with the weights P(a, b) and the strict semi-order corresponding to the empty relation on A with the weight:

$$1 - \sum_{a,b \in A} P(a, b).$$

□

Since most exploitation techniques give a similar result when applied to P or when applied to a relation Q such that Q = αP, α ∈]0 ; 1[, this proposition might prove useful in building "counter-examples" for exploitation techniques to be coupled with the GSC method.

IV- Conclusion.

These few remarks concerning the construction and exploitation of outranking relations leave many important questions open. We mention here what we consider to be interesting directions for future research.

We already mentioned that propositions 1 and 2 were far from exhausting all the links between construction and exploitation techniques that would be interesting to investigate in order to obtain a good "interface" between them. The problem of the interpretation of "valuations" in methods leading to valued outranking relations and its corollary in terms of "admissible operations" on these valuations remains widely open. A profitable line of research would consist in trying to entirely characterize and/or analyze a pair consisting of a method of construction and a method of exploitation. In the valued case, it seems that only such an analysis would settle the already-mentioned questions of the "interpretation of the valuations" and of the "admissible operations" on them.

Concerning the characterization of outranking relations that can be obtained with a given construction method, many questions are still pending. We already noticed some open interesting combinatorial problems concerning proposition 1. Moreover a lot of work remains to be done in order to go beyond proposition 2. Fishburn (1987) mentions a long list of open problems concerning the problem of representability. To this list we can add the ones mentioned in section 3-2: problem of the representation in the set of strict semi-orders, strict interval orders and strict partial orders when m ≥ 4.

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