RANKING ALTERNATIVES ON THE BASIS OF PREFERENCE RELATIONS: A PROGRESS REPORT WITH SPECIAL EMPHASIS ON OUTRANKING RELATIONS[1]

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Abstract

This paper is devoted to the study of techniques allowing to rank order the elements of a finite set on the basis of a non-necessarily complete or transitive binary relation. In the area of MCDM, such a problem occurs with Outranking Methods. We review a number of theoretical results concerning this problem and show how they may be useful in order to guide the choice of a particular technique.

Keywords : MCDM, Outranking Relations, Ranking rules.

I. Introduction.

Suppose that you want to rank order a finite set of alternatives X evaluated on n criteria g_1, g_2, …, g_n. A common way to do so (see, *e.g.*, Keeney and Raiffa (1976)) is to attach a number v(a) to each alternative $a \in X$ reflecting its "desirability" and to rank order the alternatives according to these numbers. The number v(a) attached to alternative a is generally defined on the basis of its evaluations $g_1(a)$, $g_2(a)$, …, $g_n(a)$ on the n criteria and some "inter-criteria" information (*e.g.,* weights or tradeoffs) leading to an aggregation function V such that $v(a) = V(g_1(a), g_2(a), …, g_n(a))$. Once the aggregation function V is defined, the alternatives are automatically rank ordered. However, the definition of such an aggregation function is not an easy task and, in particular, requires very rich inter-criteria information. Furthermore, this approach is usually implemented in such a way as to:

-allow small "advantages" on a number of criteria to compensate for a large "disadvantage" on another,

-consider that all differences between the evaluations of the alternatives on the several criteria are significant regardless of the imprecision, the uncertainty or inaccurate determination that may affect these evaluations,

which may be open to criticism.

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Using the so-called "Outranking Methods" (see Vincke (1992a) and Roy and Bouyssou (1993) for a detailed presentation) a similar ranking problem can be addressed in a very different way. Within these methods it is customary to distinguish two different steps:

-the "construction step" in which one or several "outranking relations" are built,

-the "exploitation step" in which (in this particular problem formulation) outranking relations are used to derive a ranking of the alternatives.

Starting with a finite set of alternatives evaluated on several criteria, the "construction step" consists in comparing alternatives by pairs taking all criteria into account. This is usually done in such a way as to disallow an unlimited compensation of "large disadvantages" and to take into account the fact that small differences between evaluations are not always significant. This leads to one or several binary relations – the so-called "outranking relations". Outranking relations, in most methods, are built using a concordance-discordance principle. This principle leads to declaring that an alternative is "at least as good as" another if:

-a "sufficient majority" of criteria supports this proposition (concordance principle) and

-the opposition of the "minority" is not "too strong" (non-discordance principle).

It is well-known that this principle may lead to binary relations that are intransitive and/or incomplete (transitivity or completeness can only be obtained in rather exceptional cases – see Bouyssou (1992a) and Perny (1992)). Thus rank ordering alternatives on the basis of such information is far from being an easy task. This is the *raison d'être* of the exploitation step.

Many techniques for rank ordering alternatives on the basis of a non-necessarily transitive or complete binary relation have been proposed in the literature. Most of the techniques that have been proposed in the area of MCDM were justified on an *ad hoc* basis. To what extent are they really satisfactory ? How to compare them ? Can we think of "better" techniques ? Such are some of the questions that are central to this paper. Its aim is twofold. First we review, omitting technical details, a number of theoretical results connected to the analysis of exploitation techniques in a ranking problem formulation. This review is not exhaustive. Second we wish to show that such results may prove useful for the practitioner by shedding some light on the respective strengths and weaknesses of exploitation techniques, therefore illustrating the type of research advocated in Bouyssou *et al*. (1993). In our opinion, such theoretical analyses do not aim at defining the "best" technique for obtaining a ranking on the basis of a binary relation. More modestly, they should aim at providing guidelines to the analyst when deciding which exploitation technique to use in a particular situation – needless to say that these guidelines are not the only possible ones and that, in a real-world study, other guidelines, such as simplicity or acceptability, may be helpful as well. They may also prove useful in suggesting new techniques having appealing properties.

It should be mentioned that the problem of deriving a ranking on the basis of a non-necessarily transitive or complete binary relation is far from being specific to outranking methods. It occurs in a variety of contexts such as voting procedures, inquiries, direct pairwise comparisons of alternative or stimuli, etc. Though we shall mainly interpret the results presented in this paper in the context of outranking methods, they may also have an interest in these other settings (some of the properties we shall use should then be re-interpreted and/or modified accordingly).

This paper will illustrate two approaches to the theoretical analysis of ranking techniques. The first one consists in defining a list of properties that seem "desirable" for an exploitation technique to possess. Given such a list of properties one may then try:

-to analyze whether or not they are satisfied by a number of techniques,

-to establish "impossibility theorems", *i.e.* subsets of properties that cannot be simultaneously fulfilled,

-to determine, given the above-mentioned impossibility theorems, the techniques that satisfy the greatest possible numbers of properties.

We illustrate this type of analysis in section 4 for ranking techniques based on crisp relations. The second one consists in trying to find a list of properties that would "characterize" a given technique, *i.e.* a list of properties that this technique would be the only one to satisfy. This allows to emphasize the specific features of an exploitation technique and, thus, to compare it more easily with other ones. This is illustrated in section 5 for ranking techniques based on valued relations. These two types of analysis are not unrelated: ideally they should merge in the end, the characterizing properties exhibited by the second type of analysis being members of the list of "desirable" properties used in the first type of analysis. As we shall see much work remains to be done in order to attain such a result. Let us also mention that both types of analysis have their own difficulties. In the first one, the main problem consists in defining the list of "desirable" properties. These properties should indeed cover every aspects of what seems to be constitutive of an "appropriate" technique. In the second one, the characterizing properties will only be useful if they have a clear and simple interpretation which may not always be the case when analyzing a complex technique.

The paper is organized as follows. We present our notations and definitions in section 2. In section 3, we show why it makes sense, in the context of outranking relations, to separate the analysis of an exploitation technique from that of a construction technique. Section 4 illustrates the first type of analysis for exploitation techniques designed to rank order alternatives on the basis of a crisp relation. Section 5 illustrates the second one for exploitation techniques dealing with valued relations. We conclude in section 6 and mention some directions for future research.

II. Definitions and Notations.

Throughout this paper X will denote a finite set. A *valued* (binary) *relation* T on X is a function from XxX into [0, 1]. It is said to be *reflexive* if T(a, a) = 1, for all a ∈ X. If Y ⊆ X and T is a valued relation on X, we denote by T/Y the restriction of T to Y, *i.e.* the valued relation on Y such that for all a, b ∈ Y, T/Y(a, b) = T(a, b). A valued relation T on X such that T(a, b) ∈ {0, 1}, for all a, b ∈ X, is said to be *crisp*. We often write a T b instead of T(a, b) = 1 and Not(a T b) instead of T(a, b) = 0 when T is a crisp relation. We denote by $\mathcal{F}_X$ (resp. $\mathcal{U}_X$) the set of all valued (resp. crisp) reflexive relations on X.

Let T be a crisp relation on X. This relation is said to be:
–*complete* if [a T b or b T a],
–*transitive* if [a T b and b T c ⇒ a T c],
–*asymmetric* if [a T b ⇒ Not(b T a)],
for all a, b, c ∈ X.
A *weak order* is a crisp, complete and transitive binary relation. Let T be a crisp relation on X. We denote by G(T, X) the set of *greatest elements* of X given T, *i.e.*, G(T, X) = {a ∈ X: a T b for all b ∈ X}. Notice that G(T, X) may well be empty. When T is a weak order on X, it is easy to see that G(T, X) is always nonempty and consists in the first equivalence class of T.

Consider a particular technique of obtaining a binary relation on the finite set X, *e.g.* a voting procedure, a pair comparison method or an outranking method. The characteristics of this technique will often confer some "structural" properties to the binary relations it produces. Given a particular construction technique K, we denote by $\mathcal{K}_X$ the subset of all binary relations that can be obtained on the finite set X using K. A **$\mathcal{K}$-ranking procedure** $\succsim_X$ on the finite set X is a function assigning a weak order $\succsim_X$(R) on X to any R ∈ $\mathcal{K}_X$. A **$\mathcal{K}$-ranking rule $\succsim$** is a function associating with each finite set X a $\mathcal{K}$-ranking procedure $\succsim_X$ on X. Thus, a $\mathcal{K}$-ranking rule allows to build a weak order on the basis of any relation that can be obtained with the construction technique K on a finite set.

We respectively denote by $\sim_X$(R) and $\succ_X$(R) the symmetric and asymmetric parts of $\succsim_X$(R), *i..e.*, the relations such that, for all a, b ∈ X, [a $\sim_X$(R) b iff (a $\succsim_X$(R) b and b $\succsim_X$(R) a)] and [a $\succ_X$(R) b iff (a $\succsim_X$(R) b and Not b $\succsim_X$(R) a)].

III. The "structural properties" of outranking relations.

In general, it is not possible to analyze a ranking rule without the knowledge of the construction technique to which it is designed to be coupled. Most properties that are useful for analyzing ranking rules involve either specifying the result of the rule when applied to relations having special properties or linking the result obtained with a relation R with the one obtained with a relation R′ derived from R. Therefore, without a precise knowledge of the set $\mathcal{K}_X$ of the relations that can be obtained on a set X, these properties may well be meaningless (*e.g.* because they impose a constraint on the result of the rule when applied to a relation that cannot be obtained with the construction technique K on the set X). In this section, we characterize the sets $\mathcal{K}_X$ for two well-known construction techniques, generalizing classical results concerning the method of majority decisions (see McGarvey (1952) or Deb (1976)).

The construction technique of ELECTRE I (Roy (1968)) builds a crisp relation starting with a set of alternatives evaluated on several "true-criteria" and "inter-criteria information". The construction technique of ELECTRE III (Roy (1978)) builds a valued relation starting with a set of alternatives evaluated on several "pseudo-criteria" and "inter-criteria information" (on the notions of true and pseudo-criterion, see Roy (1985)). They both use a concordance/discordance principle in order to build an outranking relation that is interpreted as an "at least as good as" preference relation. We know that such relations may not possess remarkable properties such as transitivity or completeness. The following proposition, proven in Bouyssou (1995), shows more: any (reflexive) relation may be obtained with such techniques.

Proposition 1.
(a) Any reflexive crisp relation on a finite set X may be obtained as the result of the construction technique of ELECTRE I with a particular choice of criteria and parameters.
(b) Any reflexive valued relation on a finite set X may be obtained as the result of the construction technique of ELECTRE III with a particular choice of criteria and parameters.

This proposition has a simple interpretation. For the axiomatic investigation of exploitation techniques to be coupled with ELECTRE I and ELECTRE III, it makes sense to suppose that they will be confronted to any reflexive preference relation (this is not the case with the construction technique used in the PROMETHEE method – see Brans *et al.* (1984) – as shown in Bouyssou (1995)). This proposition allows, to some extent, to separate the analysis of exploitation techniques from that of construction techniques (it should be noticed that the proof of proposition 1 requires to apply the construction techniques of ELECTRE I and III to situations involving a large number of criteria; for each of these construction techniques, the determination of the minimal number of criteria for which proposition 1 is true is an open – and difficult – problem).

IV. Ranking alternatives on the basis of a crisp relation.

We shall be interested in this section in $\mathcal{U}$-ranking rules, *i.e.* rules associating a weak order to every reflexive crisp relation defined on a finite set. Given proposition 1, such rules are clearly of interest for ELECTRE I. The problem of defining "reasonable" ranking rules for crisp relations is not an easy one. It has generated numerous studies, in particular in the case of complete crisp relations (which is the appropriate setting with the method of majority decisions; see Rubinstein (1980), Henriet (1985), Barthélémy and Monjardet (1988), Barthélémy *et al.* (1989)). The related problem of defining choice rules has attracted even more attention (see, *e.g.*, Fishburn (1977), Bordes (1983), Moulin (1986), Laffond *et al.* (1992)). This difficulty is largely due to the fact that when a crisp preference relation is not complete and/or has cycles in its asymmetric part, the very notion of a "good" alternative is not easy to define.

IV-1 Some properties of ranking rules for crisp relations.

A $\mathcal{U}$-ranking rule $\geq$ is said to be **neutral** if, for all finite set X and all R $\in \mathcal{U}_X$,
[σ is a permutation on X] ⇒ [a $\geq_X$(R) b ⇔ σ(a) $\geq_X$(R^σ) σ(b), for all a, b ∈ X],
where R^σ is the element of $\mathcal{U}_X$ defined by $R^\sigma(\sigma(a), \sigma(b))$ = R(a, b) for all a, b ∈ X. Neutrality expresses the fact that a ranking rule does not discriminate between alternatives just because of their labels.

A $\mathcal{U}$-ranking rule $\geq$ is said to be:
faithful if [R is a weak order on X] ⇒ R = $\geq_X$(R),
data-preserving 1 if [R is a transitive crisp relation on X] ⇒ R ⊆ $\geq_X$(R),
data-preserving 2 if [R is a crisp relation on X and its transitive closure R* is complete] ⇒ R* = $\geq_X$(R),
data-preserving 3 if [R is a crisp relation on X and its asymmetric part P is without circuit] ⇒ P ⊆ $\geq_X$(R),
covering compatible if [R(a, b) ≥ R(b, a) and for all c ∈ A\{a, b}, R(a, c) ≥ R(b, c) and R(c, a) ≤ R(c, b)] ⇒ a $\geq_X$(R) b,
for all finite set X and all R ∈ $\mathcal{U}_X$.
Faithfulness implies that a ranking rule applied to a weak order should preserve it. The three versions of data preservation are in the same vein. They say that when it is possible to obtain a weak order on the basis of R without deleting information contained in R, a ranking rule should do so. Covering compatibility is in the same spirit: when a "covers" b it seems intuitive to consider that b should not be ranked before a.

A $\mathcal{U}$-ranking rule $\geq$ is said to be:
independent of non-discriminating alternatives if [R(a, b) = k and R(b, a) = k′ for all a ∈ X\Y and all b ∈ Y] ⇒ $\geq_Y$(R/Y) = $\geq_X$(R)/Y,
independent of the best-ranked elements if $\geq_Z$(R/Z) = [$\geq_X$(R)]/Z,
where Z = X\G($\geq_X$(R), X),
for all finite set X and all R ∈ $\mathcal{U}_X$.

Independence of non-discriminating alternatives implies that when there is a subset of alternatives that are compared similarly (being either preferred, indifferent or incomparable) to all other alternatives, the ranking of the other alternatives is not affected by the presence of this subset. Independence of the best ranked alternatives implies that removing from X the alternatives that were ranked first leaves unchanged the ranking of the other alternatives.

A $\mathcal{U}$-ranking rule $\succeq$ is said to be **monotonic** if, for all finite set X and all $R \in \mathcal{U}_X$,
$[a \succeq_X(R)\ b \Rightarrow a \succeq_X(R')\ b]$ and $[a \succ_X(R)\ b \Rightarrow a \succ_X(R')\ b]$,
where R' is an element of $\mathcal{U}_X$ such that $R'/X\backslash\{a\} = R/X\backslash\{a\}$ and $[R(a, c) \leq R'(a, c), R(c, a) \geq R'(c, a)$ for all $c \in X\backslash\{a\}]$.
Monotonicity says that improving an alternative cannot decrease its position in the ranking (it is easily seen it also implies the converse property: deteriorating an alternative cannot improve its position in the ranking).

IV-2 Some ranking rules and their properties.

We first introduce three basic ranking rules that correspond to simple rules introduced in the literature. The first two are based on the "Net Flow score" of the alternatives, defined, for all finite set X, all $a \in X$ and all $R \in \mathcal{U}_X$, by:

$$S_{NF}(a, R, X) = \sum_{c \in X\backslash\{a\}} (R(a, c) - R(c, a)).$$

The "Net Flow Rule" (NFR) rank orders the alternatives according to their net flow scores and corresponds to the well-known Copeland rule (see, *e.g.*, Fishburn (1973); characterizations of this rule can be found in Rubinstein (1980) and Henriet (1985)). When it is directly applied to R we denote this rule by NFR(R). The "Repeated Net Flow Rule" (RNFR) ranks in first position the elements with the highest Net Flow score in X. These elements are then removed from X and the Net Flow scores of the remaining alternatives are computed in the reduced set. Alternatives with the highest Net Flow score in the reduced set are then ranked in second position and so on. When it is directly applied to R, we denote this rule by RNFR(R).
The "Rank Rule" (RR) consists first in reducing the circuits of R. This leads to a relation R^r without circuit on a set X^r of elements obtained by replacing all alternatives belonging to the same circuit of R by a single element; we have A R^r B as soon as there are some alternatives a, $b \in X$ such that a R b and a (resp. b) belongs to the circuit of R represented by A (resp. B). Since R^r is without circuit, X^r has maximal elements for R^r (an element $A \in X^r$ is maximal for R^r if, for all $B \in X^r$, $B \neq A$ implies Not(B R^r A)). Alternatives corresponding to a maximal element of X^r for R^r are then ranked in first position. These maximal elements are then removed from X^r. Alternatives corresponding to maximal elements for R^r in the reduced set are then ranked in second position and so on. This defines a rule denoted by $RR(R^r)$.

These three basic rules (NFR(R), RNFR(R) and RR(R^r)) are simple and intuitive. Their comparison is not obvious however. Hence the use of the properties introduced in the preceding section to distinguish between them. We sum up our results in Table 1 in which a Y indicates that a property is satisfied and a N that it is violated (see Vincke (1992b) for a proof).

[Insert Table 1 about here]

With respect with to this set of property, it is apparent that none of the three rules is dominated. This table may however prove useful since in a particular situation some properties may acquire more importance than others (*e.g.*, if alternatives are likely to become unavailable, independence of the best ranked elements may be seen as extremely desirable). A question immediately arises: can we think of "better" rules that would dominate some or even all of the rules envisaged so far ? A simple way to try to answer this question is to imagine new rules and to enter them into Table 1. New rules are easily created by applying preliminary transformations on R before using NFR, RNFR or RR. Among the possible transformations that are of interest, let us mention:

- -replacing R by R^r,
- -taking the asymmetric part P of R,
- -taking the transitive closure R* of R.

These preliminary transformations can be combined to obtain (using obvious notations) relations such as P*, P^r, R^{*r}, P^{*r} (where $R^{*r} = (R^*)^r = (R^r)^*$ and $P^{*r} = (P^*)^r = (P^r)^*$). This defines rules such as NFR(P^{*r}), RNFR(R^r) or RR(P^r). Given that RR is only defined for relations without circuit and some rules are identical (*e.g.* NFR(R) and NFR(P)), the systematic application of this process leads to 16 distinct rules including the three basic ones. The complete table exhibiting their properties can be found in Vincke (1992b and 1993). In Table 2, we only consider nondominated rules. These six rules are all distinct.

[Insert Table 2 about here]

Comparing Tables 1 and 2, it is easy to observe, for example, that NFR(R) and RNFR(R) are both dominated by RNFR(P*) while RR(R^r) remains nondominated. This process of imagining new rules possessing many properties is not endless however since it may well happen that some properties cannot be satisfied together. As an example, let us mention the following proposition proved in Vincke (1992b):

Proposition 2. There is no $\mathcal{U}$-ranking rule being at the same time data-preserving 2 and independent of non-discriminating alternatives.

This proves that the six rules included in Table 2 are non dominated rules in the set of all $\mathcal{U}$-ranking rules with respect to our set of properties. If these rules are judged unsatisfactory, one may try either to find other nondominated rules or to modify the list of properties. The reader is referred to Vincke (1992b and 1993) for more details.

V. Ranking alternatives on the basis of a valued relation.

We shall be interested in this section in $\mathcal{F}$-ranking rules, *i.e.* rules associating a weak order to every reflexive valued relation defined on a finite set. Given proposition 1, such rules are clearly of interest for ELECTRE III. The same analysis that the one presented in section 4 can be conducted for such rules. However, in order to illustrate another type of analysis, we shall concentrate here on establishing "characterization" results, *i.e.* exhibiting sets of properties satisfied by a *single* rule.

Replacing $\mathcal{U}_X$ by $\mathcal{F}_X$ whenever necessary, it is clear that all properties introduced in section 4.1 can be generalized to $\mathcal{F}$-ranking rules. We shall need the following additional ones.

Consider a sequence of valued relations on X ($R^i \in \mathcal{F}_X$, i = 1, 2, ...). We say that this sequence converges to $R \in \mathcal{F}_X$ if, for all $\varepsilon > 0$, there is an integer k such that, for all j > k and all a, b $\in$ A, $|R^j(a, b) - R(a, b)| < \varepsilon$. An $\mathcal{F}$-ranking rule $\gtrsim$ is said to be **continuous** if, for all finite set X, all $R \in \mathcal{F}_X$, all sequences ($R^i \in \mathcal{F}_X$, i = 1, 2, ...) converging to R and all a, b $\in$ X,
[a $\gtrsim_X(R^i)$ b for all R^i in the sequence] $\Rightarrow$ [a $\gtrsim_X(R)$ b].
Continuity says that "small" changes in a valued relation should not lead to radical changes in the associated ranking.

An $\mathcal{F}$-ranking rule $\gtrsim$ is said to be **greatest-faithful** if [R is a crisp relation on a finite set X and $G(R, X) \neq \emptyset$] $\Rightarrow$ $G(\gtrsim_X(R), X) \subseteq G(R, X)$.
Greatest-faithfulness says that if a crisp relation has greatest elements then the top-ranked elements should be chosen among them. In spite of names, it should be noticed that a faithful ranking rule is not necessarily greatest-faithful and *vice versa*.

An important characteristic of a ranking rule lies in the way it deals with circuits in the relation R. Consider two reflexive valued relations R and R′ on X and suppose that R′ is identical to R except that, for some distinct a, b, c $\in$ X and some $\varepsilon \in$ [-1 ; 1]:
$[R(a, b) = R'(a, b) + \varepsilon$ and $R(b, a) = R'(b, a) + \varepsilon]$ or
$[R(a, b) = R'(a, b) + \varepsilon$, $R(b, c) = R'(b, c) + \varepsilon$ and $R(c, a) = R'(c, a) + \varepsilon]$.

Thus, R and R′ are identical except on a circuit of length 2 or 3 on which a positive or negative quantity has been added. In this case we say that R and R′ are *circuit-equivalent*.
An $\mathcal{F}$-ranking rule $\gtrsim$ is **independent of circuits** if for all finite set X and all R, R′ $\in \mathcal{F}_X$,
[R and R′ are circuit-equivalent] $\Rightarrow$ $\gtrsim_X(R) = \gtrsim_X(R')$.
This property has a straightforward interpretation. When R and R′ are circuit-equivalent via a circuit of length 2, independence of circuits implies that the ranking is only influenced by the differences R(a, b) – R(b, a). When R and R′ are circuit-equivalent via a circuit of length 3, independence of circuits implies that intransitivities of the kind R(a, b) > 0, R(b, c) > 0 and R(c, a) > 0 can be "wiped out". Notice that this property makes an explicit use of the cardinal properties of the valuations of the relation R (except in the particular case in which both R and R′ are crisp).

An $\mathcal{F}$-ranking rule $\gtrsim$ is said to be **ordinal** if for all finite set X, all R $\in \mathcal{F}_X$ and all strictly increasing and one-to-one transformation ϕ on [0, 1],
$\gtrsim_X(R) = \gtrsim_X(\phi[R])$
where $\phi[R]$ is the valued relation on X such that $\phi[R](c, d) = \phi(R(c, d))$ for all c, d $\in$ X.
Ordinality implies that a ranking rule should not make use of the "cardinal" properties of the valuations.

An $\mathcal{F}$-ranking rule $\gtrsim$ is said to be **strictly monotonic** if, for all finite set X and all R $\in \mathcal{F}_X$,
[a $\gtrsim_X(R)$ b $\Rightarrow$ a $>_X(R')$ b], where R′ is an element of $\mathcal{F}_X$ such that R′/X\{a} = R/X\{a} and [R(a, c) ≤ R′(a, c), R(c, a) ≥ R′(c, a) for all c $\in$ X\{a}, at least one of these inequalities being strict].
Strict monotonicity says that improving an alternative must improve its position in the ranking. It is clear that strict monotonicity implies monotonicity.

We shall be interested here in two ranking rules. The first one is just the Net Flow Rule applied to valued relations, which we call the Valued Net Flow Rule (VNFR). The second one, the Valued Min in Favor Rule (VMIFR), is obtained by rank ordering the alternatives according to the following score:
$$S_{MinF}(a, R, X) = \min_{c \in X\setminus\{a\}} R(a, c).$$
Besides their simplicity and intuitive appeal, there are a number of reasons for being interested in these two rules. The VNFR rule has a long history in Social Choice. We already mentioned that it coincides with the rule of Copeland when R is crisp. When R(a, b) is interpreted as a percentage of voters considering that a is preferred or indifferent to b, it corresponds to the well-known rule of Borda (see Fishburn (1973); for characterizations of various versions of Borda's rule see Young (1974), Hansson and Sahlquist (1976), Nitzan and Rubinstein (1981)). The VNFR rule is also used in the PROMETHEE II outranking method (see Brans *et*

al. (1984)), the Net Flow score being also used, though in a different way, in the exploitation technique of ELECTRE III. It should be noticed that this rule makes use of the "cardinal" properties of the numbers R(a, b). On the contrary, the Min In Favor Rule is purely "ordinal" and uses the valuations R(a, b) as if they were a numerical representation of a credibility relation between pairs of alternatives.

It is not difficult to see that the VNFR is neutral, continuous, faithful, strictly monotonic and independent of circuits but neither ordinal nor greatest-faithful. The VMIFR is neutral, greatest-faithful, continuous and ordinal. It is not faithful. It is neither strictly monotonic nor independent of circuits. We sum up our observations in Table 3.

[Insert Table 3 about here]

We leave to the reader the – not very difficult – task of confronting these two rules with the properties introduced in section 4. This table contains italicized "Y". The reason for this lies in the following proposition (see Bouyssou (1992b and c) for a proof):

Proposition 3.
(a) The Valued Net Flow Rule is the only $\mathcal{F}$-ranking rule that is neutral, strictly monotonic and independent of circuits.
(b) The Valued Min In Favor Rule is the only $\mathcal{F}$-ranking rule ordinal, continuous and greatest-faithful.

Such characterization results are useful in quickly identifying the main characteristics of a ranking rule. The VNFR uses the cardinal properties of the valuations. This allows to deal with circuits in a very elegant and efficient way. On the contrary the VMIFR is purely ordinal at the cost of not being faithful. Let us mention that different systems of properties may be used to characterize a given rule. Alternative characterizations of the VNFR (resp. the VMIFR) may be found in Bouyssou (1993) (resp. Pirlot (1992 and 1995)). They use properties that, in some contexts, may be found to be more easily interpretable than the ones that we used here.

VI- Conclusions

It is worth stressing that the theoretical analyses presented here do not and cannot, in our opinion, aim at defining a "best" ranking rule. They nevertheless allow to show that, sometimes, a given rule is the only one to satisfy a set of properties. It may also happen that, *for a given set of properties*, a rule dominates another (because it satisfies the same properties plus some others) or that some rules cannot be dominated (because the addition of new property would result in a contradiction). Thus, they provide some guidelines to the analyst who has to choose a ranking rule. It should be noted however that the interpretation of the "properties" is dependent upon the particular decision-aid situation at hand and the way the binary relation was obtained. Let us also mention that it cannot be overemphasized that the type of analysis presented in this paper which focuses exclusively on exploitation techniques can only be meaningfully conducted if the set of binary relations to which these techniques will be confronted has been clearly identified.

Our review of techniques allowing to rank order alternatives on the basis of a binary relation is far from being exhaustive. Many important concepts and results can be found in the literature and especially in the literature on Social Choice where such a problem has a long history. Nevertheless many important questions are still open – many $\mathcal{U}$-ranking rules are still awaiting their characterization, a list of "desirable" properties for $\mathcal{F}$-ranking rules is still to be devised – and much work remains to be done in order to unify the two types of analysis presented here. We do hope that this review will contribute to stimulate research in this area.

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U - Ranking Rules	NFR(R)	RNFR(R)	RR(R^r)
neutrality	Y	Y	Y
faithfulness	Y	Y	Y
data - presev. 1	Y	Y	Y
data - preserv. 2	N	N	Y
data - preserv. 3	N	N	Y
covering compat.	Y	Y	Y
indep. / Non - disc alt.	Y	Y	N
indep. / Best ranked alt.	N	Y	Y
monotonicity	Y	N	Y

Table 1: The three basic rules

U - Ranking Rules	RNFR(P*) RNFR(P *^{r}) RR(P^r)	RNFR(R*) RNFR(R *^{r}) RR(R^r)
neutrality	Y	Y
faithfulness	Y	Y
data - presev. 1	Y	Y
data - preserv. 2	N	Y
data - preserv. 3	Y	Y
covering compat.	Y	Y
indep. / Non - disc alt.	Y	N
indep. / Best ranked alt.	Y	Y
monotonicity	Y	Y

Table 2: The six nondominated rules

F - Ranking Rules	VMIFR	VNFR
Neutrality	Y	***Y***
Continuity	***Y***	Y
Strict Monotonicity	N	***Y***
Faithfulness	N	Y
Greatest - Faithfulness	***Y***	N
Independence of circuits	N	***Y***
Ordinality	***Y***	N

Table 3: Two basic *F*-Ranking Rules

(see text for the explanation of italicized "Y")