On the interpretation of the interaction index between criteria in a Choquet integral model

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Abstract

Using a Choquet integral model in the context of multiple criteria decision making often involves assessing a capacity on the basis of preferences among alternatives given by a decision-maker. In such circumstances, the elicited capacity is rarely unique. This lack of uniqueness complicates the interpretation of classic indices, such as the interaction index between criteria. It is often the case that the elicitation makes only use of binary alternatives, i.e., alternative that have either a neutral or a satisfactory evaluation on each criteria. We give conditions guaranteeing that preferences expressed on such alternatives can be represented by a Choquet integral model. On the basis of these conditions, we show that a negative interaction among a group is never necessary, i.e., we can always find a capacity for which this interaction is positive. Outside the framework of binary alternatives, we propose a linear programming model allowing one to test whether the sign of the interaction index remains unchanged for all capacities that are compatible with the preferences expressed by the decision-maker.

Keywords: Choquet integral model, Multiple criteria decision making, Interaction index, Binary alternatives.

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1. Introduction

In Multiple Criteria Decision Making (MCDM), the additive value function model is popular among practitioners and have solid theoretical foundations [19]. This model implies accepting an *independence hypothesis* stating that changing a common evaluation of a criterion should not affect preferences between alternatives [3]. In some contexts, this hypothesis might be seen as restrictive [11]. Hence, several other models that do not require this independence hypothesis have been developed.

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A famous one is the Choquet integral model. Its use in MCDM was popularized through the work of Michel Grabisch [7, 8]. The Choquet integral model is presently considered as a central tool in MCDM when one wants to escape the independence hypothesis [10, 11, 12]. In this model, the importance of criteria and groups of criteria is modelled using a “capacity”.

In the literature, the particular case of 2-additive capacities has received much attention [9]. This case is often considered as a useful compromise between an additive model implying independence and a general Choquet integral model (i.e., using a capacity that is not restricted to be 2-additive) raising difficult elicitation and interpretation issues [15]. This model is often used in practice (evaluation of comfort [13], performance measurement [2, 5], systems design [23]). Nevertheless, the general case remains important. This is the subject of this paper.

When a capacity is elicited on the basis of preferences, one should expect its non-uniqueness. This clearly complicates the interpretation of the aggregation model. For instance, within the entire set of capacities that are compatible with the preference information, one capacity could exhibit a positive interaction index between a given group of criteria, while another capacity in this set might exhibit a negative one.

This question was first tackled in Mayag and Bouyssou [21] in the case of Choquet integral model using 2-additive capacities. These results were extended in [16] removing the hypothesis that the capacity is 2-additive but adding the hypothesis that there is no indifference in the preference information that is collected. We remove this last hypothesis in the present paper.

We give necessary and sufficient conditions that preferences on binary alternatives obtained from a decision maker can be represented by a Choquet integral model. This generalizes results in [22] for the case of a Choquet integral model using a 2-additive capacity. We then show that, under the same conditions, a capacity that is compatible with the preference information that was collected can always be chosen so that interaction indices between all groups of criteria are all strictly positive. Because the framework of binary alternatives is restrictive, we also propose a linear programming model allowing to test for the existence of a capacity compatible with the preference information that was collected and for the robustness of the sign of the interaction index between a group of criteria.

The rest of this paper is organized as follows. We present our framework in Section 2. The concept of necessary and possible interaction introduced in [21] is generalized in Section 3. Our main results are presented In Sections 4 and 5. Section 6 present our linear programming model allowing to go beyond the case of binary alternatives. A final section summarizes and concludes.

2. Notation and preliminaries

2.1. The framework

Let X be a set of alternatives evaluated on a set of n criteria $N = \{1, 2, \ldots, n\}$. For any subset $A \subseteq N$, throughout this paper we use the notation $A \subseteq_{\geq 2} N$ if and only if $|A| \geq 2$ (i.e., A contains at least two criteria). The set X is assumed to be the Cartesian product $X_1 \times X_2 \times \ldots \times X_n$, where X_i is the set of possible levels on criterion $i \in N$.

We will suppose that the criteria are recoded numerically using, for all $i \in N$, a function u_i from X_i into $\mathbb{R}$. Using these functions we assume that the various recoded criteria are commensurate, so that the application of the Choquet integral model is meaningful [14].

As in [14], we assume that the DM is able to identify on each criterion $i \in N$ two reference levels 0_i and 1_i:

- the level 0_i in X_i is considered as a neutral level and we set $u_i(0_i) = 0$,
- the level 1_i in X_i is considered as a good level and we set $u_i(1_i) = 1$.

For all $x = (x_1, \ldots, x_n) \in X$, we sometimes write $u(x)$ as a shorthand for $(u_1(x_1), \ldots, u_n(x_n))$. For all $S \subseteq N$, we define the binary alternative $a_S = (1_S, 0_{-S})$ in X such that $a_i = 1_i$ if $i \in S$ and $a_i = 0_i$ otherwise. We often write $a_{i_1 i_2 \ldots i_s}$ instead of $a_{\{i_1, i_2, \ldots, i_s\}}$, where $1 \leq s \leq n$. Moreover, we write a_0 instead of $a_\emptyset$. Our work makes use the set $\mathcal{B}^g$ which we define below.

Definition 1. *The general set of binary alternatives is defined by,*

$$\mathcal{B}^g = \{a_S = (1_S, 0_{-S}) : S \subseteq N\}.$$

The term "general" is used here since the set $\mathcal{B} = \{a_0, a_i, a_{ij} \colon i, j \in N\} \subseteq \mathcal{B}^g$ is already defined in the literature, under the name of binary alternatives [14, 22].

2.2. The Choquet integral

The Choquet integral [9, 10, 11] is an aggregation function known in MCDM as a tool generalizing the weighted arithmetic mean. The Choquet integral uses the notion of capacity [4, 9] defined as a function μ from the power set 2^N into $[0, 1]$ such that:

- $\mu(\emptyset) = 0$,
- $\mu(N) = 1$,
- For all $S, T \in 2^N$, $\left[S \subseteq T \Longrightarrow \mu(S) \leq \mu(T)\right]$ (monotonicity).

Definition 2. *For an alternative* $x = (x_1, \ldots, x_n) \in X$*, the expression of the Choquet integral w.r.t. a capacity* μ *is given by,*

$$C_\mu\big(u(x)\big) = \sum_{i=1}^{n} \Big[u_{\sigma(i)}(x_{\sigma(i)}) - u_{\sigma(i-1)}(x_{\sigma(i-1)})\Big]\mu\big(N_{\sigma(i)}\big),$$

where σ *is a permutation on* N *such that* $N_{\sigma(i)} = \{\sigma(i), \ldots, \sigma(n)\}$, $u_{\sigma(0)}(x_{\sigma(0)}) = 0$ *and* $u_{\sigma(1)}(x_{\sigma(1)}) \leq u_{\sigma(2)}(x_{\sigma(2)}) \leq \ldots \leq u_{\sigma(n)}(x_{\sigma(n)})$.

Remark 1. *For all* $S \subseteq N$*, we have* $C_\mu(u(a_S)) = \mu(S)$.

We suppose that the DM gives preferences by comparing some elements of X. We then obtain the binary relations P and I defined as follows.

Definition 3. *An ordinal preferential information* $\{P, I\}$ *on* X *is given by,*

$P = \{(x, y) \in X \times X$*: DM strictly prefers* x *to* y $\}$,

$I = \{(x, y) \in X \times X$*: DM is indifferent between* x *and* $y\}$.

We frequently write $a\,P\,b$ and $a\,I\,b$ instead of $(a, b) \in P$ and $(a, b) \in I$ respectively. We add to this ordinal preferential information a binary relation M modeling the monotonicity relations between the general set of binary alternatives, and allowing us to ensure the satisfaction of the monotonicity condition: $\big[S \subseteq T \implies \mu(S) \leq \mu(T)\big]$.

Definition 4. *For all* $a_S, a_T \in \mathcal{B}^g$*,* $a_S\,M\,a_T$ *if* $[S \supseteq T$ *and not*$(a_S\,(P \cup I)\,a_T)]$.

Remark 2. $a_S\,M\,a_T \Longrightarrow C_\mu(u(a_S)) \geq C_\mu(u(a_T))$.

In the sequel, we need the following two classic definitions in graph theory [20].

Definition 5. *Let* $\mathcal{R}$ *be a binary relation on* X *and* P *its asymmetric part. There exists a strict cycle in* $\mathcal{R}$ *if there exists elements* $x_0, x_1, \ldots, x_r \in X$ *such that* $x_0\,\mathcal{R}\,x_1\,\mathcal{R}\,\ldots\,\mathcal{R}\,x_r\,\mathcal{R}\,x_0$ *and for at least one* $i \in \{0, \ldots, r-1\}$*,* $x_i\,P\,x_{i+1}$.

Definition 6. *Let* $\mathcal{R}$ *be a binary relation on* X*. We write* $x\,TC_{\mathcal{R}}\,y$ *if there exists elements* $x_0, x_1, \ldots, x_r \in X$ *such that* $x = x_0\,\mathcal{R}\,x_1\mathcal{R}\,\ldots\,\mathcal{R}\,x_r = y$*. Hence,* $TC_{\mathcal{R}}$ *is the transitive closure of the binary relation* $\mathcal{R}$.

2.3. The Shapley interaction index

The following interaction index, known as the Shapley interaction index, related to a capacity was introduced in [9] in order to take into account some synergies between criteria.

Definition 7. *The interaction [9] index w.r.t. a capacity* μ *is defined for all* $A \subseteq N$ *by,*

$$I^{\mu}_{A} = \sum_{K \subseteq N \setminus A} \frac{(n-k-a)!k!}{(n-a+1)!} \sum_{L \subseteq A} (-1)^{a-\ell} \mu(K \cup L), \text{ where } \ell = |L|,\ k = |K| \text{ and } a = |A|.$$

The index I^{μ}_{A} measures interaction for a subset $A \subseteq N$. In this paper, we are interested in the interaction between several criteria, so when we talk about the index I^{μ}_{A}, it will be for $A \subseteq_{\geq 2} N$. We present below several write results that will be useful in the proof of Proposition 2.

Remark 3. *We can rewrite the interaction index of* $A \subseteq_{\geq 2} N$*, w.r.t. a capacity* μ *as follows:*

$$I^{\mu}_{A} = \sum_{K \subseteq N \setminus A} \frac{(n-k-a)!k!}{(n-a+1)!} \Delta_A \mu(K),$$

where $\ell = |L|,\ k = |K|,\ a = |A|$ *and* $\Delta_A \mu(K) = \sum_{L \subseteq A} (-1)^{a-\ell} \mu(K \cup L)$.

The next lemma gives a decomposition of $\Delta_A \mu(K)$ used in the proof of the Proposition 2 (we assume that 0 is an even number).

Lemma 1. *For all* $A \subseteq_{\geq 2} N$ *and* $K \subseteq N \setminus A$*, we have*

$$\Delta_A \mu(K) = \sum_{\substack{p=0,\\ p\ even}}^{a} \Bigg[\sum_{\substack{L \subseteq A,\\ \ell = a-p}} \mu(K \cup L) \ - \sum_{\substack{L \subseteq A,\\ \ell = a-p-1}} \mu(K \cup L) \Bigg].$$

Proof. This proof consists of a simple decomposition into positive and negative terms. We give it in [17]. □

In the next section, we extend the concept of necessary and possible interaction introduced in the case of a 2-additive Choquet integral model [21].

3. Necessary and possible interaction

We suppose that the DM compares a number of alternatives in terms of strict preferences (P) or indifference (I). The following definition tells us when this ordinal preferential information is representable by a Choquet integral model.

Definition 8. *An ordinal preferential information* $\{P, I\}$ *on* X*, is representable by a Choquet integral model if we can find a capacity* μ *such that: for all* $x, y \in X$*, we have*

$$x\, P\, y \Longrightarrow C_\mu\big(u(x)\big) > C_\mu\big(u(y)\big),$$
$$x\, I\, y \Longrightarrow C_\mu\big(u(x)\big) = C_\mu\big(u(y)\big).$$

The set of all capacities that can be used to represent $\{P, I\}$ at hand will be denoted by $C_{\text{Pref}}(P, I)$. When there is no ambiguity on the underlying ordinal preferential information, we simply write C_{Pref}. The following definition will be central in the rest of this text. It is inspired from [21] where it was given in the special case of 2-additive Choquet integral model.

Definition 9. *Let $\{P, I\}$ be an ordinal preferential information and $A \subseteq_{\geq 2} N$.*

1. *There exists a possible positive (resp. null, negative) interaction for A if there exists $\mu \in C_{Pref}$ such that $I^{\mu}_{A} > 0$ (resp. $I^{\mu}_{A} = 0$, $I^{\mu}_{A} < 0$),*
2. *There exists a necessary positive (resp. null, negative) interaction for A if $I^{\mu}_{A} > 0$ (resp. $I^{\mu}_{A} = 0$, $I^{\mu}_{A} < 0$) for all $\mu \in C_{Pref}$.*

If an interaction is possible but not necessary, then its interpretation seems difficult as its is strongly related to the capacity chosen in C_{Pref}. Indeed, the interpretation of the interaction only makes sense in the case of the necessary interaction. In the next section we present our results when there is no indifference in the ordinal preferential information of DM.

4. Results when I is empty

In this section, we start with the particular case $I = \emptyset$, and we deal the general case ($I \neq \emptyset$) in the next section. The condition $I = \emptyset$ is likely to be met in most applications: indifference is indeed much less likely between alternative than strict preference, unless alternatives have been specially designed to be indifferent.

4.1. Result on the set of alternatives X

When $I = \emptyset$, Proposition 1 shows that null interaction is never necessary. In our working paper [17], we had already obtained this result by assuming that the DM provides a linear order of preferences on a subset Y of X. Here, we drop this hypothesis.

Proposition 1. *Let $\{P, I\}$ be an ordinal preferential information on X such that $\{P, I\}$ is representable by a Choquet integral model. If the relation I is empty then there is no necessary null interaction.*

Proof. Let us suppose that $I = \emptyset$ and $\{P, I\}$ can be represented by a Choquet integral model using a capacity μ for which $I^{\mu}_{A} = 0$ where $A \subseteq_{\geq 2} N$. We show that this null interaction is not necessary.

Let us define the capacity β_ε by, $\beta_\varepsilon(S) = \begin{cases} \dfrac{1}{1+\varepsilon}\mu(S), & \text{if } \; S \subsetneq N, \\ 1, & \text{if } \; S = N. \end{cases}$

where ε is a strictly positive real number to be determined as follows. For all $x \in X$, we have:

$$
\begin{aligned}
C_{\beta_\varepsilon}\big(u(x)\big) &= \sum_{i=1}^{n}\Big[u_{\sigma(i)}(x_{\sigma(i)}) - u_{\sigma(i-1)}(x_{\sigma(i-1)})\Big]\beta_\varepsilon(N_{\sigma(i)})\\
&= u_{\sigma(1)}(x_{\sigma(1)})\beta_\varepsilon(N) + \sum_{i=2}^{n}\Big(u_{\sigma(i)}(x_{\sigma(i)}) - u_{\sigma(i-1)}(x_{\sigma(i-1)})\Big)\beta_\varepsilon(N_{\sigma(i)})\\
&= u_{\sigma(1)}(x_{\sigma(1)}) + \frac{1}{1+\varepsilon}\sum_{i=2}^{n}\Big(u_{\sigma(i)}(x_{\sigma(i)}) - u_{\sigma(i-1)}(x_{\sigma(i-1)})\Big)\mu(N_{\sigma(i)})\\
&= \frac{1}{1+\varepsilon}\Big[u_{\sigma(1)}(x_{\sigma(1)}) + \sum_{i=2}^{n}\Big(u_{\sigma(i)}(x_{\sigma(i)}) - u_{\sigma(i-1)}(x_{\sigma(i-1)})\Big)\mu(N_{\sigma(i)})\Big] + \frac{\varepsilon}{1+\varepsilon}u_{\sigma(1)}(x_{\sigma(1)})\\
&= \frac{1}{1+\varepsilon}C_\mu\big(u(x)\big) + \frac{\varepsilon}{1+\varepsilon}u_{\sigma(1)}(x_{\sigma(1)}).\\
&= \frac{1}{1+\varepsilon}\Big[C_\mu\big(u(x)\big) + \varepsilon u_{\sigma(1)}(x_{\sigma(1)})\Big].
\end{aligned}
$$

Therefore, for all $(x,y)\in P$, we then have:

$$
C_{\beta_\varepsilon}\big(u(x)\big) - C_{\beta_\varepsilon}\big(u(y)\big) = \frac{1}{1+\varepsilon}\Big[\Big(C_\mu\big(u(x)\big) - C_\mu\big(u(y)\big)\Big) + \varepsilon\Big(u_{\sigma(1)}(x_{\sigma(1)}) - u_{\gamma(1)}(y_{\gamma(1)})\Big)\Big].
$$

We are looking for ε such that $C_{\beta_\varepsilon}\big(u(x)\big) - C_{\beta_\varepsilon}\big(u(y)\big) > 0$ for all $(x,y)\in P$. We have, $C_{\beta_\varepsilon}\big(u(x)\big) - C_{\beta_\varepsilon}\big(u(y)\big) > 0 \iff \varepsilon\Big(u_{\sigma(1)}(x_{\sigma(1)}) - u_{\gamma(1)}(y_{\gamma(1)})\Big) > -\Big(C_\mu\big(u(x)\big) - C_\mu\big(u(y)\big)\Big)$.
Let us consider the set $\Omega = \{(x,y)\in P : u_{\sigma(1)}(x_{\sigma(1)}) - u_{\gamma(1)}(y_{\gamma(1)}) < 0\}$.

- If $\Omega = \emptyset$, then for all $(x,y)\in P$, $u_{\sigma(1)}(x_{\sigma(1)}) - u_{\gamma(1)}(y_{\gamma(1)}) \geq 0$. Thus for all $(x,y)\in P$, for all $\varepsilon > 0$, we have $C_{\beta_\varepsilon}\big(u(x)\big) - C_{\beta_\varepsilon}\big(u(y)\big) > 0$.
- If $\Omega \neq \emptyset$, we choose ε such that $0 < \varepsilon < \min\limits_{(x,y)\in\Omega}\left(\dfrac{C_\mu\big(u(y)\big) - C_\mu\big(u(x)\big)}{u_{\sigma(1)}(x_{\sigma(1)}) - u_{\gamma(1)}(y_{\gamma(1)})}\right)$ in such a way that $C_{\beta_\varepsilon}\big(u(x)\big) - C_{\beta_\varepsilon}\big(u(y)\big) > 0$ for all $(x,y)\in P$.

So in both cases we can choose $\varepsilon = \dfrac{1}{4}\min\limits_{(x,y)\in\Omega}\left(\dfrac{C_\mu\big(u(y)\big) - C_\mu\big(u(x)\big)}{u_{\sigma(1)}(x_{\sigma(1)}) - u_{\gamma(1)}(y_{\gamma(1)})}\right)$ such that $\{P, I\}$ is

representable by the Choquet integral model C_{β_ε}, i.e., $\beta_\varepsilon \in C_{\text{Pref}}$. Moreover we have,

$$
\begin{aligned}
I_A^{\beta_\varepsilon} &= \frac{(n-a)!}{(n-a+1)!}\sum_{L\subseteq A}(-1)^{a-\ell}\beta_\varepsilon\big((N\setminus A)\cup L\big) + \sum_{K\subsetneq N\setminus A}\frac{(n-k-a)!k!}{(n-a+1)!}\sum_{L\subseteq A}(-1)^{a-\ell}\beta_\varepsilon(K\cup L)\\
&= \frac{(n-a)!}{(n-a+1)!}\beta_\varepsilon(N) + \frac{(n-a)!}{(n-a+1)!}\sum_{L\subsetneq A}(-1)^{a-\ell}\beta_\varepsilon\big((N\setminus A)\cup L\big) +\\
&\quad \sum_{K\subsetneq N\setminus A}\frac{(n-k-a)!k!}{(n-a+1)!}\sum_{L\subseteq A}(-1)^{a-\ell}\beta_\varepsilon(K\cup L).
\end{aligned}
$$

$$I_A^{\beta_\varepsilon} = \frac{(n-a)!}{(n-a+1)!} + \frac{1}{1+\varepsilon}\frac{(n-a)!}{(n-a+1)!}\sum_{L\subsetneq A}(-1)^{a-\ell}\mu\big((N\setminus A)\cup L\big) +$$
$$\frac{1}{1+\varepsilon}\sum_{K\subsetneq N\setminus A}\frac{(n-k-a)!k!}{(n-a+1)!}\sum_{L\subseteq A}(-1)^{a-\ell}\mu(K\cup L)$$
$$= \frac{1}{1+\varepsilon}I_A^\mu + \frac{\varepsilon}{1+\varepsilon}\frac{1}{n-a+1}$$
$$= \frac{\varepsilon}{1+\varepsilon}\frac{1}{n-a+1} > 0, \text{ since } I_A^\mu = 0.$$

We deduce that there exists a possible positive interaction for A. Hence there is no null interaction for A. Hence, null interactions are never necessary when $I=\emptyset$. □

The following example illustrates the result of Proposition 1.

Example 1. *$N=\{1,2,3\}$, $X=\{a,b,c,d\}$, $a=(6,11,9)$, $b=(6,13,7)$, $c=(16,11,9)$, $d=(16,13,7)$ and $P=\{(a,b),(d,c)\}$.*
$\{P,I\}$ is representable by the capacity μ (such that$I_{13}^\mu=0$) given in Table 1 and the corresponding Choquet integral is given in Table 2. For all $i\in N$, we define the utility function u_i by $u_i(x_i)=x_i$. We recall that $I_{13}^\mu = \frac{1}{2}(\mu_{123}-\mu_{12}+\mu_{13}-\mu_{23}-\mu_1+\mu_2-\mu_3)$.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\mu(S)$	0	0	0	1	0.5	0.5	1

Table 1: A capacity $\mu\in C_{\text{Pref}}$ such that $I_{13}^\mu=0$

x	a	b	c	d
$C_\mu\big(u(x)\big)$	7.5	6.5	11	13

Table 2: The Choquet integral corresponding at the capacity μ of Table 1

We have $u_{\sigma(1)}(a_{\sigma(1)})-u_{\gamma(1)}(b_{\gamma(1)})=6-6=0$ and $u_{\sigma(1)}(d_{\sigma(1)})-u_{\gamma(1)}(c_{\gamma(1)})=7-9=-2<0$, then $\Omega=\{(d,c)\}$ and $\varepsilon=\frac{1}{2}\times\frac{C_\mu\big(u(c)\big)-C_\mu\big(u(d)\big)}{u_{\sigma(1)}(d_{\sigma(1)})-u_{\gamma(1)}(c_{\gamma(1)})}=\frac{1}{4}\times\frac{11-13}{7-9}=0.25$.

A capacity $\beta^\mu\in C_{Pref}$ such that $I_{13}^{\beta^\mu}>0$ and the corresponding Choquet integral are respectively given in Table 3 and Table 4.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\beta^\mu(S)$	0	0	0	0.8	0.4	0.4	1

Table 3: A capacity $\beta^\mu\in C_{\text{Pref}}$ such that $I_{13}^{\beta^\mu}>0$

x	a	b	c	d
$C_{\beta^\mu}\big(u(x)\big)$	7.2	6.4	10.6	11.8

Table 4: The Choquet integral corresponding at the capacity β^μ of Table 3

We have $I_{13}^{\beta^\mu} = \frac{1}{2}(\mu_{123}-\mu_{12}+\mu_{13}-\mu_{23}-\mu_1+\mu_2-\mu_3) = \frac{1}{2}(1-0.8+0.4-0.4-0+0-0) = 0.1 > 0.$ *Or, as in the proof of Proposition 1, we have* $I_{13}^{\beta^\mu} = \frac{\varepsilon}{1+\varepsilon} \times \frac{1}{n-a+1} = \frac{0.25}{1.25} \times \frac{1}{2} = 0.1 > 0.$

In the next subsection, we will restrict ourselves on the set $\mathcal{B}^g$.

4.2. Results on the general set of binary alternatives $\mathcal{B}^g$

The two following propositions are taken from [16]. We recall them to facilitate the comparison with this paper. Proposition 2 provides a necessary and sufficient condition for an ordinal preferential information on $\mathcal{B}^g$ containing no indifference to be representable by a Choquet integral model.

Proposition 2. *Let* $\{P, I\}$ *be an ordinal preferential information on* $\mathcal{B}^g$ *such that* $I = \emptyset$*. Then,* $\{P, I\}$ *is representable by a Choquet integral model if and only if the binary relation* $(P \cup M)$ *contains no strict cycle.*

Proof. See the proof of Proposition 1 in [16]. □

When $\{P, I\}$ is representable by a Choquet integral model, then there exists a representation for which all the interaction indices are strictly positive. It is the result of Proposition 3. This result shows that when there is no indifference, negative and null interactions are not necessary.

Proposition 3. *Let* $\{P, I\}$ *be an ordinal preferential information on* $\mathcal{B}^g$ *such that* $I = \emptyset$*, and* $(P \cup M)$ *containing no strict cycle. Then there exists a capacity* $\mu \in C_{Pref}$ *such that* $I_A^\mu > 0$ *for all* $A \subseteq_{\geq 2} N$*.*

Proof. See the proof of Proposition 2 in [16]. □

5. Results when I is not empty

In this section, we relax the condition $I = \emptyset$ made in the previous section and we generalize some of our previous results. We start by the following proposition which extends Proposition 2.

Proposition 4. *Let* $\{P, I\}$ *be an ordinal preferential information on* $\mathcal{B}^g$*.*
$\{P, I\}$ *is representable by a Choquet integral if and only if the binary relation* $(P \cup M \cup I)$ *contains no strict cycle.*

Proof. **Necessity**. Suppose that the ordinal preferential information $\{P, I\}$ on $\mathcal{B}^g$ is representable by a Choquet integral. So there exists a capacity μ such that $\{P, I\}$ is representable by C_μ.
If $(P \cup M \cup I)$ contains a strict cycle, then there exists $x_0, x_1, \ldots, x_r$ on $\mathcal{B}^g$ such that $x_0(P\cup M\cup I)x_1(P\cup M\cup I)\ldots(P\cup M\cup I)x_r(P\cup M\cup I)x_0$ and there exists $x_i, x_{i+1} \in \{x_0, x_1, \ldots, x_r\}$ so that $x_i P x_{i+1}$. Since $\{P, I\}$ is representable by C_μ, therefore $C_\mu(u(x_0)) \geq \ldots \geq C_\mu(u(x_i)) > C_\mu(u(x_{i+1})) \geq \ldots \geq C_\mu(u(x_0))$, then $C_\mu(u(x_0)) > C_\mu(u(x_0))$, a contradiction. Hence, $(P \cup M \cup I)$ contains no strict cycle.
Sufficiency. We assume that the graph $\big(\mathcal{B}^g,\ (P \cup M \cup I)\big)$ does not contain any strict cycle and let us define the binary relation $\mathcal{R}_{I\cup M}$ on $\mathcal{B}^g$ by, for all $x, y \in \mathcal{B}^g$, we have $x\, \mathcal{R}_{I\cup M}\, y$ if and only if $x\, TC_{I\cup M}\, y$ and $y\, TC_{I\cup M}\, x$. $\mathcal{R}_{I\cup M}$ is an equivalence relation. Let $\mathcal{B}' = \mathcal{B}^g/\mathcal{R}_{I\cup M}$ be the quotient set of $\mathcal{B}^g$ by the equivalence relation $\mathcal{R}_{I\cup M}$. Let us define on the set $\mathcal{B}'$ the preference relation P' by, for all $A,\, B \in \mathcal{B}'$, $A\, P'\, B \iff \exists a \in A,\, \exists b \in B : a\, P\, b$ or $a\, M\, b$.
The graph $\big(\mathcal{B}', P'\big)$ contains no strict cycle because the graph $\big(\mathcal{B}^g, (P \cup M \cup I)\big)$ contains no strict cycle. But the graph $\big(\mathcal{B}', P'\big)$ no indifference, so that there exists $\{\mathcal{B}'_0, \mathcal{B}'_1, \ldots, \mathcal{B}'_m\}$ a partition of $\mathcal{B}'$, built by using a suitable topological sorting on graph $\big(\mathcal{B}', P'\big)$ [6]. We construct a partition $\{\mathcal{B}'_0, \mathcal{B}'_1, \ldots, \mathcal{B}'_m\}$ as follows:
$\mathcal{B}'_0 = \{A \in \mathcal{B}' \colon \forall C \in \mathcal{B}',\text{ not } [A(P \cup M)C]\}$, $\mathcal{B}'_1 = \{A \in \mathcal{B}' \setminus \mathcal{B}'_0 \colon \forall C \in \mathcal{B}' \setminus \mathcal{B}'_0,\text{ not } [A(P \cup M)C]\}$,
$\mathcal{B}'_i = \{A \in \mathcal{B}' \setminus (\mathcal{B}'_0 \cup \ldots \cup \mathcal{B}'_{i-1}) \colon \forall C \in \mathcal{B}' \setminus (\mathcal{B}'_0 \cup \ldots \cup \mathcal{B}'_{i-1}),\text{ not } [A(P \cup M)C]\}$, for all $i = 2, 3, \ldots, m$.
Let $\mathcal{B}_i = \{x \in A \colon A \in \mathcal{B}'_i\}$ for all $i = 0, 1, \ldots, m$. Therefore $\{\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_m\}$ is a suitable topological sorting on graph $\big(\mathcal{B}^g, (P \cup M \cup I)\big)$ since $\{\mathcal{B}'_0, \mathcal{B}'_1, \ldots, \mathcal{B}'_m\}$ is a suitable topological sorting on graph $\big(\mathcal{B}', P'\big)$. Let us define the capacity $\mu \colon 2^N \longrightarrow [0, 1]$ as follows.

$$\text{For all } S \subseteq N,\ \mu(S) = \begin{cases} 0, & \text{if } a_S \in \mathcal{B}_0, \\ (2n)^\ell/(2n)^m, & \text{if } a_S \in \mathcal{B}_\ell,\ \ell \in \{1, 2, \ldots, m\}. \end{cases}$$

Let $a_S, a_T \in \mathcal{B}^g$.

a) Assume that $a_S\, I\, a_T$, therefore there exists $A \in \mathcal{B}'$ such that $a_S, a_T \in A$. Since $A \in \mathcal{B}'$, thus there exists $\ell \in \{0, 1, \ldots, m\}$ such that $A \in \mathcal{B}'_\ell$, then we have $a_S, a_T \in \mathcal{B}_\ell$.

   – If $\ell = 0$, therefore $C_\mu(u(a_S)) = 0 = C_\mu(u(a_T))$.
   – If $\ell \in \{1, \ldots, m\}$, therefore $C_\mu(u(a_S)) = (2n)^\ell/(2n)^m = C_\mu(u(a_T))$.

   In both cases, we have $C_\mu(u(a_S)) = C_\mu(u(a_T))$.

b) Assume that $a_S\, P\, a_T$, then there exists $A \in \mathcal{B}'_r$, $C \in \mathcal{B}'_q$, such that $a_S \in A$, $a_T \in C$ with $r, q \in \{0, 1, \ldots, m\}$ and $r > q$. Thus $q \in \{0, 1, \ldots, m-1\}$, $r \in \{1, \ldots, m\}$ and $C_\mu(u(a_S)) = \mu(S) = (2n)^r/(2n)^m$.

   – If $q = 0$, therefore $C_\mu(u(a_T)) = \mu(T) = 0 < (2n)^r/(2n)^m = \mu(S) = C_\mu(u(a_S))$.
   – If $q \in \{1, \ldots, m-1\}$, therefore $C_\mu(u(a_T)) = \mu(T) = (2n)^q/(2n)^m$. Since $r > q$, therefore $(2n)^r > (2n)^q$, then $(2n)^r/(2n)^m > (2n)^q/(2n)^m$, i.e., $C_\mu(u(a_S)) > C_\mu(u(a_T))$.

Hence $\{P, I\}$ is representable by C_μ, i.e., $\mu \in C_{\text{Pref}}$. $\square$

The following proposition shows that, if the DM is not indifferent between the best alternative a_N and another alternative, then a positive interaction is always possible for all subsets of at least two criteria.

Proposition 5. *Let $\{P, I\}$ be an ordinal preferential information on $\mathcal{B}^g$ such that $(P \cup M \cup I)$ contains no strict cycle. If not$(a_S \, TC_{I\cup M} \, a_N)$ for all $S \subsetneq N$, then there exists a capacity $\mu \in C_{Pref}(P, I)$ such that $I^\mu_A > 0$ for all $A \subseteq_{\geq 2} N$.*

Proof. The partitions $\{\mathcal{B}'_0, \ldots, \mathcal{B}'_m\}$ of $\mathcal{B}'$ and $\{\mathcal{B}_0, \ldots, \mathcal{B}_m\}$ of $\mathcal{B}^g$ are built as in the proof of Proposition 4. Let us define the capacity μ: $2^N \longrightarrow [0, 1]$ as follows.

$$\text{For all } S \subseteq N, \quad \mu(S) = \begin{cases} 0, & \text{if } a_S \in \mathcal{B}_0, \\ \dfrac{(2n)^\ell}{(2n)^{n+m}}, & \text{if } a_S \in \mathcal{B}_\ell,\ \ell \in \{1, 2, \ldots, m-1\}, \\ 1, & \text{if } a_S \in \mathcal{B}_m. \end{cases}$$

Let $a_S, a_T \in \mathcal{B}^g$.

a) Suppose that $a_S \, I \, a_T$, then there exists $r \in \{0, 1, \ldots, m\}$ such that $a_S, a_T \in \mathcal{B}_r$, thus $C_\mu(u(a_S)) = \mu(S) = \mu(T) = C_\mu(u(a_T))$.

b) Suppose that $a_S \, P \, a_T$, then there exists $r, q \in \{0, 1, \ldots, m\}$ such that $a_S \in \mathcal{B}_r$, $a_T \in \mathcal{B}_q$. As $a_S \, P \, a_T$, then $r > q$. We have $C_\mu(u(a_S)) = \mu(S) = \dfrac{(2n)^r}{(2n)^{n+m}}$ (if $1 \leq r \leq m-1$) or 1 (if $r = m$). Hence, we have $C_\mu(u(a_S)) \geq \dfrac{(2n)^r}{(2n)^{n+m}}$.

   - If $q = 0$, then $a_T \in \mathcal{B}_0$ and $a_S \in \mathcal{B}_r$ with $r \geq 1$. As $C_\mu(u(a_T)) = C_\mu(u(a_0)) = \mu(\emptyset) = 0$, then $C_\mu(u(a_S)) > C_\mu(u(a_T))$.
   - If $q \geq 1$, $C_\mu(u(a_T)) = \mu(T) = \dfrac{(2n)^q}{(2n)^{n+m}}$, since $1 \leq q \leq m-1$. But $r > q$ therefore $C_\mu(u(a_S)) \geq \dfrac{(2n)^r}{(2n)^{n+m}} > \dfrac{(2n)^q}{(2n)^{n+m}} = C_\mu(u(a_T))$, then $C_\mu(u(a_S)) > C_\mu(u(a_T))$.

   Hence, in both cases we have $C_\mu(u(a_S)) > C_\mu(u(a_T))$.

We deduce that $\{P, I\}$ is representable by C_μ i.e., $\mu \in C_{\text{Pref}}(P, I)$.
Let $A \subseteq_{\geq 2} N$, according to the Lemma 1 we have:

$$(n-a+1)! \times I_A^\mu = \sum_{K\subseteq N\setminus A} (n-k-a)!k! \sum_{L\subseteq A} (-1)^{a-\ell}\mu(K\cup L)$$
$$= \sum_{K\subseteq N\setminus A} (n-k-a)!k! \sum_{\substack{p=0,\\ p \text{ even}}}^{a} \Bigg[\sum_{\substack{L\subseteq A,\\ \ell=a-p}} \mu(K\cup L) - \sum_{\substack{L\subseteq A,\\ \ell=a-p-1}} \mu(K\cup L)\Bigg]$$
$$\geq \sum_{K\subseteq N\setminus A} \sum_{\substack{p=0,\\ p \text{ even}}}^{a} \Bigg[\sum_{\substack{L\subseteq A,\\ \ell=a-p}} \mu(K\cup L) - \sum_{\substack{L\subseteq A,\\ \ell=a-p-1}} \mu(K\cup L)\Bigg], \text{ since } (n-k-a)!k! \geq 1.$$

Moreover, $\sum_{K\subseteq N\setminus A} \sum_{\substack{p=0,\\ p \text{ even}}}^{a} \Bigg[\sum_{\substack{L\subseteq A,\\ \ell=a-p}} \mu(K\cup L) - \sum_{\substack{L\subseteq A,\\ \ell=a-p-1}} \mu(K\cup L)\Bigg]$

$= \sum_{K\subseteq N\setminus A} \sum_{\substack{p=0,\\ p \text{ even}}}^{a} \sum_{\substack{L\subseteq A,\\ \ell=a-p}} \mu(K\cup L) - \sum_{K\subseteq N\setminus A} \sum_{\substack{p=0,\\ p \text{ even}}}^{a} \sum_{\substack{L\subseteq A,\\ \ell=a-p-1}} \mu(K\cup L)$

$= \mu(N) + \sum_{\substack{p=2,\\ p \text{ even}}}^{a} \sum_{\substack{L\subseteq A,\\ \ell=a-p}} \mu((N\setminus A)\cup L) + \sum_{K\subsetneq N\setminus A} \sum_{\substack{p=0,\\ p \text{ even}}}^{a} \sum_{\substack{L\subseteq A,\\ \ell=a-p}} \mu(K\cup L)$ - $\sum_{K\subseteq N\setminus A} \sum_{\substack{p=0,\\ p \text{ even}}}^{a} \sum_{\substack{L\subseteq A,\\ \ell=a-p-1}} \mu(K\cup L)$

$\geq \mu(N) - \sum_{K\subseteq N\setminus A} \sum_{\substack{p=0,\\ p \text{ even}}}^{a} \sum_{\substack{L\subseteq A,\\ \ell=a-p-1}} \mu(K\cup L).$

Therefore, $(n-a+1)!k! \times I_A^\mu \geq \mu(N) - \sum_{K\subseteq N\setminus A} \sum_{\substack{p=0,\\ p \text{ even}}}^{a} \sum_{\substack{L\subseteq A,\\ \ell=a-p-1}} \mu(K\cup L)$. We still have to prove that $\mu(N) - \sum_{K\subseteq N\setminus A} \sum_{\substack{p=0,\\ p \text{ even}}}^{a} \sum_{\substack{L\subseteq A,\\ \ell=a-p-1}} \mu(K\cup L) > 0$.

Let $K \subseteq N\setminus A$, $p \in \{0,\ldots,a\}$ even number and $L \subseteq A$ with $\ell = a-p-1$. We have $L \subsetneq A$, therefore $K\cup L \subsetneq N$, then by hypothesis not$(a_{K\cup L} TC_{I\cup M} a_N)$. Thus $a_N \in \mathcal{B}_m$ and there exists $\ell_{K\cup L} \in \{0,1,\ldots,m-1\}$ such that $a_{K\cup L} \in \mathcal{B}_{\ell_{K\cup L}}$. Then $\mu(K\cup L) = \dfrac{(2n)^{\ell_{K\cup L}}}{(2n)^{n+m}}$ or $\mu(K\cup L) = 0$, hence in both cases we have $\mu(K\cup L) \leq \dfrac{(2n)^{\ell_{K\cup L}}}{(2n)^{n+m}}$. Therefore,

$$\sum_{K\subseteq N\setminus A} \sum_{\substack{p=0,\\ p \text{ even}}}^{a} \sum_{\substack{L\subseteq A,\\ \ell=a-p-1}} \mu(K\cup L) \leq \sum_{K\subseteq N\setminus A} \sum_{\substack{p=0,\\ p \text{ even}}}^{a} \sum_{\substack{L\subseteq A,\\ \ell=a-p-1}} \frac{(2n)^{\ell_{K\cup L}}}{(2n)^{n+m}}$$
$$\leq \sum_{K\subseteq N\setminus A} \sum_{\substack{p=0,\\ p \text{ even}}}^{a} \sum_{\substack{L\subseteq A,\\ \ell=a-p-1}} \frac{(2n)^{m-1}}{(2n)^{n+m}} = \frac{(2)^{n-1}(2n)^{m-1}}{(2n)^{n+m}}. \text{ Moreover,}$$

$\dfrac{(2)^{n-1}(2n)^{m-1}}{(2n)^{n+m}} = \dfrac{2^{n-1}(2n)^{m-1}}{(2n)^{n}(2n)^{m}} = \dfrac{2^{n-1}(2n)^{m-1}}{2(2)^{n-1}(n)^{n}(2n)(2n)^{m-1}} = \dfrac{1}{4n^{n+1}} < 1 = \mu(N)$. Hence we have $\mu(N) > \left[\sum_{K\subseteq N\setminus A}\sum_{\substack{p=0,\\ p \text{ even}}}^{a}\sum_{\substack{L\subseteq A,\\ \ell=a-p-1}}\mu(K\cup L)\right]$, i.e., $\mu(N) - \left[\sum_{K\subseteq N\setminus A}\sum_{\substack{p=0,\\ p\ even}}^{a}\sum_{\substack{L\subseteq A,\\ \ell=a-p-1}}\mu(K\cup L)\right] > 0$.
We can therefore conclude that $(n-a+1)! \times I^{\mu}_{A} > 0, i.e., I^{\mu}_{A} > 0$. □

Our Lemma 2 shows that the condition not($a_{N\setminus\{i\}}\ TC_{I\cup M}\ a_N$) for all $i \in N$ is equivalent to the condition not($a_S\, TC_{I\cup M}\, a_N$) for all $S \subsetneq N$ used above. This new condition forbids to be indifferent between the best alternative a_N and another alternative $a_{N\setminus\{i\}}$, obtained from a_N by degrading the satisfaction of the DM from the reference level 1_i, to the reference level 0_i on criterion i, while remaining unchanged for the other criteria.

Lemma 2. *Let $\{P, I\}$ be an ordinal preferential information on $\mathcal{B}^g$ such that $(P\cup M\cup I)$ contains no strict cycle. The condition [not($a_S\ TC_{I\cup M}\ a_N$) for all $S \subsetneq N$] is equivalent to [not($a_{N\setminus\{i\}}\ TC_{I\cup M}\ a_N$) for all $i \in N$].*

Proof. **Necessity**. Suppose that not($a_S\ TC_{I\cup M}\ a_N$) for all $S \subsetneq N$, then not($a_{N\setminus\{i\}}\ TC_{I\cup M}\ a_N$) for all $i \in N$ since $N \setminus \{i\} \subsetneq N$ for all $i \in N$.
Sufficiency. Assume that not($a_{N\setminus\{i\}}\ TC_{I\cup M}\ a_N$) for all $i \in N$.
We suppose that there exists $S \subsetneq N$ such that $a_S\ TC_{I\cup M}\ a_N$. Since $S \subsetneq N$, then there exists $i_0 \in N \setminus S$ such that $S \subseteq N \setminus \{i_0\}$. We have $a_N\,(P\cup M\cup I)\,a_{N\setminus\{i_0\}}\,(P\cup M\cup I)\,a_S\,TC_{I\cup M}\ a_N$.

- If $a_N\,P\,a_{N\setminus\{i_0\}}$ or $a_{N\setminus\{i_0\}}P\,a_S$, then $(P\cup I\cup M)$ contains a strict cycle. A contradiction.
- Else, we have $a_N\,(I\cup M)\,a_{N\setminus\{i_0\}}\,(I\cup M)\,a_S\,TC_{I\cup M}\ a_N$, then $a_{N\setminus\{i_0\}}\,TC_{I\cup M}\ a_N$. A contradiction.

We deduce that not($a_S\ TC_{I\cup M}\ a_N$) for all $S \subsetneq N$. □

We conjecture that the condition not($a_{N\setminus\{i\}}\ TC_{I\cup M}\ a_N$) is necessary and sufficient to obtain a possible positive interaction for all $A \subseteq_{\geq 2} N$. Proposition 6 shows that the conjecture is true for a small number of criteria ($n \leq 5$). This proposition is useful in practice since most of decision problems have less than 6 criteria. At this stage, we do not have neither a proof for the general case ($n \geq 6$), nor a counter-example indicating that this condition is not necessary.

Proposition 6. *Let N be a set of n criteria, with $2 \leq n \leq 5$. Let $\{P, I\}$ be an ordinal preferential information on $\mathcal{B}^g$ such that $(P\cup M\cup I)$ contains no strict cycle. If there exists a capacity $\mu \in C_{Pref}$ such that $I^{\mu}_{A} > 0$ for all $A \subseteq_{\geq 2} N$, then we have not($a_{N\setminus\{i\}}\,TC_{I\cup M}\,a_N$) for all $i \in N$.*

Proof. We assume that $I^{\mu}_{A} > 0$, for all $A \subseteq_{\geq 2} N$ and there exists $i \in N$ such that $a_{N\setminus\{i\}}\,TC_{I\cup M}\,a_N$.
For all $A \subseteq_{\geq 2} N$, let us consider $J^{\mu}_{A} = (n-a+1)! I^{\mu}_{A}$. So we have $J^{\mu}_{A} > 0$ since $I^{\mu}_{A} > 0$.

- Case $n = 2$.

  Following the hypothesis, we have $a_1 \, TC_{I\cup M} \, a_{12}$ or $a_2 \, TC_{I\cup M} \, a_{12}$. Hence we obtain $\mu_{12} = \mu_1$ or $\mu_{12} = \mu_2$ implying $J^{\mu}_{12} = -\mu_2 \leq 0$ or $J^{\mu}_{12} = -\mu_1 \leq 0$. A contradiction since $J^{\mu}_{12} > 0$.

- Case $n = 3$.

  Without loss of generality, we assume that $i = 3$, i.e., $\mu_{123} = \mu_{12}$. Hence we have,

  $J^{\mu}_{23} = \mu_{123} - \mu_{12} - \mu_{13} + \mu_{23} + \mu_1 - \mu_2 - \mu_3$ and $J^{\mu}_{123} = \mu_{123} - \mu_{12} - \mu_{13} - \mu_{23} + \mu_1 + \mu_2 + \mu_3$.

  This leads to $J^{\mu}_{23} + J^{\mu}_{123} = -2(\mu_{13} - \mu_1) \leq 0$. Which is impossible because $J^{\mu}_{23}, J^{\mu}_{123} > 0$.

- Case $n = 4$.

  Without loss of generality, we assume that $i = 4$, i.e., $\mu_{1234} = \mu_{123}$. Hence we have:

  $J^{\mu}_{34} = 2\mu_{1234} - 2\mu_{123} - 2\mu_{124} + \mu_{134} + \mu_{234} + 2\mu_{12} - \mu_{13} - \mu_{14} - \mu_{23} - \mu_{24} + 2\mu_{34} + \mu_1 + \mu_2 - 2\mu_3 - 2\mu_4$

  $J^{\mu}_{134} = \mu_{1234} - \mu_{123} - \mu_{124} + \mu_{134} - \mu_{234} + \mu_{12} - \mu_{13} - \mu_{14} + \mu_{23} + \mu_{24} - \mu_{34} + \mu_1 - \mu_2 + \mu_3 + \mu_4$

  $J^{\mu}_{234} = \mu_{1234} - \mu_{123} - \mu_{124} - \mu_{134} + \mu_{234} + \mu_{12} + \mu_{13} + \mu_{14} - \mu_{23} - \mu_{24} - \mu_{34} - \mu_1 + \mu_2 + \mu_3 + \mu_4$

  $J^{\mu}_{1234} = \mu_{1234} - \mu_{123} - \mu_{124} - \mu_{134} - \mu_{234} + \mu_{12} + \mu_{13} + \mu_{14} + \mu_{23} + \mu_{24} + \mu_{34} - \mu_1 - \mu_2 - \mu_3 - \mu_4$.

  This leads to $2J^{\mu}_{34} + 3(J^{\mu}_{134} + J^{\mu}_{234}) + 2J^{\mu}_{1234} = -12(\mu_{124} - \mu_{12}) \leq 0$. A contradiction since $J^{\mu}_{34}, J^{\mu}_{134}, J^{\mu}_{234}, J^{\mu}_{1234} > 0$.

- Case $n = 5$.

  Like in the previous cases, without loss of generality, we assume that $i = 5$, i.e., $\mu_{12345} = \mu_{1234}$. We then have:

  $J^{\mu}_{1245} = \mu_{12345} - \mu_{1234} - \mu_{1235} + \mu_{1245} - \mu_{1345} - \mu_{2345} + \mu_{123} - \mu_{124} - \mu_{125} + \mu_{134} + \mu_{135} - \mu_{145} + \mu_{234} + \mu_{235} - \mu_{245} + \mu_{345} + \mu_{12} - \mu_{13} + \mu_{14} + \mu_{15} - \mu_{23} + \mu_{24} + \mu_{25} - \mu_{34} - \mu_{35} + \mu_{45} - \mu_1 - \mu_2 + \mu_3 - \mu_4 - \mu_5$.

  $J^{\mu}_{1345} = \mu_{12345} - \mu_{1234} - \mu_{1235} - \mu_{1245} + \mu_{1345} - \mu_{2345} + \mu_{123} + \mu_{124} + \mu_{125} - \mu_{134} - \mu_{135} - \mu_{145} + \mu_{234} + \mu_{235} + \mu_{245} - \mu_{345} - \mu_{12} + \mu_{13} + \mu_{14} + \mu_{15} - \mu_{23} - \mu_{24} - \mu_{25} + \mu_{34} + \mu_{35} + \mu_{45} - \mu_1 + \mu_2 - \mu_3 - \mu_4 - \mu_5$.

  $J^{\mu}_{2345} = \mu_{12345} - \mu_{1234} - \mu_{1235} - \mu_{1245} - \mu_{1345} + \mu_{2345} + \mu_{123} + \mu_{124} + \mu_{125} + \mu_{134} + \mu_{135} + \mu_{145} - \mu_{234} - \mu_{235} - \mu_{245} - \mu_{345} - \mu_{12} - \mu_{13} - \mu_{14} - \mu_{15} + \mu_{23} + \mu_{24} + \mu_{25} + \mu_{34} + \mu_{35} + \mu_{45} + \mu_1 - \mu_2 - \mu_3 - \mu_4 - \mu_5$.

  $J^{\mu}_{145} = 2\mu_{12345} - 2\mu_{1234} - 2\mu_{1235} + \mu_{1245} + \mu_{1345} - \mu_{2345} + 2\mu_{123} - \mu_{124} - \mu_{125} - \mu_{134} - \mu_{135} + 2\mu_{145} + 2\mu_{234} + 2\mu_{235} - \mu_{245} - \mu_{345} + u_{12} + \mu_{13} - 2\mu_{14} - 2\mu_{15} - 2\mu_{23} + \mu_{24} + \mu_{25} + \mu_{34} + \mu_{35} - 2\mu_{45} + 2\mu_1 - \mu_2 - \mu_3 + 2\mu_4 + 2\mu_5$.

  $J^{\mu}_{245} = 2\mu_{12345} - 2\mu_{1234} - 2\mu_{1235} + \mu_{1245} - 2\mu_{1345} + \mu_{2345} + 2\mu_{123} - \mu_{124} - \mu_{125} + 2\mu_{134} + 2\mu_{135} - \mu_{145} - \mu_{234} - \mu_{235} + 2\mu_{245} - \mu_{345} + \mu_{12} - 2\mu_{13} + \mu_{14} + \mu_{15} + \mu_{23} - 2\mu_{24} - 2\mu_{25} + \mu_{34} + \mu_{35} - 2\mu_{45} - \mu_1 + 2\mu_2 - \mu_3 + 2\mu_4 + 2\mu_5$.

$J^\mu_{345} = 2\mu_{12345} - 2\mu_{1234} - 2\mu_{1235} - 2\mu_{1245} + \mu_{1345} + \mu_{2345} + 2\mu_{123} + 2\mu_{124} + 2\mu_{125} - \mu_{134} - \mu_{135} - \mu_{145} - \mu_{234} - \mu_{235} - \mu_{245} + 2\mu_{345} - 2\mu_{12} + \mu_{13} + \mu_{14} + \mu_{15} + \mu_{23} + \mu_{24} + \mu_{25} - 2\mu_{34} - 2\mu_{35} - 2\mu_{45} - \mu_1 - \mu_2 + 2\mu_3 + 2\mu_4 + 2\mu_5$.

$J^\mu_{45} = 6\mu_{12345} - 6\mu_{1234} - 6\mu_{1235} + 2\mu_{1245} + 2\mu_{1345} + 2\mu_{2345} + 6\mu_{123} - 2\mu_{124} - 2\mu_{125} - 2\mu_{134} - 2\mu_{135} + 2\mu_{145} - 2\mu_{234} - 2\mu_{235} + 2\mu_{245} + 2\mu_{345} + 2\mu_{12} + 2\mu_{13} - 2\mu_{14} - 2\mu_{15} + 2\mu_{23} - 2\mu_{24} - 2\mu_{25} - 2\mu_{34} - 2\mu_{35} + 6\mu_{45} + 2\mu_1 + 2\mu_2 + 2\mu_3 - 6\mu_4 - 6\mu_5$.

Therefore we have $J^\mu_{45} + 2(J^\mu_{145} + J^\mu_{245} + J^\mu_{345}) + 2(J^\mu_{1245} + J^\mu_{1345} + J^\mu_{2345}) = -24(\mu_{1235} - \mu_{123}) \leq 0$. A contradiction since J^μ_{45}, J^μ_{145}, J^μ_{245}, J^μ_{345}, J^μ_{1245}, J^μ_{1345}, $J^\mu_{2345} > 0$.

□

The following example illustrates the result of Proposition 2.

Example 2. *$N = \{1, 2, 3, 4\}$, $P = \{(a_{23}, a_1), (a_{234}, a_{123}), (a_2, a_{13})\}$ and $I = \{(a_{14}, a_{23})\}$. $(P \cup M \cup I)$ contains no strict cycle, and hence $\{P, I\}$ is representable by a Choquet integral model. A suitable topological sorting of $(P \cup M \cup I)$ is given by $\mathcal{B}_0 = \{a_0\}$, $\mathcal{B}_1 = \{a_1, a_3, a_4\}$, $\mathcal{B}_2 = \{a_{13}, a_{34}\}$, $\mathcal{B}_3 = \{a_2\}$, $\mathcal{B}_4 = \{a_{12}, a_{14}, a_{23}, a_{24}\}$, $\mathcal{B}_5 = \{a_{123}, a_{124}, a_{134}\}$, $\mathcal{B}_6 = \{a_{234}\}$ and $\mathcal{B}_7 = \{a_N\}$. Table 5 gives a capacity $\mu \in C_{Pref}$ and the corresponding interaction indices I^μ_S.*

S	$8^7 \times \mu(S)$	I^μ_S
$\emptyset$	0	–
$\{1\}$, $\{3\}$, $\{4\}$	8	–
$\{2\}$	8^3	–
$\{1, 2\}$	8^4	0.29
$\{1, 3\}$	8^2	0.29
$\{1, 4\}$	8^4	0.29
$\{2, 3\}$, $\{2, 4\}$	8^4	0.34
$\{3, 4\}$	8^2	1.04
$\{1, 2, 3\}$, $\{1, 2, 4\}$	8^5	0.42
$\{1, 3, 4\}$	8^5	0.43
$\{2, 3, 4\}$	8^6	0.53
N	8^7	0.83

Table 5: A capacity $\mu \in C_{\text{Pref}}$ and the corresponding interaction indices I^μ_S

We can see that for all $S \subseteq_{\geq 2} N$, we have $I^\mu_S > 0$.

6. A LP model testing for necessary interaction

This section builds on [21]. We drop the hypothesis that we only ask an ordinal preferential information on $\mathcal{B}^g$. We show how to test the existence of some necessary positive and negative interactions. We assume that the DM provides a strict preference P and an indifference I relations on a set X (that is not necessarily $\mathcal{B}^g$). Let A be a subset of at least two criteria. Our approach consists in testing first, in two steps, the compatibility of $\{P, I\}$ with a Choquet integral model, and then, in the third step, the existence of necessary positive or negative interaction for A.

We start this section with a remark giving a simplified version of the monotonicity condition. We use it in our linear programs, to reduce the number of constraints.

Remark 4. *The monotonicity condition is equivalent to the following condition: for all* $S \subsetneq N$, $\forall i \in N \setminus S,\ \mu(S) \leq \mu(S \cup \{i\})$.

6.1. The process

Step 1. The following linear program LP_1 models each relation in $\{P, I\}$ by introducing two nonnegative slack variables α^+_{xy} and α^-_{xy} in the corresponding constraint (Equation $(1a)$ or $(1b)$). Equation $(1c)$ (resp. $(1d)$) ensures the normalization (resp. monotonicity) of capacity μ. The objective function Z_1 minimizes all the nonnegative variables introduced in $(1a)$ and $(1b)$.

Minimize $Z_1 = \sum_{(x,y)\in P\cup I} (\alpha^+_{xy} + \alpha^-_{xy})$ $\quad LP_1$

Subject to

$$C_\mu(u(x)) - C_\mu(u(y)) + \alpha^+_{xy} - \alpha^-_{xy} \geq \varepsilon \ \ \forall x, y \in X \text{ such that } x\ P\ y \quad (1a)$$

$$C_\mu(u(x)) - C_\mu(u(y)) + \alpha^+_{xy} - \alpha^-_{xy} = 0 \ \ \forall x, y \in X \text{ such that } x\ I\ y \quad (1b)$$

$$\alpha^+_{xy} \geq 0,\ \ \alpha^-_{xy} \geq 0 \ \ \forall x, y \in X \text{ such that } x(P \cup I)y$$

$$\mu(N) = 1 \quad (1c)$$

$$\mu(S \cup \{i\}) \geq \mu(S) \ \ \forall S \subsetneq N,\ \forall i \in N \setminus S. \quad (1d)$$

$$\varepsilon \geq 0.$$

$C_\mu(u(x))$ depends linearly on the decision variables $\mu(S)$. The linear program LP_1 is always feasible due to the introduction of the nonnegative variables α^+_{xy} and α^-_{xy}. There are two possible cases:

1. If the optimal solution of LP_1 is $Z_1^* = 0$, then we can conclude that, depending on the sign of the variable ε ($\varepsilon = 0$ or $\varepsilon > 0$), $\{P, I\}$ may be represented by a Choquet integral. The next step of the procedure, Step 2 hereafter, will confirm or not this possibility.

2. If the optimal solution of LP_1 is $Z_1^* > 0$, then there is no Choquet integral model compatible with $\{P, I\}$.

Step 2. Here, the linear program LP_2 ensures the existence of a Choquet integral model compatible with $\{P, I\}$, when the optimal solution of LP_1 is $Z_1^* = 0$. Compared to the previous linear program, in this formulation, we only removed the nonnegative variables α^+_{xy} and α^-_{xy} (or put them equal to zero) and change the objective function by maximizing the value of the variable ε, in order to satisfy the strict preference relation.

Maximize $Z_2 = \varepsilon$ $\qquad LP_2$

Subject to

$C_\mu(u(x)) - C_\mu(u(y)) \geq \varepsilon \;\; \forall x, y \in X$ such that $x\,P\,y$ $\qquad (2a)$

$C_\mu(u(x)) - C_\mu(u(y)) = 0 \;\; \forall x, y \in X$ such that $x\,I\,y$ $\qquad (2b)$

$\mu(N) = 1$ $\qquad (2c)$

$\mu(S \cup \{i\}) \geq \mu(S) \;\; \forall S \subsetneq N, \; \forall i \in N \setminus S$ $\qquad (2d)$

$\varepsilon \geq 0.$

Notice that LP_2 is solved only if $Z_1^* = 0$. Hence, the linear program LP_2 is always feasible and it does not have an unbounded solution (it is not restrictive to suppose that $C_\mu(u(x)) \in [0,1]$, $\forall x \in X$). Hence, we have one of the following two cases.

1. If the linear program LP_2 is feasible with optimal solution $Z_2^* = 0$, then there is no Choquet integral model compatible with $\{P, I\}$.
2. If the optimal solution of is LP_2 is $Z_2^* > 0$, then $\{P, I\}$ is representable by a Choquet integral model.

Step 3. At this step, we assume that $\{P, I\}$ is representable by a Choquet integral model, i.e., $Z_2^* > 0$. To test the necessary negative (resp. positive) interaction for the subset $A \subseteq_{\geq 2} N$, we add to the previous linear program LP_2 the constraint $(2e)$ and we obtain the following linear program denoted by LP^A_{NN} (resp. LP^A_{NP}).

Maximize $Z_3 = \varepsilon$ $\qquad LP^A_{NN}$ (resp. LP^A_{NP})

Subject to

$C_\mu(u(x)) - C_\mu(u(y)) \geq \varepsilon \;\; \forall x, y \in X$ such that $x\,P\,y$ $\qquad (2a)$

$C_\mu(u(x)) - C_\mu(u(y)) = 0 \;\; \forall x, y \in X$ such that $x\,I\,y$ $\qquad (2b)$

$\mu(N) = 1$ $\qquad (2c)$

$\mu(S \cup \{i\}) \geq \mu(S) \;\; \forall S \subsetneq N, \; \forall i \in N \setminus S$ $\qquad (2d)$

$I^\mu_A \geq 0$ (resp. $I^\mu_A \leq 0$) $\qquad (2e)$

$\varepsilon \geq 0.$

I^μ_A depends linearly on the decision variables $\mu(S)$. After solving the linear program, we have one of the following three possible conclusions:

1. If LP^A_{NN} (resp. LP^A_{NP}) is not feasible, then the interaction for A is necessarily negative (resp. positive). Indeed, as the program LP_2 is feasible with an optimal solution $Z_2^* > 0$, the contradiction about the representation of $\{P, I\}$ only comes from the introduction of the constraint $I^\mu_A \geq 0$ (resp. $I^\mu_A \leq 0$).

2. If LP_{NN}^A (resp. LP_{NP}^A) is feasible and the optimal solution $Z_3^* = 0$, then the constraint $C_\mu(u(x)) - C_\mu(u(y)) \geq \varepsilon \quad \forall x, y \in X$ such that $x\,P\,y$ is satisfied with $\varepsilon = 0$, i.e., it is not possible to model strict preference by adding the constraint $I_A^\mu \geq 0$ (resp. $I_A^\mu \leq 0$) in LP_{NN}^A (resp. LP_{NP}^A). Therefore, we can conclude that the interaction for A is necessarily negative (resp. positive).

3. If LP_{NN}^A (resp. LP_{NP}^A) is feasible and the optimal solution $Z_3^* > 0$, then the interaction for A is not necessarily negative (resp. positive).

For each of the previous linear programs, we have $n(2^{n-1} - 1)$ constraints of monotonicity. Furthermore, the Table 6 give the list of the decision variables and Table 7 give an idea of number of variables and number of constraints of monotonicity.

	Decision variables
LP_1	ε, α_{xy}^+, α_{xy}^-, $\mu(S)$ $(\emptyset \subsetneq S \subsetneq N)$
LP_2	ε, $\mu(S)$ $(\emptyset \subsetneq S \subsetneq N)$
LP_{NN}^A	ε, $\mu(S)$ $(\emptyset \subsetneq S \subsetneq N)$
LP_{NP}^A	ε, $\mu(S)$ $(\emptyset \subsetneq S \subsetneq N)$

Table 6: Decision variables

	Number of variables $\mu(S)$	Number of constraints of monotonicity
$n = 3$	6	9
$n = 4$	14	28
$n = 5$	30	75
$n = 6$	62	186
$n = 7$	126	441
$n = 8$	254	1 016
$n = 9$	510	2 295
$n = 10$	1 022	5 110
$n = 11$	2 046	11 253
$n = 12$	4 094	24 564

Table 7: Number of variables $\mu(S)$ and number of constraints of monotonicity with $3 \leq n \leq 12$

In practice, the number of criteria generally does not exceed 12. Thus, with a common solver, we are able to solve these linear programs.

6.2. Example

In this section, we illustrate our decision procedure with an example, inspired from [1]. Let us consider a recruitment problem, where the executive manager of a company looks for engaging a new young employee. The manager takes into account the following four criteria.

1. Educational degree (abbreviated: Ed);
2. Professional experience (abbreviated: Ex);
3. Age (abbreviated: Ag);
4. Job interview (abbreviated: In).

In this example, $X = \{$Arthur, Bernard, Charles, Daniel, Esther, Felix, Germaine, Henry, Irene$\}$ and $N = \{1,\ 2,\ 3\,,4\}$.
The candidates, evaluated by the executive manager and their scores for every criterion on a $[0,\ 10]$ scale are presented in Table 8. We suppose that the criteria have to be maximized.
Now, suppose that the executive manager (the DM) on the basis of her preference structure is able to give only the following partial information on the reference actions $X' = \{$Arthur, Bernard, Charles, Germaine, Irene$\}$.

(a) The candidate Charles (C) is preferred to candidate Bernard (B),

(b) The candidate Germaine (G) is preferred to candidate Arthur (A),

(c) The candidates Charles (C) and Irene(I) are indifferent.

	A	B	C	D	E	F	G	H	I
Ed	8	3	10	5	8	5	8	5	0
Ex	6	1	9	9	0	9	10	7	10
Ag	7	10	0	2	8	4	5	9	2
In	5	10	5	9	6	7	7	4	8

Table 8: Evaluation matrix

Step 1. Linear program LP_1 with nonnegative slack variables α^+_{CB}, α^-_{CB}, α^+_{GA}, α^-_{GA}, α^+_{CI} and α^-_{CI}.
Minimize $Z_1 = \alpha^+_{CB} + \alpha^-_{CB} + \alpha^+_{GA} + \alpha^-_{GA} + \alpha^+_{CI} + \alpha^-_{CI}$
Subject to
$C_\mu(C) - C_\mu(B) + \alpha^+_{CB} - \alpha^-_{CB} \geq \varepsilon$
$C_\mu(G) - C_\mu(A) + \alpha^+_{GA} - \alpha^-_{GA} \geq \varepsilon$
$C_\mu(C) - C_\mu(I) + \alpha^+_{CI} - \alpha^-_{CI} = 0$

$C_\mu(A) = 5 + \mu_{123} + \mu_{13} + \mu_1$
$C_\mu(B) = 1 + 2\mu_{134} + 7\mu_{34}$
$C_\mu(C) = 5\mu_{124} + 4\mu_{12} + \mu_1$
$C_\mu(D) = 2 + 3\mu_{124} + 4\mu_{24}$
$C_\mu(E) = 6\mu_{134} + 2\mu_{13}$
$C_\mu(F) = 4 + \mu_{124} + 2\mu_{24} + 2\mu_2$
$C_\mu(G) = 5 + 2\mu_{124} + \mu_{12} + 2\mu_2$
$C_\mu(H) = 4 + \mu_{123} + 2\mu_{23} + 2\mu_3$
$C_\mu(I) = 2\mu_{234} + 6\mu_{24} + 2\mu_2$

$\mu_{12} \geq \mu_1$; $\mu_{12} \geq \mu_2$
$\mu_{13} \geq \mu_1$; $\mu_{13} \geq \mu_3$
$\mu_{14} \geq \mu_1$; $\mu_{14} \geq \mu_4$
$\mu_{23} \geq \mu_2$; $\mu_{23} \geq \mu_3$
$\mu_{24} \geq \mu_2$; $\mu_{24} \geq \mu_4$
$\mu_{34} \geq \mu_3$; $\mu_{34} \geq \mu_4$
$\mu_{123} \geq \mu_{12}$; $\mu_{123} \geq \mu_{13}$; $\mu_{123} \geq \mu_{23}$
$\mu_{124} \geq \mu_{12}$; $\mu_{124} \geq \mu_{14}$; $\mu_{124} \geq \mu_{24}$
$\mu_{134} \geq \mu_{13}$; $\mu_{134} \geq \mu_{14}$; $\mu_{134} \geq \mu_{34}$
$\mu_{234} \geq \mu_{23}$; $\mu_{234} \geq \mu_{24}$; $\mu_{234} \geq \mu_{34}$
$\mu_{1234} \geq \mu_{123}$; $\mu_{1234} \geq \mu_{124}$; $\mu_{1234} \geq \mu_{134}$; $\mu_{1234} \geq \mu_{234}$
$\mu_{1234} = 1$
$\varepsilon \geq 0$
α^+_{CB}, α^-_{CB}, α^+_{GA}, α^-_{GA}, α^+_{CI} and $\alpha^-_{CI} \geq 0$.
The linear program LP_1 is feasible and optimal solution of LP_1 is $Z_1^* = 0$, then we can conclude that, depending on the sign of the variable ε, $\{P, I\}$ may be represented by a Choquet integral. The next step of the procedure, Step 2 hereafter, will confirm or not this possibility.

Step 2. The linear program corresponding to the test of the existence of a capacity μ compatible with $\{P, I\}$ is the following:
Maximize $Z_2 = \varepsilon$
Subject to
$C_\mu(C) - C_\mu(B) \geq \varepsilon$
$C_\mu(G) - C_\mu(A) \geq \varepsilon$
$C_\mu(C) - C_\mu(I) = 0$

$C_\mu(A) = 5 + \mu_{123} + \mu_{13} + \mu_1$
$C_\mu(B) = 1 + 2\mu_{134} + 7\mu_{34}$
$C_\mu(C) = 5\mu_{124} + 4\mu_{12} + \mu_1$
$C_\mu(D) = 2 + 3\mu_{124} + 4\mu_{24}$
$C_\mu(E) = 6\mu_{134} + 2\mu_{13}$
$C_\mu(F) = 4 + \mu_{124} + 2\mu_{24} + 2\mu_2$
$C_\mu(G) = 5 + 2\mu_{124} + \mu_{12} + 2\mu_2$
$C_\mu(H) = 4 + \mu_{123} + 2\mu_{23} + 2\mu_3$
$C_\mu(I) = 2\mu_{234} + 6\mu_{24} + 2\mu_2$

$\mu_{12} \geq \mu_1$; $\mu_{12} \geq \mu_2$
$\mu_{13} \geq \mu_1$; $\mu_{13} \geq \mu_3$
$\mu_{14} \geq \mu_1$; $\mu_{14} \geq \mu_4$
$\mu_{23} \geq \mu_2$; $\mu_{23} \geq \mu_3$
$\mu_{24} \geq \mu_2$; $\mu_{24} \geq \mu_4$
$\mu_{34} \geq \mu_3$; $\mu_{34} \geq \mu_4$
$\mu_{123} \geq \mu_{12}$; $\mu_{123} \geq \mu_{13}$; $\mu_{123} \geq \mu_{23}$
$\mu_{124} \geq \mu_{12}$; $\mu_{124} \geq \mu_{14}$; $\mu_{124} \geq \mu_{24}$
$\mu_{134} \geq \mu_{13}$; $\mu_{134} \geq \mu_{14}$; $\mu_{134} \geq \mu_{34}$
$\mu_{234} \geq \mu_{23}$; $\mu_{234} \geq \mu_{24}$; $\mu_{234} \geq \mu_{34}$
$\mu_{1234} \geq \mu_{123}$; $\mu_{1234} \geq \mu_{124}$; $\mu_{1234} \geq \mu_{134}$; $\mu_{1234} \geq \mu_{234}$
$\mu_{1234} = 1$
$\varepsilon \geq 0$.

The linear program LP_2 is feasible and optimal solution of LP_2 is $Z_2^* = 3.8 > 0$, then we can conclude that $\{P, I\}$ is representable by a Choquet integral model. Moreover, the results obtained by solving LP_2 are given in Tables 9 and 10.

S	$\mu(S)$
$\emptyset$, $\{1\}$, $\{3\}$, $\{4\}$, $\{1,3\}$, $\{1,4\}$, $\{3,4\}$	0
$\{2\}$, $\{2,3\}$, $\{2,4\}$, $\{2,3,4\}$	0.9
$\{1,2\}$, $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, N	1

Table 9: A capacity $\mu \in C_{\text{Pref}}$

x	A	B	C	D	E	F	G	H	I
$C_\mu(x)$	6	3	9	8.6	6	8.6	9.8	6.8	9

Table 10: The corresponding Choquet integral at the capacity μ of Table 9

Step 3. In order to know if the interaction is necessarily negative for $\{1,2,3\}$, we obtain the LP_{NN}^{123} by adding at the previous linear program LP_2 the constraints $I_{123}^{\mu} \geq 0$ with $I_{123}^{\mu} = \mu_{1234} + \mu_{123} - \mu_{124} - \mu_{134} - \mu_{234} - \mu_{12} - \mu_{13} + \mu_{14} - \mu_{23} + \mu_{24} + \mu_{34} + \mu_1 + \mu_2 + \mu_3 - \mu_4$.

The linear program LP_{NN}^{123} is feasible and the optimal solution is $Z_3^* = 3.8 > 0$. Then interaction is not necessarily negative for {Educational degree, Professional experience, Age}. Moreover, the results obtained by solving LP_{NN}^{123} are given in Tables 11 and 12 (with $I_{123}^{\mu} = 1 > 0$).

S	$\mu(S)$
$\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1,2\}$, $\{1,3\}$, $\{1,4\}$,$\{2,3\}$, $\{3,4\}$, $\{1,3,4\}$	0
$\{2,4\}$, $\{2,3,4\}$	0.125
$\{1,2,3\}$, $\{1,2,4\}$	0.2
N	1

Table 11: A capacity $\mu \in C_{\text{Pref}}$ such that $I_{123}^{\mu} > 0$.

x	A	B	C	D	E	F	G	H	I
$C_\mu(x)$	5.2	1	1	3.1	0	4.45	5.4	4.2	1

Table 12: The corresponding Choquet integral at capacity μ of Table 11

7. Conclusion

When a capacity is elicited on the basis of preferences obtained from a decision maker, it is unlikely to be unique (this contrasts with the "continuous case" studied in Timonin [24, 25]). In particular, this non-uniqueness complicates the interpretation of the interaction index. Indeed, we give examples in which the sign of the interaction index depends upon the arbitrary choice of a capacity within the set of all capacities compatible with the preferences that were obtained. We generalize the concept of necessary and possible interaction introduced in [21] outside the case of 2-additive capacities. Necessary interactions are the only interactions that can safely be interpreted since their sign does not vary within the set of all compatible capacities. We have given conditions under which preferences on the general set of binary alternatives can be represented using a capacity in a Choquet integral model. We do the same adding the extra condition that the representing capacity has strictly positive interaction indices for all groups of criteria. These results seem to call for much care in the interpretation of the interaction index.

Our results leave some important questions open. The first one would be to derive results similar to ours but for the case of strictly negative interaction indices. Our proof technique does not seem easy to adapt to cover this case. Results of this type using the non-additivity index introduced in Wu and Beliakov [26], instead of the interaction index have been proposed in Kaldjob Kaldjob et al. [18]. The second would be to develop tools allowing to analyze "necessary interactions" for a large class of aggregation models, including the Choquet integral model. Finally, the study of aggregation models using bipolar scales [10] seems to be promising.

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Response to the comments of reviewer 2

We would first like to thank you for your new valuable comments about our paper. We have taken into account all your minor comments. In particular, we have modified our notation for subsets of N containing more than 2 elements, as you suggested (comment made on page 2). We have also changed the type of the paper to *"Full Length Article"* (FLA), because *"research article"* does not appear among the article types.

For the two major comments, we propose the following responses.

C.1 P.13: sufficiency part of the proof of Lemma 2: you write "therefore $a_{N\setminus i}\, M\, a_S$". But it could be that $a_{N\setminus i}\, P\, a_S$, so that $a_{N\setminus i}\, M\, a_S$ is not true (if I understand well the definition of M).

Thank you for your comment, you are right. The idea was to complete the graph with the monotonicity relation for the pairs (a_S, a_T), with $S \supseteq T$, on which the decision maker gave no preference. We have modified our proof of Lemma 2 (see page 13).

C.2 Prop 6: I appreciate the effort although the result is limited, but I think the proof is not correct: in $n = 3$, I cannot see how you can find $J_{23} + J_{123} = -2(\mu_{13} - \mu_1)$. Clearly the coefficient of μ_{123} is not zero. Same with $n = 4$: the coefficient of μ_{1234} cannot be zero in $2J_{34} + 3(J_{134} + J_{234}) + 2J_{1234}$. Also, there is a mistake in J_{34} : the coeff. of μ_1 is 1, not 2.

Thank you for your comment. You are right, in J^{μ}_{34}, the coefficient of μ_1 is 1, not 2. But, our results remain valid. Indeed, we have:

- for $n = 3$, the coefficient of μ_{123} is not zero in J_{23}, neither in J_{123}, since we have:

$$\begin{cases} J^{\mu}_{23} = \mu_{123} - \mu_{12} - \mu_{13} + \mu_{23} + \mu_1 - \mu_2 - \mu_3 \\ \text{and} \\ J^{\mu}_{123} = \mu_{123} - \mu_{12} - \mu_{13} - \mu_{23} + \mu_1 + \mu_2 + \mu_3 \end{cases}$$

But we assume that $\mu_{123} = \mu_{12}$, therefore we have:

$$\begin{cases} J^{\mu}_{23} = -\mu_{13} + \mu_{23} + \mu_1 - \mu_2 - \mu_3 \\ \text{and} \\ J^{\mu}_{123} = -\mu_{13} - \mu_{23} + \mu_1 + \mu_2 + \mu_3 \end{cases}$$

Adding members to members, we obtain $J^\mu_{23} + J^\mu_{123} = -2(\mu_{13} - \mu_1) \leq 0$.

- For $n = 4$, the coefficient of μ_{1234} is not zero in J_{34}, neither in J_{134}, neither in J_{234}, neither in J_{1234}, since we have:

$J^\mu_{34} = 2\mu_{1234} - 2\mu_{123} - 2\mu_{124} + \mu_{134} + \mu_{234} + 2\mu_{12} - \mu_{13} - \mu_{14} - \mu_{23} - \mu_{24} + 2\mu_{34} + \mu_1 + \mu_2 - 2\mu_3 - 2\mu_4$
$J^\mu_{134} = \mu_{1234} - \mu_{123} - \mu_{124} + \mu_{134} - \mu_{234} + \mu_{12} - \mu_{13} - \mu_{14} + \mu_{23} + \mu_{24} - \mu_{34} + \mu_1 - \mu_2 + \mu_3 + \mu_4$
$J^\mu_{234} = \mu_{1234} - \mu_{123} - \mu_{124} - \mu_{134} + \mu_{234} + \mu_{12} + \mu_{13} + \mu_{14} - \mu_{23} - \mu_{24} - \mu_{34} - \mu_1 + \mu_2 + \mu_3 + \mu_4$
$J^\mu_{1234} = \mu_{1234} - \mu_{123} - \mu_{124} - \mu_{134} - \mu_{234} + \mu_{12} + \mu_{13} + \mu_{14} + \mu_{23} + \mu_{24} + \mu_{34} - \mu_1 - \mu_2 - \mu_3 - \mu_4$.

But we assume that $\mu_{1234} = \mu_{123}$, therefore we have:

$J^\mu_{34} = -2\mu_{124} + \mu_{134} + \mu_{234} + 2\mu_{12} - \mu_{13} - \mu_{14} - \mu_{23} - \mu_{24} + 2\mu_{34} + \mu_1 + \mu_2 - 2\mu_3 - 2\mu_4$
$J^\mu_{134} = -\mu_{124} + \mu_{134} - \mu_{234} + \mu_{12} - \mu_{13} - \mu_{14} + \mu_{23} + \mu_{24} - \mu_{34} + \mu_1 - \mu_2 + \mu_3 + \mu_4$
$J^\mu_{234} = -\mu_{124} - \mu_{134} + \mu_{234} + \mu_{12} + \mu_{13} + \mu_{14} - \mu_{23} - \mu_{24} - \mu_{34} - \mu_1 + \mu_2 + \mu_3 + \mu_4$
$J^\mu_{1234} = -\mu_{124} - \mu_{134} - \mu_{234} + \mu_{12} + \mu_{13} + \mu_{14} + \mu_{23} + \mu_{24} + \mu_{34} - \mu_1 - \mu_2 - \mu_3 - \mu_4$.

This leads to $2J^\mu_{34} + 3(J^\mu_{134} + J^\mu_{234}) + 2J^\mu_{1234} = -12(\mu_{124} - \mu_{12}) \leq 0$.