Ranking methods for valued preference relations: A characterization of a method based on leaving and entering flows[1]

Denis Bouyssou and Patrice Perny

Abstract.
In this paper we study a particular method that builds a partial ranking on the basis of a valued preference relation. This method which is used in the MCDM method PROMETHEE I, is based on "leaving" and "entering" flows. We show that this method is characterized by a system of three independent axioms.

I- Introduction

Suppose that a number of decision alternatives are to be compared taking into account different points of view, *e.g.* several criteria or the opinion of several voters. As argued in Barrett *et al.* (1990) and Bouyssou (1990), a common practice in such situations is to associate with each ordered pair (a, b) of alternatives, a number indicating the strength or the credibility of the proposition "a is at least as good as b", *e.g.* the sum of the weights of the criteria favoring a or the percentage of voters declaring that a is preferred or indifferent to b. In this paper we study a particular method allowing to build a partial ranking, *i.e.* a reflexive and transitive binary (crisp) relation[2], on A given such information. Since a partial ranking is not necessarily complete, the method considered in this paper will allow two alternatives to be declared incomparable. Though this may seem strange, it must not be forgotten that the available information may be very poor or conflictual. Declaring that a and b are incomparable thus means that it seems difficult to take, at least at this stage of the study, a definite position on the comparison of a and b.

Let A be a finite set of objects called "alternatives" with at least three elements. We define a valued (binary) relation[3] on A as a function R associating with each ordered pair of alternatives $(a, b) \in A^2$ with $a \neq b$ an element of [0, 1]. A method $\succsim$ building a partial ranking, or, for short, a partial ranking method, is a function assigning a partial ranking $\succsim(R)$ on A to any valued relation R on A.

In this paper, we study a partial ranking method used in PROMETHEE I (see, *e.g.*, Brans *et*

1 We wish to thank Marc Pirlot and Philippe Vincke for their helpful comments on earlier drafts of this text.

2 A (crisp) binary relation S on A is reflexive if a S a, for all $a \in A$. It is transitive if for all $a, b, c \in A$, a S b and b S c imply a S c. It is complete if for all $a, b \in A$, a S b or b S a.

3 From a technical point of view, the condition $a \neq b$ could be omitted from this definition at the cost of a minor modification of our axioms. However, since it is clear that the values R(a, a) are immaterial in order to rank the alternatives, we will use this definition throughout the paper.

al. (1984) or Brans and Vincke (1985)) and defined by:

$$a \succsim_{L/E}(R)\ b \ \text{ iff } \ [L(a, R) \geq L(b, R) \text{ and } E(a, R) \leq E(b, R)] \qquad (1)$$

where:

$$L(a, R) = \sum_{c \in A\setminus\{a\}} R(a, c) \qquad \text{[Leaving flow]}$$

and

$$E(a, R) = \sum_{c \in A\setminus\{a\}} R(c, a) \qquad \text{[Entering flow]}$$

It is easily checked that the method defined by (1) is indeed a partial ranking method and that $\succsim_{L/E}(R)$ is not necessarily complete[4].

We will refer to the partial ranking method defined by (1) as the L/E Method. Besides its use in PROMETHEE I, the interest of the L/E method lies in its simplicity and intuitive appeal. The L/E method generalizes, through the use of entering and leaving flows, to the valued case the idea of declaring that a is preferred to b if a "beats" more alternatives than b and is "beaten" by less alternatives.

It should be emphasized that the L/E Method makes use of the "cardinal" properties of the valuations. In fact, it is obvious from (1) that we may well have:

$$\succsim_{L/E}(R) \neq \succsim_{L/E}(R_\phi)$$

where R_ϕ is defined by $R_\phi(a, b) = \phi(R(a, b))$ for all $a, b \in A$ and ϕ is a strictly increasing transformation on the real line such that $\phi(0) = 0$ and $\phi(1) = 1$. Thus this method does not seem to be appropriate when the comparisons of the valuations only have an ordinal meaning in term of credibility.

The purpose of this paper is to present an axiomatic characterization of the L/E method. The axioms and the characterization are presented in the next section. In a final section we present our proofs and show how the characterization of the L/E method can be extended to a much wider class of partial ranking methods.

II- The main result

Throughout the paper, we note $\sim(R)$ and $\succ(R)$ the symmetric and asymmetric parts of $\succsim(R)$, *i..e.* for all $a, b \in A$, [$a \sim(R)\ b$ iff $a \succsim(R)\ b$ and $b \succsim(R)\ a$] and [$a \succ(R)\ b$ iff $a \succsim(R)\ b$ and Not ($b \succsim(R)\ a$)].

We say that a partial ranking method $\succsim$ is non-discriminatory if for all valued relation R on A and all $a, b \in A$,

$[R(a, b) = R(b, a)$ and $R(a, c) = R(b, c)$, $R(c, a) = R(c, b)$ for all $c \in A\setminus\{a, b\}] \Rightarrow a \sim(R)\ b$.

Non-discrimination says that if two alternatives are compared similarly *vis-à-vis* any other alternatives then they should be considered indifferent. It seems rather an unobjectionable property in this context. It is obvious that the L/E Method is non-discriminatory.

4 This will only happen if R has some special properties, *e.g.* if $R(c, d) + R(d, c)$ is constant for all $c, d \in A$.

Non-discrimination has strong connections with the classical property of neutrality (see, *e.g.*, Henriet (1985) and Rubinstein (1980)). A partial ranking method $\succsim$ is said to be neutral if, for all permutation σ on A, for all valued relation R on A and all a, b $\in$ A:

$$a \succsim(R)\, b \Leftrightarrow \sigma(a) \succsim(R^{\sigma})\, \sigma(b)$$

where R^{σ} is defined by $R^{\sigma}(\sigma(a), \sigma(b)) = R(a, b)$ for all a, b $\in$ A.

Neutrality expresses the fact that a partial ranking method does not discriminate between alternatives just because of their labels. It is easily checked that neutrality implies non-discrimination for partial ranking methods always leading to a complete binary relation. When incomparability is tolerated, neutrality implies that for all valued relation R on A and all a, b $\in$ A,

[R(a, b) = R(b, a) and R(a, c) = R(b, c), R(c, a) = R(c, b) for all c $\in$ A\{a, b}] $\Rightarrow$

a $\sim$(R) b or (Not[a $\succsim$(R) b] and Not[b $\succsim$(R) a]).

Non-discrimination excludes the latter case.

A ranking method is said to be monotonic if it does not respond "in the wrong direction" to a modification of R. More formally, $\succsim$ is monotonic if, for all valued relation R on A and all a, b $\in$ A:

$$a \succsim(R)\, b \quad \Rightarrow \quad a \succsim(R')\, b$$

where R' is identical to R except that [R(a, c) < R'(a, c) or R(c, a) > R'(c, a) for some c $\in$ A\{a}] or [R(b, d) > R'(b, d) or R(d, b) < R'(d, b) for some d $\in$ A\{b}].

A partial ranking method is strongly monotonic if it responds "in the right direction" to a modification of R. More formally, $\succsim$ is strongly monotonic if, for all valued relation R on A and all a, b $\in$ A:

$$a \succsim(R)\, b \quad \Rightarrow \quad a \succ(R')\, b,$$

where R' is as before.

As defined here, monotonicity seems rather an unobjectionable property in the context of partial ranking methods. Strong monotonicity is much more demanding, excluding, in particular, the use of any threshold in the treatment of the valuations. However, it is obvious that the L/E Method is strongly monotonic and thus monotonic.

In order to introduce our final axiom let us recall some well-known definitions used in Graph Theory. A digraph consists in a set of nodes X and a set of arcs $U \subseteq X^2$. We say that a is the initial extremity and b is the final extremity of the arc u = (a, b) $\in$ U.

A cycle of length q (abbreviated as a q-cycle) in a digraph is an ordered collection of arcs (u_1, u_2, ..., u_q) such that for i = 1, 2, ..., q, $u_i \neq u_{i+1}$, one of the extremities of u_i is an extremity of u_{i-1} and the other an extremity of u_{i+1}, where u_0 is interpreted as u_q and u_{q+1} as u_1. A cycle is elementary if each node being the extremity of one arc in the cycle is the extremity of exactly two arcs in the cycle. An arc u_i in a cycle is forward if its common extremity with u_{i-1} is its initial extremity and backward otherwise. A cycle is said to be alternated if every forward arc in the cycle is followed by a backward arc and *vice versa*. Thus, the length of an alternated cycle is necessarily even.

Define A^+ and A^- as disjoint duplications of A. We note a^+ (resp. a^-) the element of A^+ (resp. A^-) corresponding to $a \in A$. Consider a digraph G which set of nodes is $X=A^+ \cup A^-$ and which set of arcs is $U = \{(x^+, y^-) \in X^2 : x^+ \in A^+, y^- \in A^- \text{ and } x \neq y\}$. It is obvious that there is a one-to-one correspondence between valued relations on A and valuations between 0 and 1 of the arcs of G. In the sequel, we identify a valued relation R with its associated valued digraph in which for all $a, b \in A$ the valuation $v_R(u)$ of the arc $u = (a^+, b^-)$ is R(a, b). It should be noticed that all cycles in G are alternated by construction.

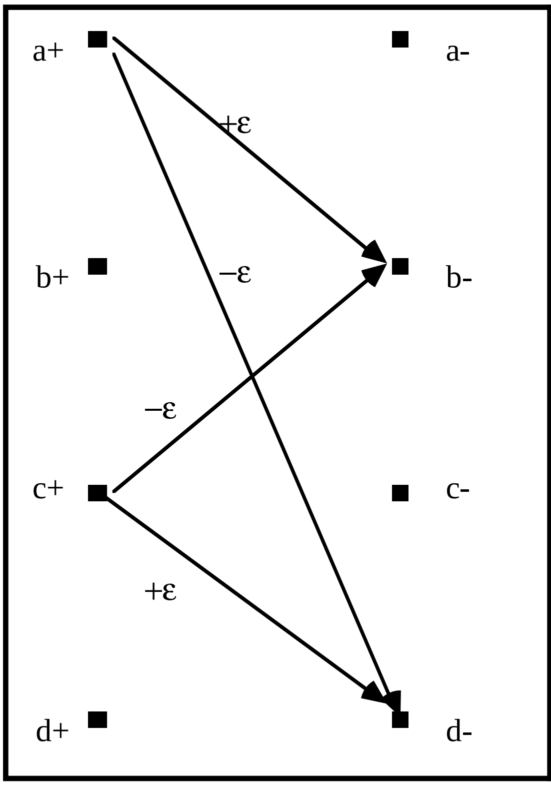
A transformation on a elementary cycle consists in adding a positive or negative quantity to the valuations of the forward arcs in the cycle and subtracting it from the valuations of the backward arcs. A transformation on an elementary cycle is admissible if all the transformed valuations are still between 0 and 1. When we apply an admissible transformation to the graph associated with a valued relation R, we obtain another valued relation R' and we say that R' has been obtained from R through an admissible transformation.

A partial ranking method is independent of alternated cycles if for all valued relations R and R':

[R' can be obtained from R through an admissible transformation on an elementary alternated 4-cycle or 6-cycle] $\Rightarrow$ $\geq(R) = \geq(R')$.

It is easy to see that if R' can be obtained from R through an admissible transformation on an elementary alternated cycle then L(a, R) = L(a, R') and E(a, R) = E(a, R') for all $a \in A$ so that the L/E Method is independent of alternated cycles (see Figure 1).

Figure 1 : An admissible transformation on an elementary alternated 4-cycle.



R	a	b	c	d
a		+ε		−ε
b				
c		−ε		+ε
d				

It is easy to see that the L/E method is not the only partial ranking method that is non-discriminatory, monotonic and independent of alternated cycles. This is also the case for the method which, for all a, b $\in$ A and all valued relation R on A, always declares that a $=$(R) b. This method, however, is not strongly monotonic. Unfortunately, the L/E method is not the only partial ranking method that is non-discriminatory, strongly monotonic and independent of alternated cycles. For instance, this is also the case for the following method based on net flows:

a $\geq$(R) b iff E(a, R) - L(a, R) $\geq$ E(b, R) - L(b, R),

which has been characterized by Bouyssou (1990).

Nevertheless, it turns out that partial ranking methods that are non-discriminatory, (strongly) monotonic and independent of alternated cycles have strong connections with the L/E Method and we have the following:

Theorem.

If a partial ranking method $\geq$ is non-discriminatory, monotonic and independent of alternated cycles then, for all valued relation R on A and all a, b $\in$ A, [a $\geq_{L/E}$(R) b $\Rightarrow$ a $\geq$(R) b]. Furthermore, if $\geq$ is strongly monotonic then, for all valued relation R on A and all a, b $\in$ A, [a $>_{L/E}$(R) b $\Rightarrow$ a $>$(R) b].

This theorem says that the L/E method is the smallest (in the sense of inclusion) partial ranking method that is non-discriminatory, monotonic and independent of alternated cycles. If $\geq$ is non-discriminatory, monotonic and independent of alternated cycles, it may happen that a $>_{L/E}$(R) b and a $=$(R) b. The second part of the theorem says that such a situation is impossible if $\geq$ is strongly monotonic. Thus the L/E method "imposes" its indifferences and strict preferences to every partial ranking method that is non-discriminatory, strongly monotonic and independent of alternated cycles. These partial ranking methods differ from the L/E method by comparing in terms of indifference or strict preference alternatives that were declared incomparable with the L/E Method.

We already noticed that the L/E Method is non-discriminatory, strongly monotonic and independent of alternated cycles. The proof of the theorem appears in the next section. Let us first observe that these three axioms are independent as shown by the following examples:

i- Let Φ : A $\rightarrow$ {1, 2, ..., |A|} be a one-to-one function.

Define $\geq$ as:

a $\geq$(R) b iff [L_1(a, R) $\geq$ L_1(b, R) and E(a, R) $\leq$ E(b, R)]

where L_1(c, R) = L(c, R).Φ(c), for all c $\in$ A.

This partial ranking method is strongly monotonic (and, thus, monotonic) and independent of alternated cycles but not non-discriminatory.

ii- Define $\geq$ as:

a $\geq$(R) b iff E(a, R) $\geq$ E(b, R) and L(a, R) $\leq$ L(b, R).

This partial ranking method is non-discriminatory and independent of alternated cycles but not monotonic (and, thus, not strongly monotonic).

iii- Define $\succeq$ as:

a $\succeq$(R) b iff $[L_3(a, R) \geq L_3(b, R)$ and $E_3(a, R) \leq E_3(b, R)]$

where $L_3(c, R) = \sum_{d \in A\setminus\{c\}} R(c, d)^2$ and $E_3(c, R) = \sum_{d \in A\setminus\{c\}} R(d, c)^2$, for all $c \in A$.

This partial ranking method is non-discriminatory and strongly monotonic but not independent of alternated cycles.

III- Proofs and remarks

Lemma 1. For all valued relations R and R', if [R' can be obtained from R through an admissible transformation on an elementary alternated cycle] then [R' can be obtained from R through a finite number of admissible transformations on elementary alternated 4-cycles and/or 6-cycles].

Proof of Lemma 1.

The proof is by induction on k where 2k is the length of an elementary alternated cycle in G. If k = 2 or 3, then the lemma is proved. Suppose now that the lemma is true for $k \geq 3$ and let us show that it is true for k+1.

Consider an elementary alternated cycle C of length 2(k+1) in G, *i.e.*, an ordered collection of ordered pairs of alternatives $((x_i^+, y_i^-) ; (x_{i+1}^+, y_i^-) : i = 1, 2, ..., k+1)$ with for all $i, j \in \{1, 2, ..., k+1\}$:

$x_i \neq y_i$, $x_{i+1} \neq y_i$ (because arcs of the type (a^+, a^-) are not in G) (2)

and

$x_i \neq x_j$ and $y_i \neq y_j$ (because the cycle is elementary) (3)

where x_{k+2} is interpreted as x_1.

Let us show that any admissible transformation on C can be obtained through a finite number of admissible transformations on elementary alternated cycles of length greater than 4 and smaller than 2k. In order to show this, we claim that for some $j \in \{1, 2, ..., k+1\}$,

$C_j = ((x_1^+, y_1^-), (x_2^+, y_1^-), (x_2^+, y_2^-), (x_3^+, y_2^-), ..., (x_j^+, y_j^-), (x_1^+, y_j^-))$

and

$C'_j = ((x_1^+, y_j^-), (x_{j+1}^+, y_j^-), (x_{j+1}^+, y_{j+1}^-), (x_{j+2}^+, y_{j+1}^-), ..., (x_{k+1}^+, y_{k+1}^-), (x_1^+, y_{k+1}^-))$

both correspond to elementary alternated cycles in G.

Condition (2) implies that we must look for candidates in {2, 3, ..., k}. From (3), we know that {2, 3, ..., k} contains at most one element t such that $x_1 = y_t$. Let J be the set obtained by removing t, if such a t exists, from {2, 3, ..., k}. We have $|J| \geq (k-1)-1 = k-2$. Since $k \geq 3$, J is not empty and the claim is proved.

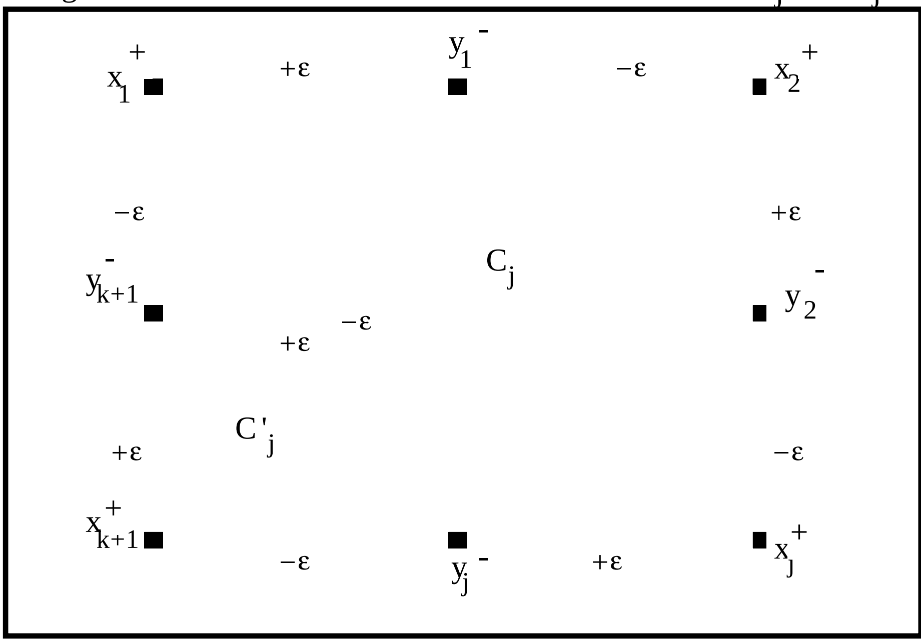
By construction, C_j and C'_j are both of length greater than 4 and smaller than 2k (see Figure 2). These two elementary alternated cycles have only the arc (x_1^+, y_j^-) in common.

This arc is backward in C_j and forward in C'_j.

Suppose now that R' has been obtained from R through an admissible transformation of ε on C. If ε = 0, there is nothing to prove. Suppose now that ε > 0 (the other case being symmetric). If $R(x_1, y_j) > 0$ then we can find a sufficiently large integer n such that performing a transformation of ε/n on C_j is an admissible transformation. After this first transformation, performing a transformation of ε/n is an admissible transformation on C'_j. It is easily seen that, after having repeated n times these transformations, we obtain R'.

If $R(x_1, y_j) = 0$, then performing a transformation of ε on C'_j is an admissible transformation. After this first transformation, performing a transformation of ε on C_j is an admissible transformation. We obtain R' after these two transformations. This completes the proof of lemma 1. □

Figure 2 : A transformation on C via transformations on C_j and C'_j.



The following lemma establishes a crucial link between admissible transformations on elementary alternated cycles and Leaving and Entering flows.

Lemma 2. For all valued relations R and R',

[L(a, R) = L(a, R') and E(a, R) = E(a, R') for all a ∈ A] ⇔

[R' can be obtained from R through a finite number of admissible transformations on elementary alternated cycles].

Proof of lemma 2.

The ⇐ part is obvious. In order to prove the ⇒ part, suppose that for some R and R' and for all c ∈ A we have L(c, R) = L(c, R') and E(c, R) = E(c, R'). If R = R' the lemma is proved. If R ≠ R' then R(a, b) ≠ R'(a, b) for some a, b ∈ A with a ≠ b and we suppose for definiteness that $R(a, b) > R'(a, b)$ (the other case being symmetric). We claim that $R(a, d) < R'(a, d)$ for some $d \in A \setminus \{a\}$, for otherwise $R(a, d) \geq R'(a, d)$ for all $d \in A \setminus \{a, b\}$ and $R(a, b) > R'(a, b)$ would contradict $L(a, R) = L(a, R')$. Using a similar argument, there is a $c \in A \setminus \{d\}$ such that $R(c, d) > R'(c, d)$. This process leads to the construction of an ordered collection of arcs in G $[(a^+, b^-), (a^+, d^-), (c^+, d^-)]$. Repeating the same process will lead to the creation of an elementary cycle in G since the number of alternatives is finite. Let Δ be the minimum over the arcs (s^+, t^-) in the cycle of $|R(s, t) – R'(s, t)|$. It is easily checked that adding Δ to the arcs in the cycle such that $R(x, y) < R'(x, y)$ and subtracting it from the arcs in the cycle such that $R(x, y) > R'(x, y)$ is an admissible transformation on the cycle. After performing this transformation, we thus obtain a valued relation R_1. If $R_1 = R'$ the lemma is proved. If not, we can repeat the same argument starting with R_1 instead of R.

Because A is finite, there is only a finite number of arcs such that $R(x, y) \neq R'(x, y)$. Since, at each step the number of arcs on which the current relation and R' are different is decreased by at least one unit, this process will terminate after a finite number of steps, which completes the proof of lemma 2. □

Proof of the Theorem.

In order to prove the first part of the theorem, we have to show that $\geq$ is non-discriminatory, monotonic and independent of alternated cycles then:

$L(a, R) \geq L(b, R)$ and $E(a, R) \leq E(b, R) \Rightarrow a \geq(R)\ b$.

Let us first show that if $\geq$ is non-discriminatory, monotonic and independent of alternated cycles then:

$L(a, R) = L(b, R)$ and $E(a, R) = E(b, R) \Rightarrow a = (R)\ b$. (4)

In order to prove (4) consider a valued binary relation R on A such that $L(a, R) = L(b, R)$ and $E(a, R) = E(b, R)$ for some $a, b \in A$. Define $\underline{R}$ by:

$\underline{R}(a, b) = \underline{R}(b, a) = (R(a, b) + R(b, a))/2$,

$\underline{R}(a, c) = \underline{R}(b, c) = (R(a, c) + R(b, c))/2$ for all $c \in A \setminus \{a, b\}$,

$\underline{R}(c, a) = \underline{R}(c, b) = (R(c, a) + R(c, b))/2$ for all $c \in A \setminus \{a, b\}$,

$\underline{R}(c, d) = R(c, d)$ for all $c, d \in A \setminus \{a, b\}$.

It is easily checked that $\underline{R}$ is a valued relation on A.

We have $\underline{R}(a, b) = \underline{R}(b, a)$, $\underline{R}(a, c) = \underline{R}(b, c)$ and $\underline{R}(c, a) = \underline{R}(c, b)$ for all $c \in A \setminus \{a, b\}$. Thus non-discrimination implies $a = (\underline{R})\ b$.

We also have $L(c, \underline{R}) = L(c, R)$ and $E(c, \underline{R}) = E(c, R)$ for all $c \in A$. Given lemma 2, we know that $\underline{R}$ can be obtained from R through a finite number of transformations on elementary alternated cycles. Given lemma 1, independence of alternated cycles implies $\geq(R) = \geq(\underline{R})$. Thus $a = (R)\ b$ which establishes (4).

Let us now show that if $\geq$ is non-discriminatory, monotonic and independent of alternated cycles then:

$L(a, R) \geq L(b, R)$ and $E(a, R) \leq E(b, R)$, at least one of these inequalities being strict,

$\Rightarrow a \succeq(R)\, b$ (5)

which will complete the proof of the first part of the theorem.

In order to prove (5) suppose that $L(a, R) \geq L(b, R)$ and $E(a, R) \leq E(b, R)$, at least one of these inequalities being strict. We note $\delta = L(a, R) - L(b, R)$ and $\zeta = E(b, R) - E(a, R)$.

We define the following sets of alternatives:

$A_1 = \{c \in A \setminus \{a, b\} : R(a, c) > 0\}$, $A_2 = \{d \in A \setminus \{a, b\} : R(b, d) < 1\}$,

$A_3 = \{e \in A \setminus \{a, b\} : R(e, a) < 1\}$, $A_4 = \{f \in A \setminus \{a, b\} : R(f, b) > 0\}$,

and denote by B_i the complement of A_i in $A\setminus\{a, b\}$. If:

$$\delta \leq \sum_{c \in A_1} R(a, c) + \sum_{d \in A_2} (1 - R(b, d)) \quad (6)$$

and

$$\zeta \leq \sum_{f \in A_4} R(f, b) + \sum_{e \in A_3} (1 - R(e, a)) \quad (7)$$

it is easy to see that it is possible to obtain a valued relation R' identical to R except on the ordered pairs of alternatives (a, c) with $c \in A_1$, (e, a) with $e \in A_3$, (b, d) with $d \in A_2$ and (f, b) with $f \in A_4$, such that $L(a, R') = L(b, R')$ and $E(a, R') = E(b, R')$. Thus (4) implies $a = (R')\, b$ and repeated applications of monotonicity lead to $a \succeq(R)\, b$.

Let us show that (6) holds, the proof being similar for (7). We have:

$$L(a, R) = \sum_{c \in A_1} R(a, c) + R(a, b).$$

and

$$L(b, R) = \sum_{d \in A_2} R(b, d) + R(b, a) + |B_2|.$$

thus,

$$\delta = \sum_{c \in A_1} R(a, c) + R(a, b) - \sum_{d \in A_2} R(b, d) - R(b, a) - |B_2|$$

and we have to show that:

$$\sum_{c \in A_1} R(a, c) + R(a, b) - \sum_{d \in A_2} R(b, d) - R(b, a) - |B_2| \leq \sum_{c \in A_1} R(a, c) + \sum_{d \in A_2} (1 - R(b, d)),$$

i.e., $|A_2| + |B_2| \geq R(a, b) - R(b, a)$.

Noticing that $|A_i| + |B_i| = |A| - 2$, it is easy to see that (6) holds as soon as $|A| \geq 3$, which completes the proof of the first part of the theorem.

In order to prove the second part of the theorem, we have to show that if $\succeq$ is non-discriminatory, strongly monotonic and independent of alternated cycles then:

$L(a, R) \geq L(b, R)$ and $E(a, R) \leq E(b, R)$, at least one of these inequalities being strict

$\Rightarrow a \succ(R)\, b$ (8)

Since strong monotonicity implies monotonicity, we know that (4) holds. Then, using strong monotonicity instead of monotonicity in the proof of (5) shows that (8) holds which completes the proof of the theorem. □

We conclude this paper by pointing out a straightforward extension of our results. Let:

$$L_\phi(a, R) = \sum_{c \in A\setminus\{a\}} \phi(R(a, c)) \text{ and } E_\phi(a, R) = \sum_{c \in A\setminus\{a\}} \phi(R(c, a))$$

where ϕ is a strictly increasing transformation on the real line such that $\phi(0) = 0$ and $\phi(1) = 1$. It is not difficult to see that a similar method of proof can be used to characterize the partial ranking method defined by:

$a \succeq(R)\, b$ iff $[L_\phi(a, R) \geq L_\phi(b, R)$ and $E_\phi(a, R) \leq E_\phi(b, R)]$,

by keeping non-discrimination and strong monotonicity unchanged and replacing our third axiom by:

[R'_ϕ can be obtained from R_ϕ through an admissible transformation on an elementary alternated 4-cycle or 6-cycle] $\Rightarrow \succeq(R) = \succeq(R')$

where R_ϕ and R'_ϕ are defined by $R_\phi(a, b) = \phi(R(a, b))$ and $R'_\phi(a, b) = \phi(R'(a, b))$ for all $a, b \in A$.

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