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Structures and Duality in Combinatorial Programming

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Abstract

We study minimal forbidden structures which are sophisticated homologous of odd-cycles in signed graphs. We start with the structures associated with bicliques, our approach turns out yet to be more general. It links several complicated graph problems.

Our approach leads to a chromatic version of two well-known Gallai identities. Using these identities we can move Lovász SDP relaxation toward the chromatic number and the LP relaxation toward the clique number. This works for a special coloring problem as well, namely Max-Coloring. We show a link between the stable set polytope and that of clique-connecting forests, which are at the basis of the chromatic Gallai identities.

For another special coloring problem, namely the partition coloring problem, it is challenging to have a nice characterization of conformality, that is, recognizing efficiently clique-vertex matrices. After proving that, for all 0-1 matrices coming from clustered graphs, a local conformality suffices to imply the total conformality, we characterize the graphs with a strong perfectness property defined not only for graphs but for clustered graphs.

Concerning the edge-coloring problem we give a best-possible min-max relation describing the edge-star polytope. Another min-max relation, involving multiflows, is given for characterizing series-parallel graphs, and it improves a previous one.

Finally, we investigate minimal forbidden structures coming from another field than graph theory, namely election problems in social choice. We list some questions that we have left open.

Contents

Chapter 1

Introduction

An hypergraph $H = (E, \mathcal{C})$ is a *clutter* if no hyperedge contains another one. Take for instance, E the edge-set of a graph and $\mathcal{C}$ its odd-circuits (considered as edge sets in this report).

Many combinatorial problems can be formulated as finding

$$\begin{array}{rll} \alpha_w(H) := & \max\{w^\top x : Ax \leq \mathbf{1},\, x \geq \mathbf{0},\, x \text{ integer}\} & \text{(packing)} \\ \text{or} \quad \tau_w(H) := & \min\{w^\top x : Ax \geq \mathbf{1},\, x \geq \mathbf{0},\, x \text{ integer}\} & \text{(covering)} \end{array}$$

where A is the incidence matrix[1] of a clutter H. Also, algorithms designed to solve combinatorial problems can include subroutines, such as cutting-plane or column generation, solving a subproblem with such a formulation.

Today, the main questions[2] are solved concerning $\alpha_w(H)$ but only partially solved concerning $\tau_w(H)$ and even then only if H is binary, see [10, 24, 25, 37], and, for a very recent survey, see [4]. A lot of results of optimization in graphs, including on max-cut see [3, 37], or on graph coloring, are unified under this so-called packing/covering framework.

Some other complicated graphs problem, as finding a maximum vertex-induced (or complete) bipartite subgraph, seem to escape this framework. So does any problem which consists in maximizing the linear weight $w(B) := \sum_{e \in B} w_e$ over particular subsets B of edges which are not defined from an hereditary property. In fact, several of these problems can be linked together by minimal forbidden structures, which are sophisticated homologous of odd-circuits in signed-graphs. All of them are NP-hard. Some are also theoretically difficult in the sense that they consist, essentially, in finding $\tau_w(H)$ for some non-binary H where, except for trivial classes, the continuous relaxation for $\tau_w(H)$ is fractional.

The graph coloring problem has a natural covering formulation but it is not clear which polynomial size formulation is the most natural. When we apply the approach for nonhereditary problems to the particular edge clique-partition problem, it lead us to a way of removing edges of the line-graph $L(G)$ of G so that the stable sets of the remaining subgraph of $L(G)$ are in 1-to-1

[1] The *incidence matrix of* $H = (E, \mathcal{C})$ is the matrix A with row set $\mathcal{C}$, column set E, and entries

$$A_{C,e} = \begin{cases} 1 & \text{if } e \in C \\ 0 & \text{if } e \notin C \end{cases} \qquad \text{(each row is the \textit{characteristic vector} } \chi^C \text{ of } C \subseteq E)$$

[2] The "main questions", for a linear system $(P) := \{Ax \leq b\}$ in general, are to characterizing when

- the polyhedron $P := \{x : Ax \leq b\}$ *is integer*, that is,

$$\max\{w^\top x : x \in P\} = \max\{w^\top x : x \in P,\, x \text{ integer}\} \quad (\forall w)$$

- the system (P) *is totally dual integral (for short, TDI)*, that is,

$$\min\{y^\top b : y^\top A \geq w^\top\} = \min\{y^\top b : y^\top A \geq w^\top,\, y \text{ integer}\} \quad (\forall w \text{ integer, when feasible})$$

(If b is integer and (P) is TDI, then P is integer).

correspondence with the colorings of G. This gives a natural packing formulation for the coloring problem which extends to the special Max-coloring problem as well. In fact, we obtain a chromatic version of two well-known Gallai identities. Using these identities we can move the polynomial SDP relaxation by Lovász (see [32]) toward the chromatic number, and the LP relaxation toward the clique number.

For the continuous relaxation of $\alpha_w(H)$, integrality and TDIness coincide, and it happens only when A is conformal[3]. This can be checked in a time polynomial in the size of A but for the special partition-coloring problem, the matrix A is encoded by a clustered graph. We will study the conformality of such matrices.

Most of the time, the polyhedron defined by the continuous relaxation of $\alpha_w(H)$ or $\tau_w(H)$ is not integer but, when the problem is polynomial, one can sometimes cut-off all fractional extreme points by adding specific valid inequalities. The seminal success for this so-called polyhedral approach was the work of Jack Edmonds for $\alpha_w(H)$ when H is a line-graph, or in other words, for the matching problem, and it was extended to H a quasi line-graph in [26]; which tends to show that easy problems have a nice linear description. As the matchings, the stars are very natural objects to look at. Surprisingly, a linear description of the so-called substar polytope is given only for bipartite graphs in the giant book by Schrijver [37] (see the very last subject at page 1871 in the index).

Note that, even if the relaxation is integer, the linear system might not be TDI. Moreover, a TDI system leads to a so-called min-max theorem that might not be best-possible if the system is not minimally-TDI[4]. One can make the linear system become minimally-TDI by adding well-chosen redundant inequalities like Sebő [38] for $\tau_w(H)$ when H is the T-cut clutter of a graft (G, T).

We will give a TDI system leading to a best-possible min-max theorem describing the (sub) star polytope. Another TDI system will be given which describes the multicut polytope and allows to improve a previous characterization of series-parallel graphs.

Our work raised many questions including that of exporting the approach by forbidden structures to other fields than graph theory, such as election problems in (computational) social choice theory (see [7] for an introduction to this field).

[3] A matrix is called conformal if it is the incidence matrix of the cliques of some graph, see [24] (Theorem 3.9, p. 39) or [37] (Theorem 82.2, p. 1431) for a characterization.

[4] That is, any proper subsystem which describes the same set is not TDI, see Chapter 22.3, p. 315 in [36].

Chapter 2

Minimal forbidden structures in linear programs

For a survey on the minimum biclique cover problem see [34], for recent results and applications (e.g. biology) see [1]. We reveal the structure associated to this problem which allows to formulate it naturally. A strong link with the clique number makes the structure interesting for itself, moreover, it is algorithmically under control. This will give in fact a general method.

2.1 The minimum biclique cover problem

Let $G = (V, E)$ be a simple undirected graph and let us denote by $K_{p,q}$ the complete bipartite graph with p and q vertices on each side. Figure 2.1 shows (a) $K_{2,2}$ with two isolated vertices (ignore the bold edges for now), (b) $K_{2,3}$ with one isolated vertex, and (c) $K_{3,3}$.

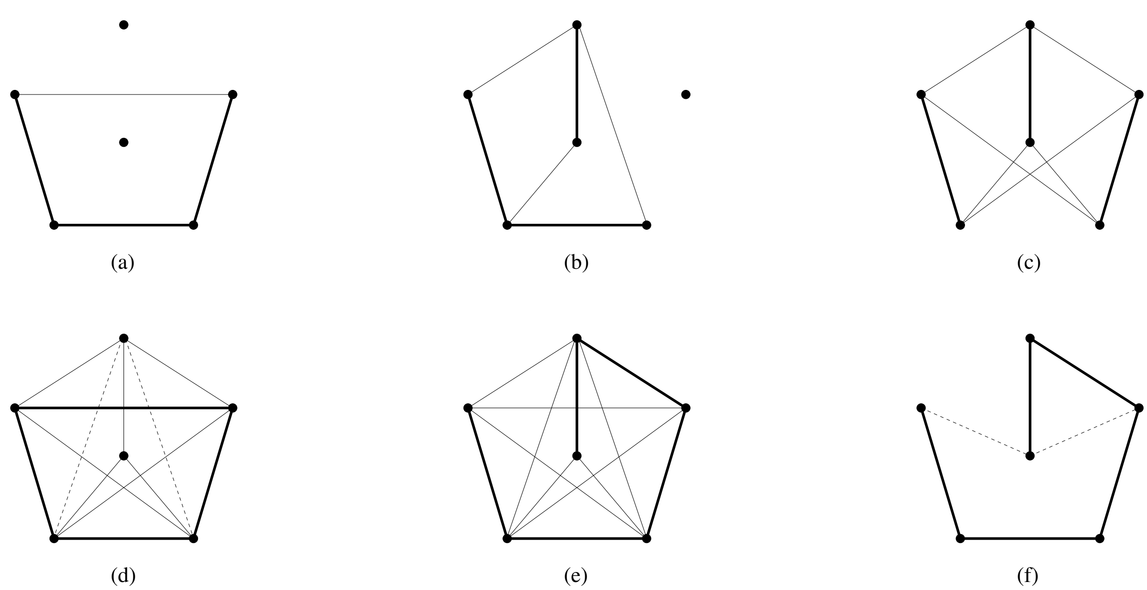


Figure 2.1: Biclique and non-biclique sets

The *minimum biclique cover problem (BIC, for short)* is to find a minimum collection of complete bipartite partial subgraphs of G which covers the edge-set of G (that is, each edge must be in at least one subgraph). The graph of Figure 2.1(d) for instance can be covered using a $K_{1,2}$ (dot lines) and a $K_{2,2}$ (bold lines) disjoint, and covering the seven remaining edges with a $K_{3,3}$ (two edges of which are already covered).

Definition 1 *Given a graph $G = (V, E)$, a subset $B \subseteq E$ is a* biclique set *of G if there is a complete bipartite partial subgraph of G with edge set $E' \subseteq E$ such that $B \subseteq E'$.*

For instance in Figure 2.1(a)-(c) the set of edges in bold is a biclique set included in a $K_{2,2}$, a $K_{2,3}$, and a $K_{3,3}$, respectively. Solving BIC amounts to cover, or equivalently to partition, the edge-set of G into a minimum number of biclique sets. So Figure 2.1(d) shows a partition into three biclique sets (one in dot lines, one in bold lines, the last in normal lines).

The minimum biclique cover problem has been mainly studied in bipartite graphs and few was known about the general case. In bipartite graphs, and in fact in triangle-free graphs, BIC is a particular case of the graph coloring problem because of the following property:

Fishburn and Hammer's theorem [27] *Let $G = (V, E)$ be a triangle-free graph. A subset $B \subseteq E$ is a biclique set if and only if every subset $B' \subseteq B$ with $|B'| = 2$ is a biclique set.*

In the auxiliary graph $B(G)$ with vertex set $V(B(G)) = E(G)$ where two vertices $e, f \in E(G)$ are linked if $\{e, f\}$ is not a biclique set (that is, e and f are neither adjacent, nor in a same C_4), then the stable sets of $B(G)$ and the biclique sets of G are in one-to-one correspondence, and hence $\chi(B(G))$ is the solution of BIC. If G is a triangle, then $\chi(B(G)) = 1$ but we need two bicliques to cover G. In essence, Fishburn and Hammer's theorem is a characterization of all graphs for which we can be sure that $\chi(B(G))$ equals the minimum number of bicliques for covering G.

Fishburn and Hammer said in [27]: *"The situation is substantially complicated by the presence of triangles, and we do not pursue the matter here."* For instance any three edges among the set $B = \{12, 34, 56, 78\}$ in the *8-antihole* $\overline{H}_8$ of Figure 2.2(c) (the four edges in bold) form a biclique set, e.g. $B' = \{12, 34, 56\}$ is contained in the complete bipartite subgraph induced by $\{135, 246\}$. However, no complete bipartite subgraph contains B, e.g. the bipartite subgraph induced by $\{1357, 2468\}$ is not complete. Similarly, in Figure 2.2(a)-(b), the set of the edges in bold is not in some biclique while any proper subset of this set is.

The following generalizes Fishburn and Hammer's theorem to all graphs:

Theorem 1 (C. and Fonlupt [16]) *Let $G = (V, E)$ be a graph. A subset $B \subseteq E$ is a biclique set if and only if every subset $B' \subseteq B$ with $|B'| \leq \omega(G)$ is a biclique set.* *(ω is the clique number)*

In Figure 2.2(c), since $|B| = 4 = \omega(\overline{H}_8)$, the theorem predicted that B might not be a biclique set although every subset of cardinality < 4 was. The proof of Theorem 1 (radically different from the case-checking approach of [27]) has three steps.

The first step is the recognition of biclique sets. The main definition of this chapter is:

Definition 2 *Let $B \subseteq E$ be a subset of edges of $G = (V, E)$ with vertex set $V(B) \subseteq V$ and let*

$$\overline{E}[V(B)] := \{uv : u, v \in V(B),\, u \neq v,\, uv \notin E\}$$

The rooted graph *of (G, B) is the graph with vertex-set $V(B)$ and edge-set $B \cup \overline{E}[V(B)]$.*

Figure 2.1(f) shows the rooted graph corresponding to set of bold edges B of Figure 2.1(e).

Now the key idea is:

The biclique lemma. *Let $B \subseteq E$ be a subset of edges of G. Then B is a biclique set if and only if the rooted graph of (G, B) has no cycle C with $|C \cap B|$ odd.*

One can check that the set B of bold edges in Figure 2.1(e) is not a biclique-set (it is a *non-biclique set*).

The biclique lemma follows from two points:

1. The extremities of an edge in B must be in different sides of the complete bipartite graph they belong to, while
2. The extremities of an edge in $\overline{E}[V(B)]$ must be in a same side (because of the completeness).

So if we contract the edges in $\overline{E}[V(B)]$ in the rooted graph, one should get a bipartite graph.

Triangles complicated indeed the situation because the set C of the three edges of a triangle form a *minimal non-biclique set*, that is, C is not a biclique set but any of its proper subsets is a biclique set.

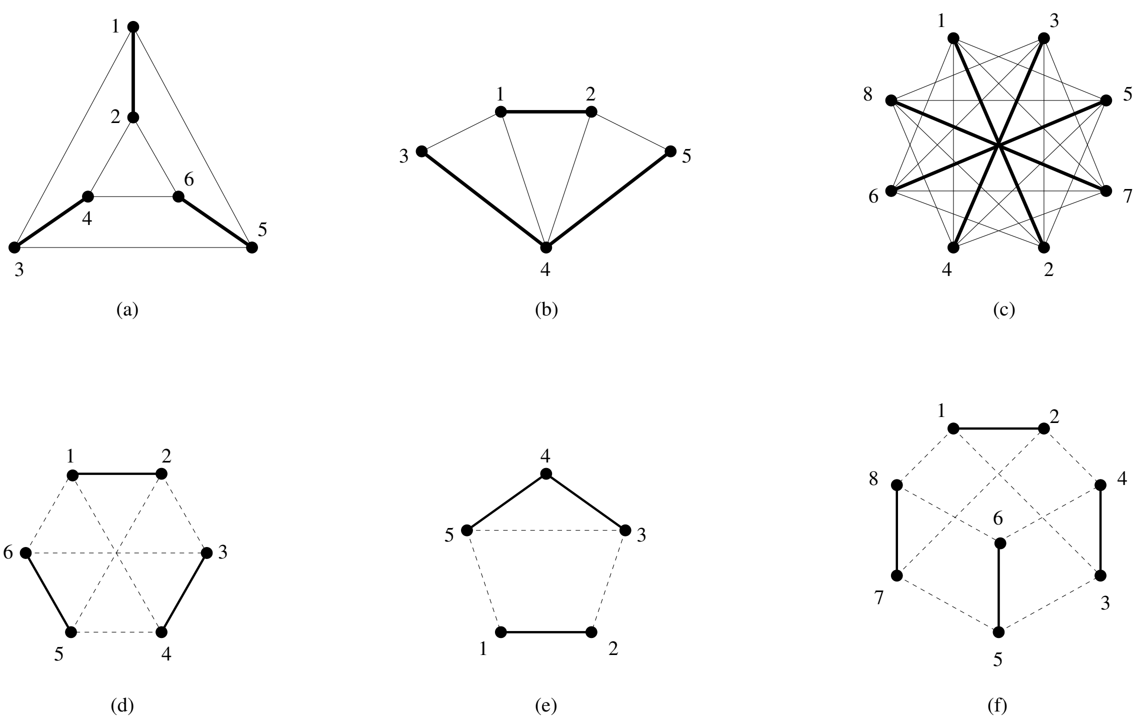


Figure 2.2: Minimal non-biclique sets and their rooted graphs

In a general graph $G = (V, E)$, BIC amounts to finding the chromatic number $\chi(H)$ of the clutter $H = (E, \mathcal{C})$ where $\mathcal{C}$ is the set of all minimal non-biclique sets of G. So, H is a graph if and only if $\omega(G) \leq 2$, by Fishburn and Hammer's theorem. Observe that, if $G = K_{2k+1}$ is a complete graph of odd order, then the edge-set of any Hamiltonian circuit C_{2k+1} of G is a minimal non-biclique set. So Theorem 1 is best-possible.

Odd-circuits inducing cliques are not the only minimal non-biclique sets. Take for instance the three edges in bold in the *prism graph* of Figure 2.2(a) or that in the *fan graph* of Figure 2.2(b). The four edges in bold of Figure 2.2(c) form a minimal non-biclique set as well. Their rooted graphs are represented, respectively, in Figure 2.2(d)-(f).

The second step in the proof of Theorem 1 is to describe the structure of the rooted graphs of a minimal non-biclique set B. For this we consider a cycle C of the rooted graph of (G, B) with $|C \cap B|$ maximum. For instance, in Figure 2.2, C has vertex-sequence (a) $1, 2, 3, 4, 5, 6$, (b) $1, 2, 3, 4, 5$, and (c) $1, 2, 4, 3, 5, 7, 8$. We prove that then:

(i) The edges in $B \setminus C$ form a matching disjoint from edges in $B \cap C$, and

(ii) Any edge of the rooted graph not belonging to $C \cup B$ is either:

(a) a *diagonal*, that is, a chord of C of the form of 14 in Figure 2.2(c);

(b) a *short-chord*, that is, a chord of C of the form of 35 in Figure 2.2(d); or

(c) a *wing*, that is, an edge of the form 46 in Figure 2.2(e).

The last step of the proof is to find in the rooted graph a subset of vertices of size $|B|$ which is a clique in the original graph of G. Figure 2.3 shows the rooted graph of a minimal non-biclique set where, in the original graph G, the white vertices form a clique of size $|B|$.

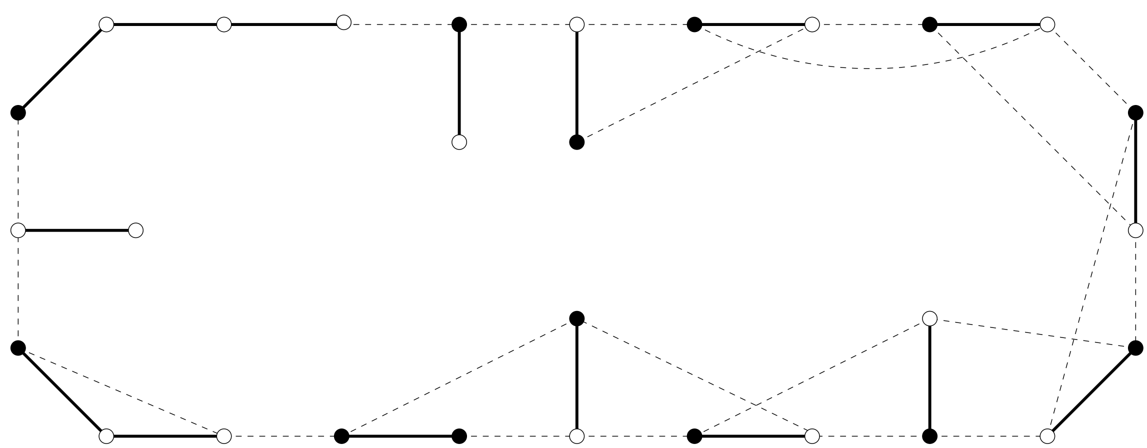


Figure 2.3: The clique of size $|B|$ of a minimal non-biclique set B

2.2 A general method: Separation algorithms

Let $G = (V, E)$ be a graph and let $\mathcal{B} = \{B_1, \ldots, B_{|\mathcal{B}|}\}$ be a set of edge subsets $B_i \subseteq E$.
The problem of covering E with sets in $\mathcal{B}$ can be formulated naturally by

$$\min\{y^\top \mathbf{1} : y^\top A \geq \mathbf{1}^\top, \, y \geq \mathbf{0}, \, y \text{ integer}\} \quad \text{where } A \text{ is the incidence matrix of the clutter } (E, \mathcal{B})$$

The matrix A may have exponentially many rows and, since the objective function is unit, the pricing problem of the relaxation amounts to finding some $B \in \mathcal{B}$ maximizing $\bar{\mu}(B)$ for a nonnegative weight vector $\bar{\mu} \in \mathbb{Q}_+^E$.

Definition 3 *Given $\mathcal{B}$, we denote $\mathcal{C}(\mathcal{B})$ the set of all $C \subseteq E$ such that no $B \in \mathcal{B}$ contains C but there exists a $B \in \mathcal{B}$ which contains C', for every proper subset $C' \subset C$.*

Well-known instances are

- if $\mathcal{B}$ is the set of bipartite subgraphs, then $\mathcal{C}(\mathcal{B})$ is the set of odd-circuits; and
- if $\mathcal{B}$ is the set of forests, then $\mathcal{C}(\mathcal{B})$ is the set of circuits.

In general, unlike these two well-known instances, the collection $\mathcal{B}$ is not hereditary. It follows that a set B' containing no $C \in \mathcal{C}(\mathcal{B})$ is not necessarily in $\mathcal{B}$. However, $B' \subseteq B$ for some $B \in \mathcal{B}$, hence finding a maximum $B \in \mathcal{B}$ is equivalent to finding a maximum B' containing no $C \in \mathcal{C}(\mathcal{B})$.

Thus the pricing is to find a minimum T intersecting all $C \in \mathcal{C}(\mathcal{B})$ which is formulated by

$$\min\{\mathbf{1}^\top x : Mx \geq \mathbf{1}, \, x \geq \mathbf{0}, \, x \text{ integer}\} \quad \text{where } M \text{ is the incidence matrix of } (E, \mathcal{C}(\mathcal{B}))$$

The matrix M has exponentially many rows. The following theorem shows that the separation problem of the relaxation is polynomial for several types of set $\mathcal{B}$.

Theorem 2 (C. and Fonlupt [16], C. [12], C. and Mahjoub [21]) *Let $G = (V, E)$ be a graph, let $\mathcal{B}$ be some set of edge subsets of G, and let $\bar{x} \in \mathbb{Q}_+^E$ be a cost vector. The problem of finding a set $C \in \mathcal{C}(\mathcal{B})$ minimizing its cost $\bar{x}(C)$ can be solved in a polynomial time when $\mathcal{B}$ is defined as the set of*

(a) edge sets of complete bipartite subgraphs of G,

(b) edge sets of complete multipartite subgraphs of G,

(c) edge sets of vertex-induced bipartite subgraphs of G.

In the proof, after characterizing the minimal structures (as explained in the next section), we build an auxiliary graph $A(\mathcal{B})$ of order $O(|V|)$. Of course $A(\mathcal{B})$ is specific to the particular problem studied, but each time it allows then to reduce the separation problem to a shortest path problem.

Theorem 2 shows that ideas that were developed to study bicliques can be applied to other problems. One of the main tool for bicliques was that sophisticated concept of odd-circuit of Figure 2.3. There is a sophisticated concept of *loop* which appears useful to study complete multipartite subgraphs. This is explain in the next section.

2.2.1 Maximum complete multipartite subgraph

Let us denote by $K_{p_1,\ldots,p_q}$ the complete multipartite graph with q shores of size $p_1, \ldots, p_q$. Call *complete multipartite* any graph which is (isomorphic to) $K_{p_1,\ldots,p_q}$ for some q. For instance the graph which appears in Figure 2.1(d)-(e) was $K_{1,1,2,2}$ (this graph is redrawn in Figure 2.4(a) where it can be seen complete multipartite by taking $\{12; 34; 5; 6\}$ as vertex-partition).

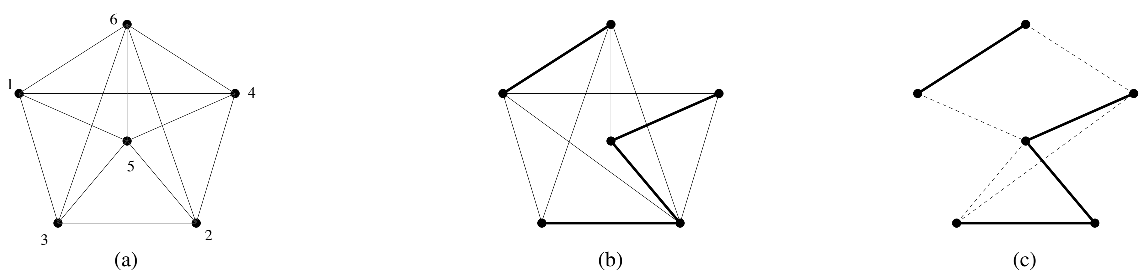


Figure 2.4: Multiclique and non-multiclique sets

Definition 4 *A subset $B \subseteq E$ is a* multiclique set *of $G = (V, E)$ if $B \subseteq E'$ where $(V(E'), E')$ is a complete multipartite partial subgraph of G.*

So every subset of edge in Figure 2.4(a) is a multiclique set. Let us remove some edges. Is the subset B of edges in bold in Figure 2.4(b) a multiclique set ? The answer is given by the key observation of [12]:

The multiclique lemma. *Let $B \subseteq E$ be a subset of edges of G. Then B is a multiclique set if and only if the rooted graph of (G, B) has no cycle C with $|C \cap B| = 1$.*

Looking at its rooted graph in Figure 2.4(c) one can conclude, using the multiclique lemma, that the set B of Figure 2.4(b) is not a multiclique set.

The proof of the multiclique lemma follows the same lines as that of the biclique lemma except that after contracting the edges in $\overline{E}[V(B)]$ in the rooted graph one should get, not a bipartite graph, but a multipartite graph, that is a graph without loop. (So in a sense, the concept of loop is here as useful as that of odd-cycle).

Recall that Fishburn and Hammer's theorem says that triangle-free graphs is exactly the class of graphs in which we are sure that the minimum number of complete bipartite subgraphs needed for covering the edge-set is equal to $\chi(B(G))$. Theorem 3 below, which offers a second generalization of Fishburn and Hammer's theorem, characterizes the class of graphs for which we know that the minimum number of complete multipartite subgraphs needed for covering the edge-set is equal to $\chi(B(G))$.

We proved Theorem 3, by using a complete description of the structure of rooted graphs of minimal non-multiclique sets (as for the proof of Theorem 1 with minimal non-biclique sets). Figure 2.5 shows such a rooted graph.

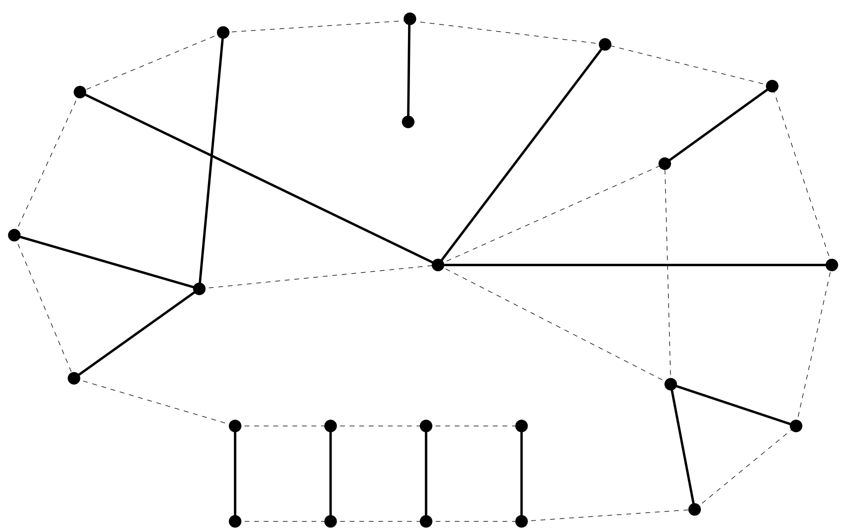


Figure 2.5: Rooted graph of a minimal non-multiclique set

A graph is *{fan,prism}-free* if none of its induced subgraph is isomorphic to the graph of Figure 2.2(a) (the prism) or to the graph of Figure 2.2(b) (the fan). We used fully the description to prove that:

1. If G has a minimal non-multiclique set C with $|C| \geq 3$, then G has minimal non-multiclique set D with $|D| = |C| - 1$; and
2. If G has a minimal non-multiclique set C with $|C| = 3$, then G has either a fan or a prism as induced subgraph.

This clearly implies Theorem 3.

Theorem 3 (C. [12]) *The cardinality of any minimal non-multiclique set of G is two if and only if G is a {fan,prism}-free graph.*

Theorem 3 is indeed a generalization of Fishburn and Hammer's result since:

1. Every triangle-free graph is {fan,prism}-free;
2. In a triangle-free graph, biclique sets and multiclique sets coincide.

2.2.2 Vertex-induced subgraphs with edge-weight

Maximum induced bipartite subgraph with edge weight

Given a graph $G = (V, E)$ with edge weight $w \in \mathbb{Z}^E$, the *maximum induced bipartite subgraph with edge weight problem* is to find a vertex-induced bipartite subgraph $(V', E(V'))$ maximizing $w(E(V'))$.

This problem can be linked to the problem of finding a maximum biclique set, together with Ridha Mahjoub, we give *"rise to some structural relations between* [...] *both problems"* in [21]. The general idea is that, if the edge-weights are non-negative, we are looking for a maximum *induced-bipartite set*, that is, a subset B of edges such that $B \subseteq E'$ where $(V(E'), E')$ is an induced bipartite subgraph of G (i.e., $E' = E(V(E'))$).

The structure of a subgraph $G[V(B)]$ induced by the vertex-set $V(B)$ of minimal non-induced-bipartite set B has some similarity with the rooted graphs of a minimal non-biclique sets where $E(V(B)) \setminus B$ plays a role similar as $\overline{E}(V(B))$. Formally, a minimal non-induced-bipartite set is a subset $B \subseteq E$ such that:

(α) $G[V(B)]$ contains an odd-cycle,

(β) for every odd-cycle C of $G[V(B)]$ and for every edge e of B, there exists a vertex v of C which is an extremity of e and of no other edge in B.

Indeed, if (β) holds, then for every odd-cycle C of $G[V(D)]$ there is a vertex v which does not belong to the subgraph $G[V(B \setminus \{e\})]$, and hence, $G[V(B \setminus \{e\})]$ is bipartite, for any $e \in B$. Otherwise, if (β) does not hold, there are an edge $e \in B$ and an odd-cycle C any vertex v of which belongs to $G[V(B \setminus \{e\})]$, and hence $B \setminus \{e\}$ is not bipartite, hence B is not minimal.

Let $G = (V, E)$ be the graph on vertex-set $\{1, \ldots, 6\}$ with edge-set composed of a 5-cycle $12, 23, 34, 45, 15$ and a pending edge 16, as in Figure 2.6.(a).

If the weight is one for edges $16, 23, 45$, and zero for all other edges, then an integer quadratic formulation for this instance of our problem is

$$\begin{aligned} \max\ & x_1x_6 + x_2x_3 + x_4x_5 \\ & x_1 + \ldots + x_5 \leq 4 \\ & 0 \leq x_1, \ldots, x_6 \leq 1 \\ & x_1, \ldots, x_6 \text{ integer} \end{aligned}$$

A classic approach is then to apply the so-called (RLT1) Sherali-Adams linearization [40] to its relaxation; one obtains

$$\begin{aligned} & \max\ y_{16} + y_{23} + y_{45} \\ 4x_1 + x_2 + x_3 + x_4 + x_5 - 4 \leq\ & y_{12} + y_{13} + y_{14} + y_{15} \leq 3x_1 \\ x_1 + 4x_2 + x_3 + x_4 + x_5 - 4 \leq\ & y_{12} + y_{23} + y_{24} + y_{25} \leq 3x_2 \\ x_1 + x_2 + 4x_3 + x_4 + x_5 - 4 \leq\ & y_{13} + y_{23} + y_{34} + y_{35} \leq 3x_3 \\ x_1 + x_2 + x_3 + 4x_4 + x_5 - 4 \leq\ & y_{14} + y_{24} + y_{34} + y_{45} \leq 3x_4 \\ x_1 + x_2 + x_3 + x_4 + 4x_5 - 4 \leq\ & y_{15} + y_{25} + y_{35} + y_{45} \leq 3x_5 \\ x_1 + x_2 + x_3 + x_4 + x_5 + 4x_6 - 4 \leq\ & y_{16} + y_{26} + y_{36} + y_{46} + y_{56} \leq 4x_6 \\ x_i + x_j - 1 \leq\ & y_{ij} \leq x_i, x_j \quad (1 \leq i < j \leq 6) \\ 0 \leq\ & x_1, \ldots, x_6 \leq 1 \end{aligned}$$

It admits the following feasible solution leading to a value $\frac{7}{3}$ for the objective function

$$\begin{aligned} x_1, x_6, y_{16} &= 1 \\ x_2, \ldots, x_5, y_{12}, \ldots, y_{15}, y_{26}, \ldots, y_{56}, y_{23}, y_{45} &= \tfrac{2}{3} \qquad \text{(see Figure 2.6.(b))} \\ \text{all other variables} &= \tfrac{1}{3} \end{aligned}$$

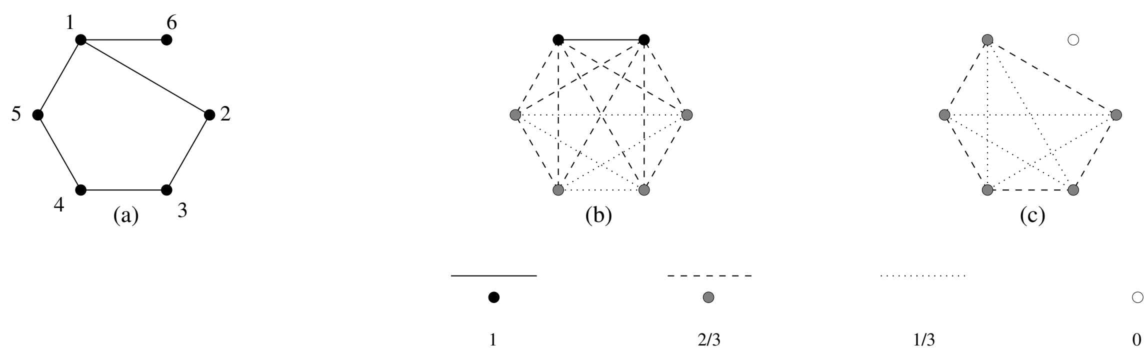


Figure 2.6: Sherali-Adams vs. minimal structures

A natural pure edge formulation is

$$\begin{cases} \max\ w^\top y \\ y(D) \le |D| - 1 \quad \text{for each minimal non-induced-bipartite set } D \\ y \in \{0,1\}^E \end{cases}$$

For our particular example, it is stronger than Sherali-Adams since $\frac{7}{3} > 2 = |D| - 1$ with $D = \{16, 23, 45\}$. Note that to handle arbitrary weights it suffices to add the constraints

$$y_{su} - y_{uv} + y_{vt} \le 1 \quad \text{for } su, uv, vt \in E \quad \text{(with possibly } s = t\text{)}$$

There is another pure edge formulation if, instead of the minimal non-induced-bipartite set inequalities, one uses

$$y(C) \le |C| - 2 \quad \text{for each odd-cycle } C$$

If now we change the objective function of our example and we maximize $y(C = \{12, 23, 34, 45, 15\})$, then it is stronger than Sherali-Adams since $|C| - 2 = 3 < \frac{10}{3}$ which is achieved with

$$\begin{aligned} x_6, y_{16}, \ldots, y_{56} &= 0 \\ x_1, \ldots, x_5, y_{12}, y_{23}, y_{34}, y_{45}, y_{15} &= \tfrac{2}{3} \qquad \text{(see Figure 2.6.(c))} \\ \text{all other variables} &= \tfrac{1}{3} \end{aligned}$$

Both pure edge formulations cannot be compared to Sherali-Adams but they both dominates the basic linearizations by Glover and Woolsey [30], and by Glover [29]. (They are contained in the projection onto the y-space).

Minimal arcs sets vertex-inducing dicycles

In (α)-(β), if we replace "odd-cycle" by "cycle", then B becomes a minimal subset of edges the vertices of which induce a graph which is not a forest. Induced trees have been studied in [11].

Among all the minimal structures we studied, except for minimal non-co-multiclique sets, the minimal non-induced-forest sets are the most easy to describe.

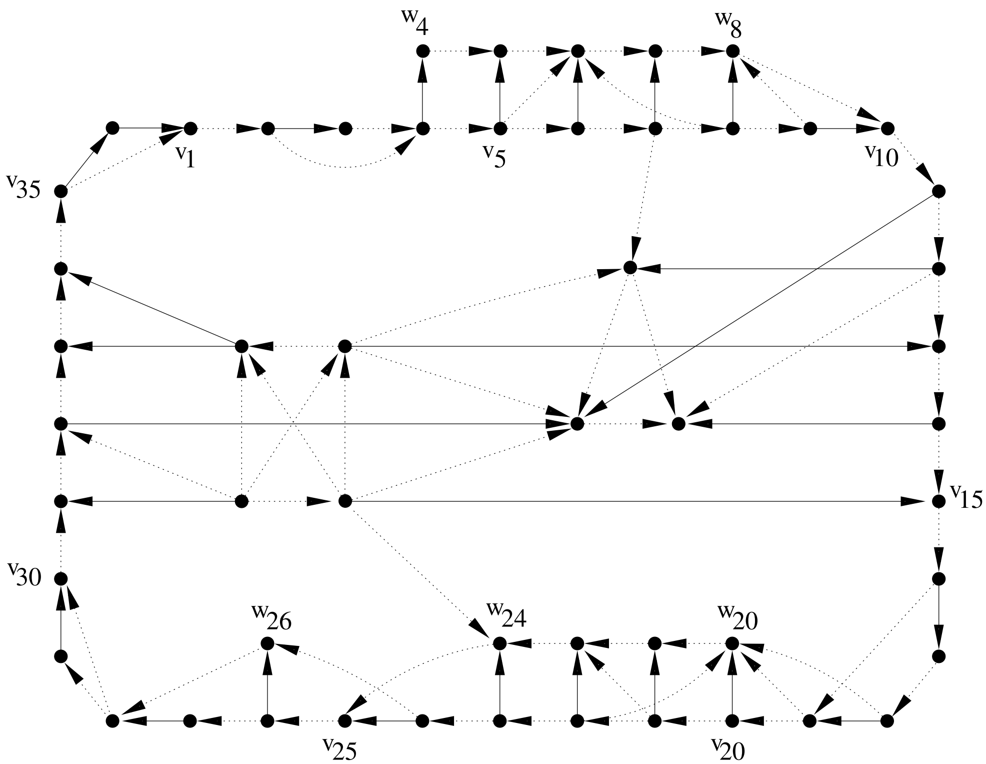


Figure 2.7: A minimal arc-set vertex-inducing a dicycle

In contrast, if one considers directed graphs and if replace everywhere "cycle" by "dicycle", one gets a rich structure of minimal forbidden arc sets C, see [20]. A digraph induced by the vertices of C is depicted by Figure 2.7 where the arcs outside C are in dot lines.

2.2.3 Disjoint cliques

Call *co-biclique set of* $G = (V, E)$ any subset $B \subseteq E$ contained in the edge set of a $K_p + K_q$, that is, two vertex-disjoint complete subgraphs of G. In [13], we used the following lemma whose proof is similar to that of the biclique lemma.

The co-biclique lemma. *Let $B \subseteq E$ be a subset of edges of G. Then B is a co-biclique set if and only if the rooted graph of (G, B) has no cycle C with $|C \setminus B|$ odd.*

The structure of the rooted graph of a minimal non-co-biclique set is close to that of rooted graphs of minimal non-biclique sets but with other types of edges allowed. (So it is slightly more complex).

Now call *co-multiclique set of* $G = (V, E)$ any subset $B \subseteq E$ contained in the edge set of several vertex-disjoint complete subgraphs of G. One has a co-multiclique lemma similar to the multiclique lemma.

The co-multiclique Lemma. *Let $B \subseteq E$ be a subset of edges of G. Then B is a co-multiclique set if and only if the rooted graph of (G, B) has no cycle C with with $|C \setminus B| = 1$.*

Observe that the rooted graph of a minimal non-co-multiclique set B must be exactly a cycle with one edge in $\overline{E}$, that is, $G[V(B)]$ is isomorphic to $K_{|B|+1} \setminus e$. This structure is very simple in comparison with all other similar structures of this chapter. (Because of its simplicity it has no interest for itself, and the separation problem reduces trivially to finding a minimum length path, however, it has led us to different things related to graph coloring and to multiflows that we will explain in the next chapters).

To conclude this chapter, we make a short remark about applying the approach to any covering problem with a pure quadratic objective function and a polynomial number of constraints, that is, of the form

$$\text{(QCOV)} \begin{cases} \min \sum_{i<j} w_{ij} x_i x_j \\ \quad Ax \geq \mathbf{1} \\ \quad x \in \{0,1\}^n \end{cases}$$

where w is non-negative and A is a 0-1 matrix with a polynomial number of rows.

Let be the graph $G = (V, E)$ where $V = \{1, \ldots, n\}$ and $E = \{ij \text{ with } w_{ij} > 0\}$. Given a row $a^\top x \geq 1$ of (QCOV), define $V_a := \{v \in V : a_v = 1\}$.

Observe that (QCOV) amounts to finding $\tau_w(H)$ for the clutter $H = (E, \mathcal{C})$ where $C \in \mathcal{C}$ if there is a V_a such that C is a minimal subset of edges spanning V_a. The separation problem for the relaxation of $\tau_w(H)$ amounts to a finding a minimum cost C. Here it is the particular case of the b-edge cover problem where $b \in \{0,1\}^V$ instead of $b \in \mathbb{Z}_+^V$. Thus it can be solved in a polynomial time (see [37], Theorem 34.15 p. 581).

Chapter 3

Graph coloring

Let be a simple connected graph $G = (V, E)$.

An *edge-clique partition of* G is a subset $E' \subseteq E$ induced by a clique partition of V, that is, of the form $E' = E(V_1) \cup \ldots \cup E(V_p)$ where $V_1, \ldots, V_p$ is a partition of V into nonempty cliques.

For instances see the set of bold edges in Figure 3.1.(a) and (d). (Notice that if V_i is a singleton, then $E(V_i) = \emptyset$).

It is easily seen that $B \supseteq E'$ is contained in some edge-clique partition $E' \subseteq E$ if and only if the rooted graph of (G, B) has no cycle C with $|C \setminus B| = 1$. Suppose that, moreover, B has no cycle, so B is a forest every tree of which spans a clique of G. It follows that the connected components of (V, B) induces a partition of V into $|V| - |B|$ cliques, and in fact a partition into cliques. If $|B|$ is maximum, then

$$\overline{\chi}(G) = |V| - |B| \qquad \text{(Recall that } \overline{\chi}(G) = \chi(\overline{G}))$$

Since B can be defined as a subset without forbidden structures, the chromatic number is expressed this way as the stability number of a clutter. It appears surprisingly that one can use a clutter which is in fact a graph and that the colorings are in 1-to-1 correspondence with the stable sets of this graph. This is explain in the next section.

3.1 The chromatic Gallai identities

A forest of G each tree of which spans a clique defines a clique partition of G (that is, a coloring in the complementary graph). In order to have a 1-to-1 correspondence between special forests and clique partitions one chooses, arbitrarily, an orientation $\vec{G}$ of G such that no clique contains a directed cycle (equivalently, $\vec{G}$ has no directed 3-cycle).

In $\vec{G}$, an *out-star* is a set of arcs all of which have the same tail and an *out-stellar forest* is the vertex-disjoint union of out-stars.

Since no clique of $\vec{G}$ has a dicycle, given a clique K of G, there is exactly one out-star of $\vec{G}$ spanning K and so there is a 1-to-1 correspondence between the clique partitions of G and the *simplicial stellar forests of* $\vec{G}$, that is, an out-stellar forest of $\vec{G}$ the out-stars of which spans cliques. (The cliques of size one correspond to empty out-stars.) Take for instance the edge-clique partition of Figure 3.1.(a) and its associated simplicial stellar forest in the orientation chosen in Figure 3.1.(b). See also Figure 3.1.(d) and (e).

Definition 5 *Let $\vec{G}$ be a orientated graph. A pair of arcs $\{e, f\}$ is* simplicial *if*

1. *e and f have the same tail;*
2. *the heads of e and f are linked by an arc.*

The sandwich line-graph $S(\vec{G})$ of $\vec{G}$ *is the graph with vertex-set the arc-set of $\vec{G}$ where two vertices are linked if they correspond to adjacent arcs which are not simplicial.*

A stable set of $S(\vec{G})$ necessarily corresponds to a simplicial stellar forest of $\vec{G}$. So we have a 1-to-1 correspondence between clique partitions of G (colorings of $\overline{G}$) and stable sets of $S(\vec{G})$. Take for instance the simplicial stellar forest of Figure 3.1.(b) and its stable set defined by the white vertices in Figure 3.1.(c). Similarly see Figure 3.1.(e) and (f).

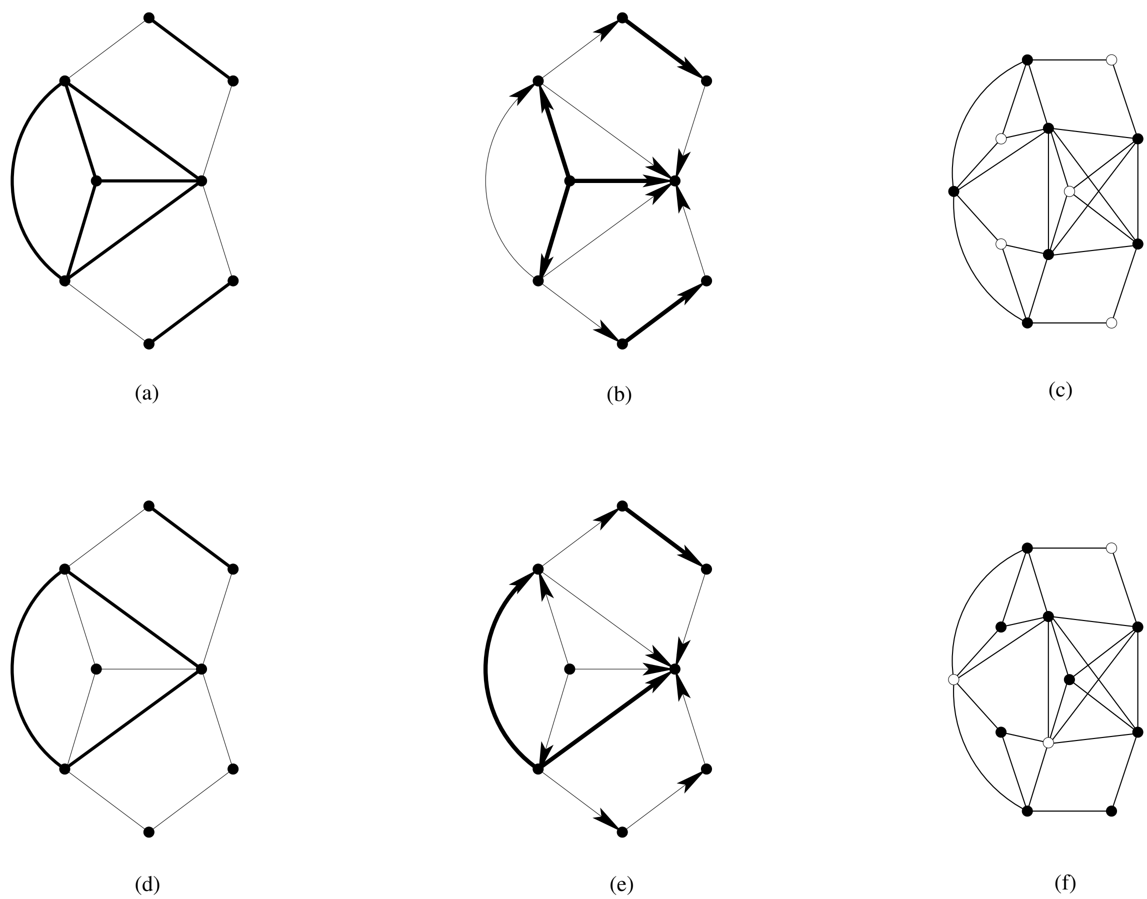


Figure 3.1: 1-to-1 correspondences between clique partitions of G, simplicial stellar forests of $\vec{G}$ and stable sets of $S(\vec{G})$

Thus for any partition into p cliques of G and its corresponding stable set S in $\vec{G}$, we have

$$p + |S| = |V(G)|,$$

and in particular $\overline{\chi}(G) = |V(G)| - \alpha(S(\vec{G}))$. Surprisingly, we also have $\alpha(G) = |V(G)| - \overline{\chi}(S(\vec{G}))$ even if, this time, there are no 1-to-1 correspondence between the stable sets of G and the clique partitions of $\vec{G}$. Indeed, if $G = K_2$, then G has three stable sets but $S(\vec{G})$ has only one clique-partition (since $\vec{G} = K_1$).

To see why this identity also holds let us define:

Definition 6 *Let $\vec{G}$ be a orientated graph. A set S of arcs is an* anti-simplicial star *of $\vec{G}$ if there exists a vertex v, called the center of S, such that:*

1. *for all $e \in S$, then v is either the tail or the head of e;*
2. *for all $e, f \in S$ with tail v, then the heads of e and f are not linked.*

An anti-simplicial stellar partition of $\vec{G}$ *is a partition of the arc-set of* $\vec{G}$ *into anti-simplicial stars.*

Given a vertex-cover T of $\vec{G}$, associate to any vertex $v \in T$ the anti-simplicial star $A(v)$ composed of all arcs entering v and all arcs from v to a vertex outside T. For instance the anti-simplicial star of $\vec{G}$ in Figure 3.2.(b) is uniquely defined by its center $v \in T$ where the set T is defined by the black vertices. With the same center but a different set T of black vertices, we obtain for instance the anti-simplicial star of Figure 3.2.(e).

Since T is a vertex-cover, the union of all $A(v)$, $v \in T$ is an anti-simplicial stellar partition of $\vec{G}$.

The anti-simplicial stellar partition of $\vec{G}$ and the clique partitions of $S(\vec{G})$ are in 1-to-1 correspondence. Figure 3.2.(c) shows the clique partition corresponding to the anti-simplicial stellar partition defined by the vertex-cover T of Figure 3.2.(a) and the orientation of Figure 3.2.(b). See also Figure 3.2.(d)-(f).

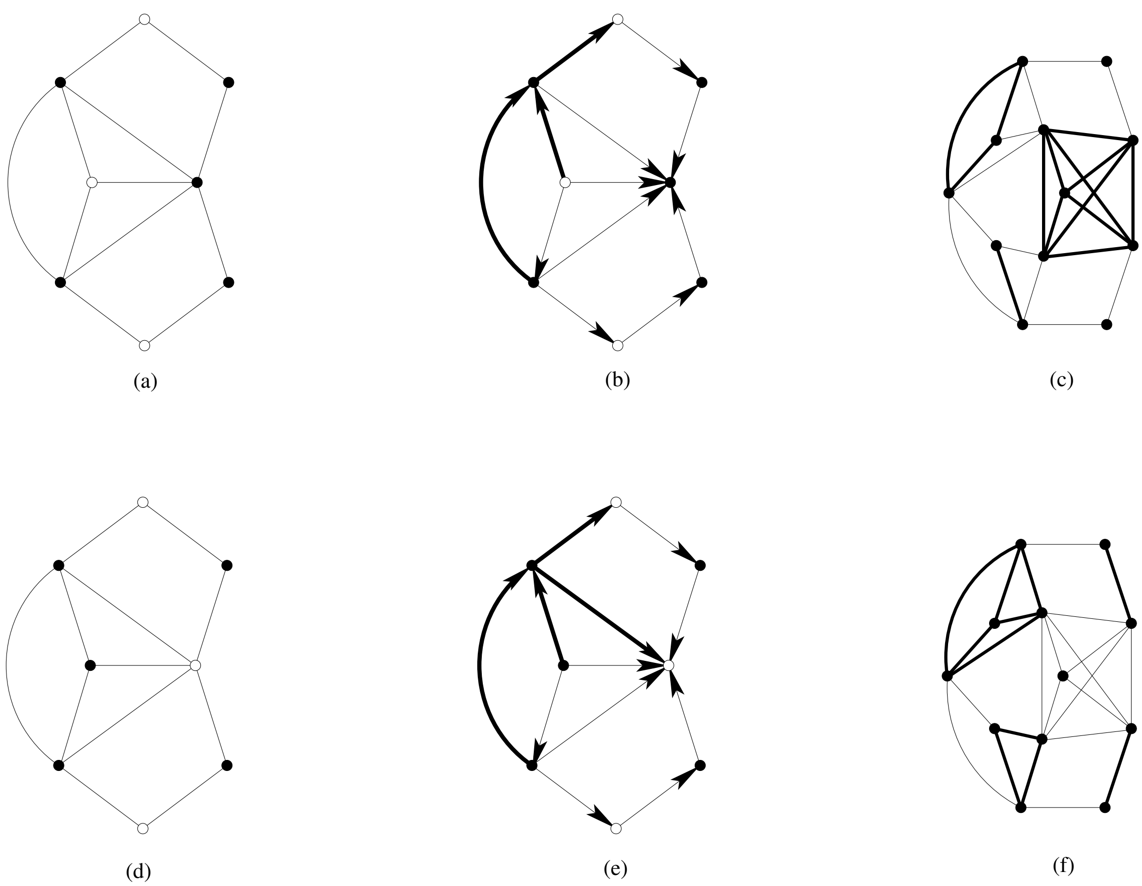


Figure 3.2: Surjection from stable sets of G onto anti-simplicial stellar partitions of $\vec{G}$ which are in 1-to-1 correspondences with the clique partitions of $S(\vec{G})$

Finally, it is not difficult to see:

Theorem 4 (Chromatic Gallai identities) (C. and Jost [19], C. and Meurdesoif [22]) *Let G be a simple graph. For any orientation $\vec{G}$ of G without 3-dicycle, then*

$$\alpha(G) + \overline{\chi}(S(\vec{G})) = |V(G)| = \alpha(S(\vec{G})) + \overline{\chi}(G)$$

where $S(\vec{G})$ is the sandwich line-graph of $\vec{G}$.

One can check the Chromatic Gallai identities in the graph $\vec{G}$ of Figure 3.1-3.2:

$$\alpha(G) = 3 \quad \overline{\chi}(S(\vec{G})) = 5 \quad |V(G)| = 8 \quad \alpha(S(\vec{G})) = 5 \quad \overline{\chi}(G) = 3$$

The name "Chromatic Gallai identities" comes from the classical Gallai identities

$$\alpha(G) + \tau(G) = |V(G)| = \nu(G) + \rho(G)$$

which can be written

$$\alpha(G) + \overline{\chi}(L(G)) = |V(G)| = \alpha(L(G)) + \overline{\chi}(G)$$

if G is triangle-free, where $L(G)$ is the line-graph which coincide in this case with $S(\vec{G})$.

The next section shows why Chromatic Gallai identities are interesting for mathematical programming (SDP and LP).

3.2 Improving Lovász ϑ number

The main application for the chromatic Gallai identities is to improve Lovász theta lower bound for coloring.

A graph parameter β associating a real number $\beta(G)$ to any graph G is called *sandwich* if

$$\alpha(G) \leq \beta(G) \leq \overline{\chi}(G) \quad \text{for all graph } G$$

The most natural sandwich parameter is the fractional clique-covering number $\overline{\chi}_f(G)$ which is the optimum of the LP relaxation

$$\overline{\chi}_f(G) := \max\{\mathbf{1}^\top x :\ Ax \leq \mathbf{1},\, x \geq \mathbf{0}\} \quad \text{where } A \text{ is the clique-vertex matrix of } G.$$

For instance, Figure 3.3.(b) shows a primal solution for $\overline{\chi}_f \geq 3.75$ and Figure 3.3.(e) shows a dual solution for $\overline{\chi}_f \leq 3.5$.

Since $\overline{\chi}_f(G)$ is NP-hard to compute, Lovász number $\vartheta(G)$ is the most interesting sandwich parameter. Indeed, it can be computed in polynomial time using SDP. There are several sandwich parameters but all that are computable in polynomial time are close to ϑ. (Unless considering hierarchies).

Definition 7 *For any graph parameter β, let*

$$\Phi_\beta(\vec{G}) := |V(G)| - \beta(S(\vec{G}))$$

where G is any graph and $\vec{G}$ is any orientation without 3-dicycle of G. If no orientation $\vec{G}$ is given, then $\Phi_\beta(G)$ is defined, not uniquely, as

$$\Phi_\beta(G) := |V(G)| - \beta(S(G))$$

where $S(G)$ is any sandwich graph of G.

The chromatic Gallai identities implies that $\Phi_\alpha = \overline{\chi}$ and that $\Phi_{\overline{\chi}} = \alpha$, moreover, Φ_β is sandwich for any sandwich graph parameter β, that is

$$\alpha(G) \leq \beta(G) \leq \overline{\chi}(G) \quad \forall G \qquad \Longrightarrow \qquad \alpha(G) \leq \Phi_\beta(G) \leq \overline{\chi}(G) \quad \forall G$$

Theorem 5 (C. and Meurdesoif [22]) *The operator Φ moves the fractional chromatic number toward the stability number and it moves Lovász number toward the clique-partition number. More precisely,*

$$\alpha(G) \leq \Phi_{\overline{\chi}_f}(G) \leq \overline{\chi}_f(G) \quad \textit{and} \quad \vartheta(G) \leq \Phi_\vartheta(G) \leq \overline{\chi}(G) \qquad \textit{(for all graph } G\textit{)}$$

In order to prove the above theorem, one needed a practical formulation for ϑ, similar to what we already have for $\overline{\chi}_f$. Using the formulations of [33, 31] and classic tools, such as Gram's theorem and the Shur complement, we proved that for any graph $G = (V, E)$

$$\left.\begin{array}{rll} \max & \sum_v \|x_v\|^2 & \\ \text{s.t.} & \|x_o\|^2 = 1 & \\ & x_o^\top x_v = \|x_v\|^2 & (\forall v) \\ & x_u^\top x_v = 0 & (\forall uv \in E) \end{array}\right\} = \vartheta(G) = \left\{\begin{array}{rll} \min & \|y_o\|^2 & \\ \text{s.t.} & \|y_v\|^2 = 1 & (\forall v) \\ & y_o^\top y_v = 1 & (\forall v) \\ & y_u^\top y_v = 0 & (\forall uv \notin E) \end{array}\right.$$

Note that Theorem 5 holds for any chosen orientation $\vec{G}$ of G. The choice of $\vec{G}$ may impact drastically the transformation of the value by Φ.

To see this, suppose that $G = W_5$ is the 5-wheel and let $\vec{G}_1$ be the orientation of Figure 3.3.(a) and $\vec{G}_2$ be the orientation of Figure 3.3.(d). The sandwich line-graphs, $S(\vec{G}_1)$ in Figure 3.3.(b) and $S(\vec{G}_2)$ in Figure 3.3.(e) are different, moreover, $\overline{\chi}_f(\vec{G}_1) \neq \overline{\chi}_f(\vec{G}_2)$ implies $\Phi_{\overline{\chi}_f}(\vec{G}_1) \neq \Phi_{\overline{\chi}_f}(\vec{G}_2)$. This works with other graph parameters. Let $\overline{\chi}'_f$ be the LP defining $\overline{\chi}_f$ strengthen with the odd-cycle inequalities. Figure 3.3.(c) shows a primal solution for $\overline{\chi}'_f(S(\vec{G}_1)) \geq 3.5$ (since $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$) and Figure 3.3.(f) shows a dual solution for $\overline{\chi}'_f(S(\vec{G}_2)) \leq 3$ (since the 5-cycle costs 2).

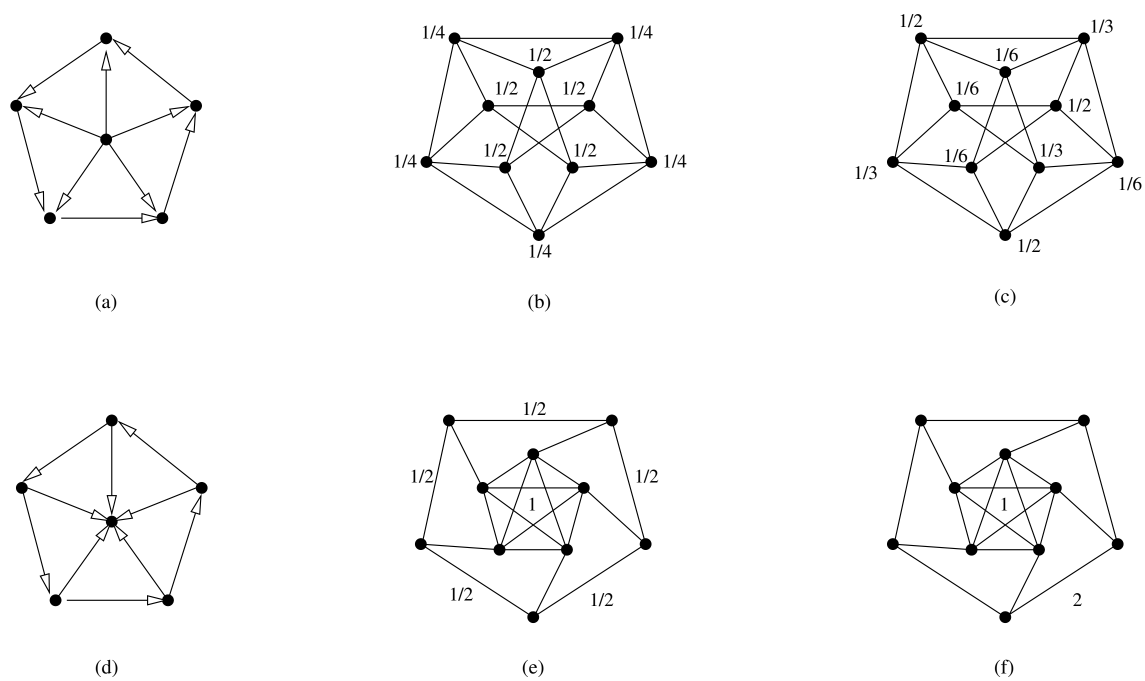


Figure 3.3: Two orientations leading to different values for $\Phi_{\overline{\chi}_f}$ and $\Phi_{\overline{\chi}'_f}$

Finally, the operator Φ has led to the following values:

$\alpha(G)$	$\overline{\chi}'_f(G)$	$\Phi_{\overline{\chi}_f}(\vec{G}_1)$	$\Phi_{\overline{\chi}_f}(\vec{G}_2) = \overline{\chi}_f(G) = \Phi_{\overline{\chi}'_f}(\vec{G}_1)$	$\Phi_{\overline{\chi}'_f}(\vec{G}_2) = \overline{\chi}(G)$
2	2.2	2.25	2.5	3

Observe that $\overline{\chi}'_f$ which is a good bound for α has led to the exact value of $\overline{\chi}$ using Φ with the orientation $\vec{G}_2$. Since ϑ is known to be also a good bound for α, one could use Φ to obtain a good polynomial bound for $\overline{\chi}$. This indeed happens for Mycielski graphs as showed by Table 3.1.

With random graphs the interest of Φ seems to decrease with the size of the graph, see Table 3.2.

	$\lvert V\rvert$	$\lvert E\rvert$	ϑ	$\overline{\chi}_f$	minΦ_ϑ	meanΦ_ϑ	maxΦ_ϑ	minρ	meanρ	maxρ
$\overline{M_3}$	5	5	2.236	2.5	–	2.764	–	–	23.6%	–
$\overline{M_4}$	11	35	2.400	2.9	3.024	3.054	3.088	26.8%	27.3%	28.7%
$\overline{M_5}$	23	182	2.529	3.245	3.200	3.228	3.274	26.5%	27.6%	29.5%
$\overline{M_6}$	47	845	2.639	3.553	3.326	3.372	3.417	26.0%	27.8%	29.5%

Table 3.1: Experimental results with ratios $\rho := (\Phi_\vartheta - \vartheta)/\vartheta$ for Mycielski graphs.

$\lvert V\rvert$	*dens*	ϑ	minΦ_ϑ	meanΦ_ϑ	maxΦ_ϑ	minρ	meanρ	maxρ
9	0.5	3.22	3.58	3.65	3.71	11.2%	13.4%	15.2%
10	0.5	3.28	3.61	3.66	3.70	10.0%	11.6%	12.8%
10	0.3	4.24	4.60	4.69	4.76	8.5%	10.6%	12.3%
15	0.3	5.43	5.90	5.94	5.98	8.7%	9.4%	10.1%

Table 3.2: Experimental results with ratios ρ for random graphs.

3.3 Special coloring problems

One chromatic Gallai identity has also an application to a batch scheduling problem called Max-coloring, this is explained in the next section.

3.3.1 Max-coloring

Let $c \in \mathbb{Z}^V$ be a cost vector on the vertex set of G.

Max-coloring is the problem of determining a clique-partition (coloring) $\mathcal{K} = K_1, \ldots, K_p$ of G which minimizes $\psi(\mathcal{K}) := \sum_{i=1}^{p} c_i$ where $c_i = \max_{v \in K_i} c(v)$, see [35]. So when c is a unit vector, $\psi(\mathcal{K}) = p$ and we meet the classical clique-partition (coloring) problem. We denote $\overline{\chi}_{\max}(G, c)$ the optimum of Max-Coloring.

Take the acyclic orientation $\vec{G}$ of G implied by the natural linear order defined by c on $V(G)$, that is, $e = uv$ is orientated from u to v if and only if $c(u) \geq c(v)$ (break ties arbitrarily). Let $S(\vec{G})$ be the sandwich line-graph of $\vec{G}$. Define $w(e) := c(v)$ for each arc $e = uv$ of $S(\vec{G})$, thus we obtain a vertex-weight vector $w \in \mathbb{Z}^{V(S(\vec{G}))}$ for the sandwich graph.

Let S be a stable set of $S(\vec{G})$. The set S is also a simplicial stellar forest in $\vec{G}$ and, moreover, observe that the weight $w(S)$ of the stable set S of $S(\vec{G})$ is equal to the weight of all leaves of this simplicial stellar forest S of $\vec{G}$. It follows that $c(V(G)) - w(S)$ is equal to the sum of the costs of the centers of S in $\vec{G}$. Furthermore, the way the orientation $\vec{G}$ was defined, the center of each out-star is the maximum cost vertex of the star, and hence the sum of the costs of the centers is equal to the objective value $\psi(\mathcal{K})$ of the clique-partition $\mathcal{K}$ of G corresponding to S. Hence, we obtain very easily:

Theorem 6 (C. and Jost [19]) $\alpha_w(S(\vec{G})) + \overline{\chi}_{\max}(G, c) = \mathbf{1}^\top c$

This reduces the sophisticated Max-coloring in G to the familiar maximum weighted stable set problem in $S(\vec{G})$.

3.3.2 Perfectness of clustered graphs

Let $\mathcal{V} = V_1, \ldots, V_p$ be a partition of the vertex-set $V(G)$.

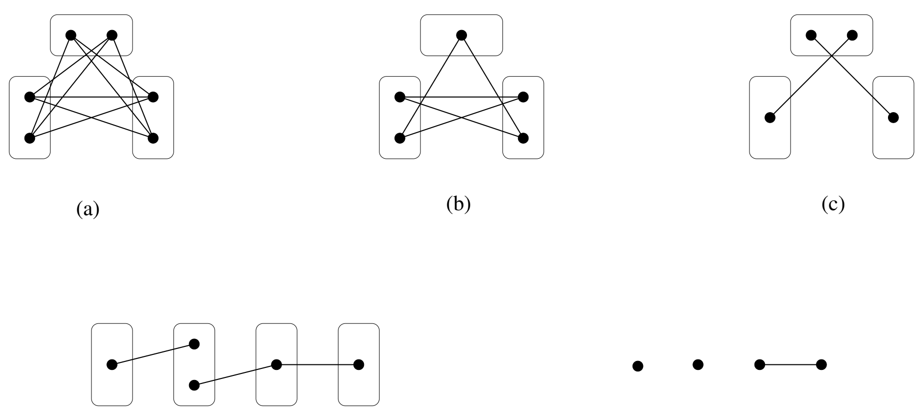
Selective-coloring, also known as *the partition coloring problem*, is to find an induced subgraph H of G with exactly one vertex in V_i minimizing its chromatic number $\chi(H)$, with $i \in I = \{1, \ldots, p\}$. So when each V_i is a singleton we meet the classical coloring problem. Selective-coloring has application in telecommunication for routing and wavelength assignment.

Call *cluster* any set V_i of $\mathcal{V}$, clearly, we can assume that each cluster is a stable set of G. For any subset $J \subseteq I$, call *cluster selection* the set of all V_j with $j \in J$.

Definition 8 *Let $(G, \mathcal{V})$ be a clustered graph.*

- *A cluster selection $\{V_j : j \in J\}$ is a* stable selection of $(G, \mathcal{V})$ *if there exists a stable set of G intersecting all V_j of the selection.*
- *The auxiliary graph $G/\mathcal{V}$ is the graph with vertices the clusters where two vertices are linked if the corresponding clusters are complete to each other.*

For instance the clustered graph in Figure 3.4.(d) has for auxiliary graph that in Figure 3.4.(e).



(a) (b) (c) (d) (e)

Figure 3.4: Clustered graph and auxiliary graph

The (maximal) stable selection–cluster matrix $M(G, \mathcal{V})$ of the clustered graph $(G, \mathcal{V})$ of Figure 3.4.(d) and the (maximal) stable set–vertex matrix $M(G/\mathcal{V})$ of its auxiliary graph are given below:

$$M(G, \mathcal{V}) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \text{ and } M(G/\mathcal{V}) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Given a 0-1 matrix A, let be the two polyhedrons

$$P(A) := \{x \in \mathbb{R}^n : Ax \leq \mathbf{1},\, x \geq \mathbf{0}\} \text{ and } D(A) := \{y \in \mathbb{R}^m : y^\top A \geq \mathbf{1}^\top,\, y \geq \mathbf{0}\}$$

So $\omega(G/\mathcal{V})$ is the max of $\mathbf{1}^\top x$ over integer vectors x in $P(M(G/\mathcal{V}))$, $\chi(G/\mathcal{V})$ is the min of $y^\top \mathbf{1}$ over integer vectors y in $D(M(G/\mathcal{V}))$, and Selective-coloring can be formulated naturally as

$$\chi_{sel}(G, \mathcal{V}) = \min\{y^\top \mathbf{1} : y \in D(M(G, \mathcal{V})),\, y \text{ integer}\}$$

One has

$$\omega(G/\mathcal{V}) \leq \chi(G/\mathcal{V}) \leq \chi_{sel}(G, \mathcal{V})$$

Notice that $\omega(G/\mathcal{V})$ is both the integer dual of $\chi(G/\mathcal{V})$ and of $\chi_{sel}(G, \mathcal{V})$. The matrix $M(G, \mathcal{V})$ can be seen as the incidence matrix of not all, but of some stable sets of $G/\mathcal{V}$ (not necessarily maximal), at least of all of size two.

A *conformal matrix* is a 0-1 matrix which is the stable set–vertex matrix of some graph (after removing the dominated rows). We have that

$$M(G, \mathcal{V}) = M(G/\mathcal{V}) \text{ if and only if } M(G, \mathcal{V}) \text{ is conformal}$$

A *perfect matrix* is a conformal matrix which is the stable set–vertex matrix of a perfect graph. Recall that the following are equivalent:

(i) for all $w \in \{0,1\}^n$, $\max\{w^\top x : x \in P(A) \cap \mathbb{Z}^n\} = \max\{w^\top x : x \in P(A)\}$,

(ii) $P(A)$ is integral;

(iii) $\{Ax \leq \mathbf{1},\, x \geq \mathbf{0}\}$ is TDI;

(iv) A is perfect.

The term "conformal" can be applied to the hypergraph of the matrix as well (or the clutter of the matrix, when it has no dominated rows). Gilmore showed that a clutter is conformal if and only if, for any three edges, there is some edge which contains the union of the three possible intersections between two edges among the three. So a triangle is the smallest example of a non-conformal clutter.

Another non-conformal matrix is

$$J_4 - I_4 := \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Observe that if one removes any column of $J_4 - I_4$ one obtains a conformal matrix, as it is isomorphic to $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ after removing dominated rows. In this sense, $J_4 - I_4$ is a minimally non-conformal matrix.

Thanks to Gilmore's characterization, it can be decided whether or not a given matrix is conformal or not in a time polynomial in the size of the matrix. However, the particular 0-1 matrices which are matrices of stable selections of a clustered graph can be encoded by $(G, \mathcal{V})$. We showed in [5] that, given $(G, \mathcal{V})$, it is NP-hard to decide whether $M(G, \mathcal{V})$ is conformal or not. On the positive side, we proved that these particular 0-1 matrices have an interesting property concerning conformality, namely:

The minimal conformality lemma. *$M(G, \mathcal{V})$ is minimally non-conformal with respect to deleting vertices of G if and only if $(G, \mathcal{V})$ is isomorphic to either (a), (b) or (c) in Figure 3.4.*

The proof of the minimal conformality lemma is in fact independent from Gilmore's characterization. This lemma can be used to characterize selective-perfection:

Definition 9 *A graph G is* selective-perfect *if the stable selection–cluster matrix $M(G, \mathcal{V})$ is perfect for all vertex partition $\mathcal{V}$.*

Since any member of $\mathcal{V}$ can be a singleton, a graph G is selective-perfect only if it is perfect. A graph $G = (V, E)$ is *threshold* if there exists $t \in \mathbb{R}^V$ such that $uv \in E \Leftrightarrow t(u) + t(v) > 0$.

We introduce a superclass of threshold graphs by extending the domain of t with the imaginary unit i of complex numbers $z = \Re(z) + i\Im(z)$.

Definition 10 *A graph $G = (V, E)$ is* i-threshold *if one can assign a complex number $t(v) \in \mathbb{R} \cup \{-i, +i\}$ to each vertex $v \in V$ such that:*

$$uv \in E \;\Leftrightarrow\; \Re(t(u) + t(v)) - \Im(t(u))\Im(t(v)) > 0.$$

Theorem 7 (Bonomo, C., Elkim, Ries [5]) *G is selective-perfect if and only if G is i-threshold.*

3.4 Polyhedral aspects

A forest F of $G = (V, E)$ is *clique-connecting* if $V(T)$ is a clique for any tree T of F.

An integer vector $x \in \mathbb{Z}^E$ is the characteristic vector of a clique-connecting forest of $G = (V, E)$ if and only if x satisfies

$$\text{(CCFO)} \begin{cases} 0 \leq x_e \leq 1 & \text{for each } e \in E, \\ x(E(U)) \leq |U| - \begin{cases} 1 & \text{if } U \text{ is a clique of } G \\ 2 & \text{otherwise} \end{cases} & \text{for each nonempty } U \subseteq V \end{cases}$$

After showing in [14] that the separation problem over (CCFO) is polynomial, we investigated the *clique-connecting forest polytope* (that is, the convex-hull of characteristic vectors of clique-connecting forests).

Since $|F \cap S| \leq 1$ for any clique-connecting forest F and any induced star S (for instance, the edges in bold of the graph in Figure 3.5(a) form an induced star), the most natural valid inequalities that one may want to use for strengthening (CCFO) are

$$x(S) \leq 1 \text{ for any induced star } S$$

There are two ways for generalizing these inequalities.

The first is to consider a more general structure than induced stars.

Definition 11 *Let K be a not necessarily maximal clique of $G = (V, E)$ with at least two vertices. Call K-*complete set *a maximal set $Q \subseteq E(K) \cup \delta(K)$ such that*

(i) $uv \in Q$, $v \notin K \Rightarrow u'v \notin E$ for some vertex $u' \in K$ incident to no edge in $Q \cap \delta(K)$;

(ii) $uv, uw \in Q$, $v, w \notin K \Rightarrow vw \notin E$.

See for instance Figure 3.5(b).

(a) (b)

Figure 3.5: K-complete sets

Observe that induced stars are precisely the K-complete sets with $|K| = 2$. The following *complete sets inequalities* are valid:

$$x(Q) \leq |K| - 1 \text{ for any } K\text{-complete set } Q$$

The second way is to use the inequalities of the clique polytope (stable set polytope). Given a vertex u of G, let us denote each edge $uv \in \delta(u)$ by e_v. Observe that any valid inequality $\sum_{v \in N(u)} a_v x_v \leq \beta$ for the clique polytope of $G[N(u)]$ induces the valid inequality

$$\sum_{e_v \in \delta(u)} a_v x_{e_v} \leq \beta$$

for the clique-connecting forest polytope of G.

Definition 12 *A non-trivial facet* $a^\top x \leq \beta$ *of the clique polytope is called* degenerate *if there exists an edge* uv *such that:*

$$a(K) = \beta \quad \Longrightarrow \quad \textit{either } u \in K \textit{ or } v \in K \qquad \textit{(for every clique } K\textit{)}$$

Theorem 8 (C. [14]) *The complete sets inequalities and the inequalities induced by non-trivial and non-degenerate facets of the clique polytope are facets of the clique-connecting forest polytope.*

Chapter 4

Min-max relations

Let $G = (V, E)$ be a graph with possibly multiple edges.

Restricted to line-graphs $L(G)$, the coloring problem amounts to find the minimum number $\chi'(G)$ of matchings in the original graph G covering the edge-set E. Its combinatorial dual is the maximum degree $\Delta(G)$ of G. One has (except if G has a triangle and no degree-3 vertex)

$$\Delta(G) = \omega(L(G)) \leq \chi(L(G)) = \chi'(G)$$

Kőnig's edge-coloring theorem (1916) is:

$$\Delta(G) = \chi'(G) \text{ if } G \text{ is bipartite.}$$

A (titanic) generalization of this is of course the strong perfect graph theorem [9]. We give in the next section a new generalization. It is independent from the notion of perfectness since it involves matrices with entries in $\{0, \frac{1}{2}, 1\}$.

4.1 The star polytope

A *star of* G is a set $S \subseteq E$ contained in $\delta(v)$ for some vertex $v \in V$. The *star polytope of* G is the convex-hull of incidence vectors its stars.

The bipartite graph $B(G)$ of $G = (V, E)$ is the graph with vertex-set two copies V_1, V_2 of V where there are two edges u_1v_2, u_2v_1 for each edge $uv \in E$. To a subset D of edges of $B(G)$ corresponds naturally a vector $x \in \{0, \frac{1}{2}, 1\}^E$ defined as

$$x := \frac{1}{2}\chi^C + \chi^M \quad \text{where} \quad \begin{array}{rl} C := & \{uv \in E : u_1v_2 \in D \text{ or } u_2v_1 \in D \text{ exclusively}\}, \text{ and} \\ M := & \{uv \in E : u_1v_2 \in D \text{ and } u_2v_1 \in D\} \end{array}$$

In particular, any matching of $B(G)$ corresponds to a vector x where C is the vertex-disjoint union of paths and cycles, and where M is a matching. Call *pcm set* such a pair (C, M) (pcm stands for path, cycle, matching).

Since $\Delta(B(G)) = \Delta(G)$, Kőnig's edge-coloring theorem implies that the star polytope of any graph G is described by the following TDI system:

$$\text{(PCM)} \left\{ \begin{array}{ll} x_e \geq 0 & \text{for all } e \in E, \\ \frac{1}{2}x(C) + x(M) \leq 1 & \text{for all pcm set } (C, M) \text{ of } G \end{array} \right.$$

So we have a linear description of the star polytope of all graph.

A TDI system with integer right-and-side is also called a "min-max relation". A *minimal-TDI* system is such that the removing of any row either describes a strict subset, or is not TDI anymore. They may have two "best-possible" min-max relations associated with a full-dimensional 0-1 polytope:

1. its so-called *Schrijver system*, which is the unique minimal-TDI system with all entries integer describing it (it may have redundant inequalities),

2. a minimal-TDI facet-defining system with integer right-hand-side (it may have fractional left-hand-side).

Given an integer k and a graph $G = (V, E)$, a *k-matching of G* is a subset $M \subseteq E$ such that $|\delta(v) \cap M| \leq k$ for all vertex $v \in V$. (A 1-matching is a matching).

A *k-matching family* is a k-tuple $(M_1, \dots, M_k)$ where M_i is a i-matching and where the M_i's are pairwise vertex-disjoint.

A *k-matching covering of G* is a collection of edge subsets M_i, $i \in I$ covering E such that M_i is a $k(i)$-matching with $k(i) \leq k$. The *cost of a k-matching covering* is $\sum_{i \in I} k(i)$.

Let us consider the following system:

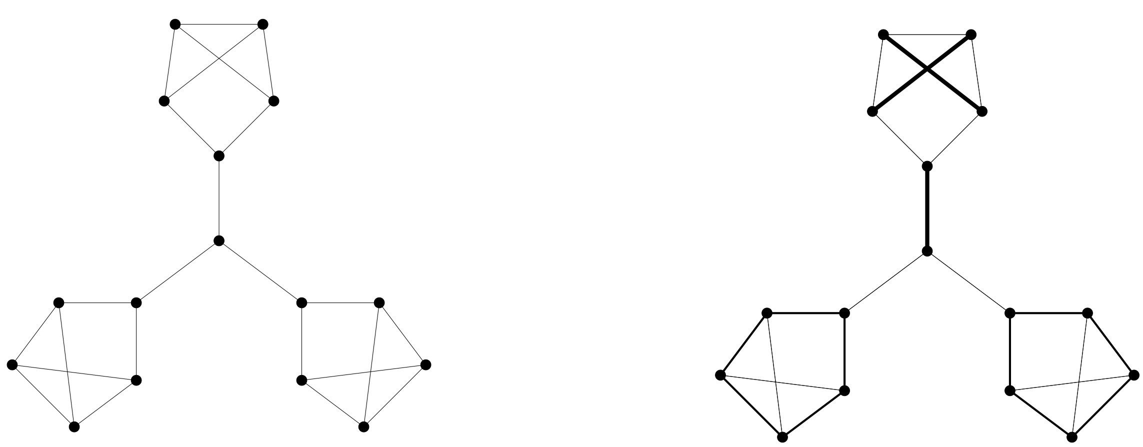
$$(k\text{-MAF}) \begin{cases} x_e \geq 0 & \text{for all } e \in E, \\ \sum_{i=1}^{k} (k - i + 1) x(M_i) \leq k & \text{for all } k\text{-matching family } (M_1, \dots, M_k) \text{ of } G \end{cases}$$

(2-MAF) is the system obtained from (PCM) by multiplying by 2 all pcm set inequalities.

Remark that

- The system (k-MAF) is TDI if and only if G (with duplication of edges allowed) admits a k-matching covering with cost $\Delta(G)$.

The 3-regular graph of Figure 4.1.(a) does not admits a 2-matching covering with cost 3. It follows that (2-MAF) is not TDI in general.



(a) (b)

Figure 4.1: 3-regular graph, without 2-matching covering of cost 3, covered by 3 ocm's

The Schrijver system of the star polytope is given by the minimum integer k such that G admits a k-matching covering with cost $\Delta(G)$. Trivially, any graph G admits a $\Delta(G)$-matching covering with cost $\Delta(G)$, and hence $1 \leq k \leq \Delta(G)$. For bipartite graphs $k = 1$ but, in general, the integer k might be so big that the min-max relation is trivial.

In fact the construction used in Figure 4.1.(a) can be extended to show that there is no fixed k such that every graph admits a k-matching covering with cost $\Delta(G)$. Figure 4.2.(a) shows a 5-regular graph which does not admit a 4-matching covering with cost 5.

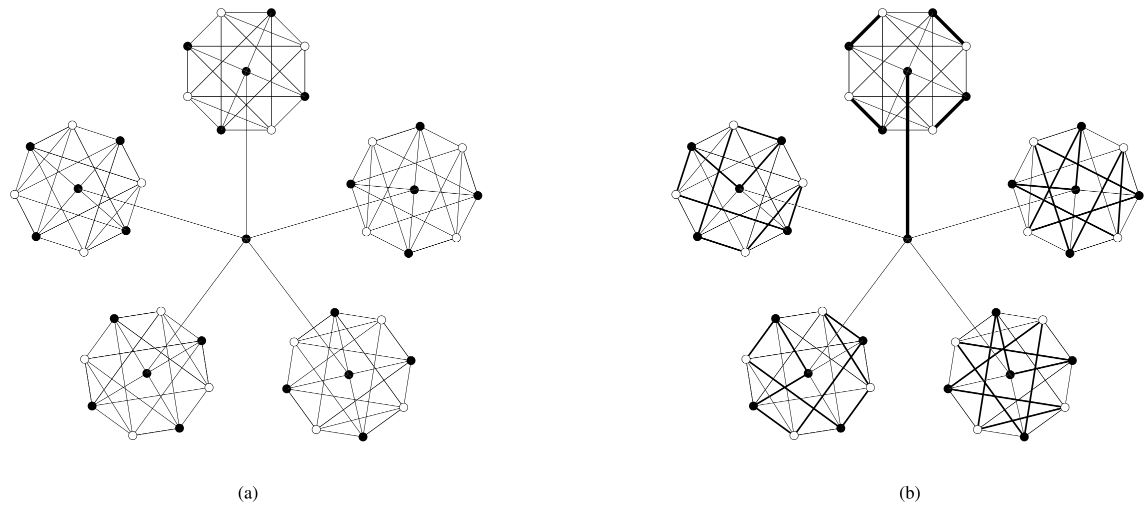


Figure 4.2: 5-regular graph, without 4-matching covering of cost 5, covered by 5 ocm's

Note that (PCM) is not facet-defining even if we take only inclusionwise maximal pcm sets. Indeed, any vector $x = \frac{1}{2}x(C) + x(M)$ where (C, M) is a pcm set is the half combination of two such vectors where (C, M) is a particular pcm set, namely an *ocm set*, that is, the vertex-disjoint union C of odd-circuits $C_1, \ldots, C_k$ and one matching M. Figure 4.1(b) shows an ocm set. See also Figure 4.2(b).

Definition 13 *An* ocm covering *is a collection of ocm sets (C_i, M_i), $i \in I$, such that, for all edge e, either $e \in M_i$ for some i, or $e \in C_i \cap C_j$ for some $i \neq j$, $i, j \in I$.*

For instance:

- The 3-regular graph in Figure 4.1(a) can be covered by the three ocm sets obtained by $2\pi/3$ rotation of the ocm set in Figure 4.1(b).
- The 5-regular graph in Figure 4.2(a) can be covered by the five ocm sets obtained by $2\pi/5$ rotation of the ocm set in Figure 4.2(b).

The following min-max relation implies that (PCM) restricted to maximal ocm sets is a minimal-TDI facet-defining system for the star polytope.

Theorem 9 (C. and Nguyen [23]) *Let G be a graph without loop and with multiple edges allowed, then the maximum size $\Delta(G)$ of a star is equal to the minimum number $|I|$ of ocm sets in an ocm covering of G.*

Theorem 9 restricted to bipartite graphs is exactly Kőnig's edge-coloring theorem. Note that it implies Kőnig's matching theorem (1931) $\tau(G) = \nu(G)$. Indeed, if $G = (V, E)$ is bipartite, then the star polytope is described by the non-negativity constraints together with $x(M) \leq 1$ for all matching M. It follows that

$$\mathbb{Q}^E \ni \bar{x} := \nu(G)^{-1} \cdot \mathbf{1} = \lambda_1 \chi^{S_1} + \ldots + \lambda_k \chi^{S_k} \qquad (\lambda_1 + \ldots + \lambda_k = 1,\ \lambda_i \geq 0)$$

for some stars $S_1, \ldots, S_k$ since $\bar{x}$ belongs to the star polytope of G. Since $\bar{x}(M) = 1 = \sum_i \lambda_i$ for any maximum matching M, then $|M \cap S_i| = 1$. Now Kőnig's matching theorem follows by induction since $\nu(G') = \nu(G) - 1$ where G' is obtained from G by removing some S_i.

Using the same polyhedral argument, one can show that there always exists an ocm set covering all maximum degree vertices. It suggests that a greedy algorithm might succeed in finding a minimum ocm covering. It is however not the case as shown by the graph of Figure 4.3(a). Indeed, suppose that we first choose the ocm set $(\{12, 13, 23\}, \emptyset)$ which misses no max-degree vertex. Then we are left with the graph of Figure 4.3(b) where the edges in dot-line are half-covered. Roughly speaking the max-degree is now 2. So $(\{23, 24, 34\}, \emptyset)$ is the only ocm set covering all max-degree vertices. But we have only one ocm set to use and it is impossible to cover the remaining even cycle of Figure 4.3(c).

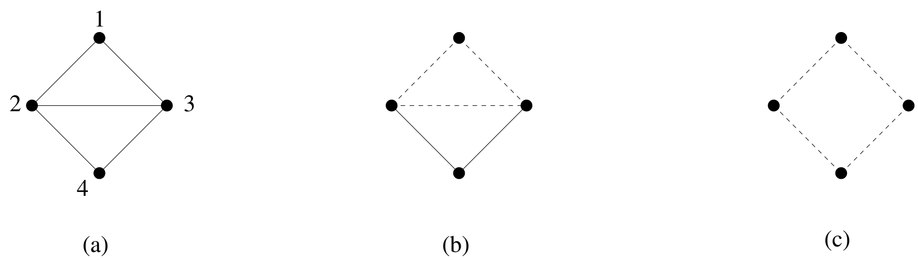


Figure 4.3: Greedy fails

4.2 Max-multiflow vs. min-multicut

In chapter 2, we have in fact considered *signed graphs*, that is, a pair (G, R) composed of a graph $G = (V, E)$ together with a subset of edges $R \subseteq E$.

In the study of complete bipartite subgraphs, the graph G was the rooted graph and R was the set of edges of the original graph and we looked for particular cycles C of G with $|C \cap R|$ is odd. For complete multipartite subgraphs we looked for cycles C of G with $|C \cap R| = 1$.

Now consider a signed graph $(G+H, R)$ where $G+H$ is the union of the supply graph $G = (V, E)$ and of the demand graph $H = (V, R)$ of the multiflow problem. An edge e of H is called a *demand*. An edge of G is called a *link* and its multiplicity is called its *capacity*. A cycle C of $G + H$ such that $C \cap R = \{e\}$ is called a *flow*; this flow satisfies the demand $e \in R$ on the path $C \setminus R$.

In the *maximum (integer) multiflow problem* one asks for the maximum number of edge-disjoint paths of G so that each path links the extremities of some edge of H.

A *multicut of* (G, H) is a subset of edges of G the removing of which disconnects any pair of vertices forming a demand.

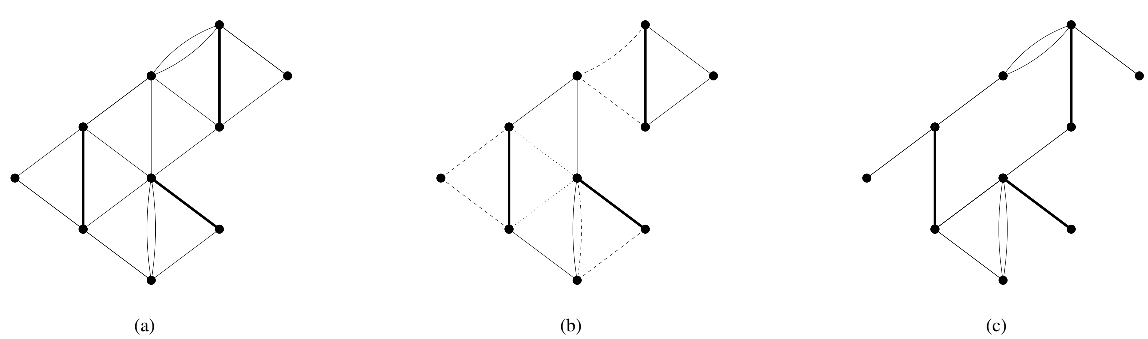


Figure 4.4: Multiflow and multicut

For instance:

- Figure 4.4(a) depicts a graph $G + H$ where the demand edges are in bold.
- Figure 4.4(b) depicts a multiflow of G, H with value 6, that is, 6 edge-disjoint paths of G each of them linking both extremities of some edge of H (in bold).
- Figure 4.4(c) depicts the graph $G + H$ after the removing of 6 edges of G letting no path of G linking both extremities of some edge of H; so the missing edges form a multicut.

We denote by max-mflow(G, H) the maximum value of a multiflow of G, H, and we denote by min-mcut(G, H) the minimum value of a multicut of G, H. As it is clear that

$$\text{max-mflow}(G, H) \leq \text{min-mcut}(G, H) \quad \text{for all } G, H$$

one can conclude that the pair G, H of Figure 4.4(a) satisfies:

$$\text{max-mflow}(G, H) = \text{min-mcut}(G, H)$$

This is of course not always the case see for instances Figure 4.5(a)-(b).

Figure 4.5: Max-multiflow is not equal to min-multicut

We proved the following:

Corollary 1 (C. [15]) *If $G + H$ is series-parallel, then* max-mflow$(G, H) =$ min-mcut(G, H).

The interest of Corollary 1 lays in the fact that it truly deals with the original multiflow problem and not only with the *feasibility version* of it, which arises if the demand is bounded.
In the feasibility version both G and H have multiple edges and the multiplicity of a demand edge e of H is called the *amount (of demand)* of e. One asks whether or not there are $|R|$ edge-disjoint *flows of* $G + H$. Such a flow collection is called a *multiflow satisfying the demand*.
The instance of Figure 4.4 is clearly feasible as there are $|R| = 3$ edge-disjoint flows. Also clearly, none of the instances of Figure 4.5(a)-(b) is feasible.

In Corollary 1, neither the number of demands is bounded, nor the *cut-condition*

$$\text{(cut-condition)} \qquad |D \setminus R| \geq |D \cap R| \quad \text{for all cut } D \text{ of } G + H$$

is asked. The cut-condition is a necessary condition for having a multiflow satisfying the demand.

4.2.1 The multicut polytope

A subset of edges $D \subseteq E(G)$ is a *multicut of* G if it is the set $D = \delta(V_1, \ldots, V_p)$ of all edges between the members of some partition $V_1, \ldots, V_p$ of $V(G)$.
The characteristic vectors of the multicuts of G are the integer vectors in

$$\text{(MCUT)} \begin{cases} 0 \leq x_e \leq 1 & \text{for all edge } e \text{ of } G, \\ x(C \setminus \{e\}) \leq x_e & \text{for all circuit } C \text{ of } G \text{ and for all edge } e \text{ of } C \end{cases}$$

Corollary 1 is a consequence of the following:

Theorem 10 (C. [15]) *A graph G is series-parallel if and only if (MCUT) is TDI.*

It was already known that (MCUT) is integral if and only if G is series-parallel [8]. Theorem 10 clearly improves this since TDIness implies integrality.

It was already known that

$$\text{(CONE)}\left\{\begin{array}{rl} x_e \geq 0 & \text{for all edge } e \text{ of } G, \\ x(C \setminus \{e\}) \leq x_e & \text{for all circuit } C \text{ of } G \text{ and for all edge } e \text{ of } C \end{array}\right.$$

is TDI if and only if G is series-parallel, see [37] (p. 505). To see that Theorem 10 improves this, let us consider a weighted graph (G, w) where $w \in \mathbb{Z}^{E(G)}$.

The weighted graph (G, w) can play the role of both unweighted graphs G, H: the edges e with $w_e > 0$ play the role of demand edges and the other edges, with non-positive weight, are links. The value $|w_e|$ is just the multiplicity of the edge e.

The cut-condition in (G, w) is that no cut has a positive weight (equivalently, no multicut). Observe that:

- the cut-condition holds if and only if the maximum of $w^\top x$ over $x \in \mathbb{Z}^{E(G)}$ in (CONE) is bounded (that is, $x = \mathbf{0}$ is the optimal solution);
- the dual of maximizing $w^\top x$ over (CONE) is feasible if and only if there is a multiflow which satisfies all the demands.
  (The multiflow is integer if the dual admits an integer solution.)

So

- (CONE) is TDI $\Leftrightarrow$ for all $w \in \mathbb{Z}^{E(G)}$, the cut-condition is sufficient for the existence of an integer multiflow satisfying the demand;
- (CONE) is integral $\Leftrightarrow$ for all $w \in \mathbb{Z}^{E(G)}$, the cut-condition is sufficient for the existence of a fractional multiflow satisfying the demand.

Besides one has

- (MCUT) is TDI $\Leftrightarrow$ for all $w \in \mathbb{Z}^{E(G)}$, the maximum weighted multicut is equal to the total amount of demand minus the maximum integer multiflow.

Hence, the TDIness of (MCUT) is stronger than that of (CONE); which shows that the new characterization of series-parallel graphs implied by Theorem 10 is stronger then the previous one.

4.2.2 The flow clutter

If the cut-condition is satisfied, then the multiflow problem becomes a packing problem on a binary hypergraph. Corollary 1 deals with the general case, that is with not necessarily binary hypergraphs. To see this let (G, R) be the signed graph with $G = (V, E)$ and $R \subseteq E$.

A circuit C of (G, R) is *odd* if $|C \cap R|$ is odd. The well-studied *odd-circuit clutter of* (G, R) is the hypergraph with vertex-set $E(G)$ and with (hyper) edge-set the odd-circuits of (G, R).

The *flow clutter of* (G, R) is the hypergraph with vertex-set $E(G)$ and with edge-set the circuits C with $|C \cap R| = 1$.

A crucial difference between the two clutters is that the odd-circuit clutter is *binary*, that is, the symmetric difference between an odd number of edges is contained in another edge, but not the flow clutter.

For instance, in the signed graph of Figure 4.6(a) one has $14 \triangle 25 \triangle 345 = 123$ where $14, 25, 345$ are edges of the flow clutter, and then also of the odd-circuit clutter, but 123 is only an edge of the odd-circuit clutter.

An hypergraph is *balanced* if no submatrix of its edge-vertex incidence matrix is the matrix of an odd-cycle. Even if G is series-parallel, the flow clutter, and then the odd-circuit clutter, might not be balanced: For the flow clutter of the signed graph of Figure 4.6(b), the vertices $4, 5, 6$ and the edges $156, 246, 345$ induce the matrix of a C_3.

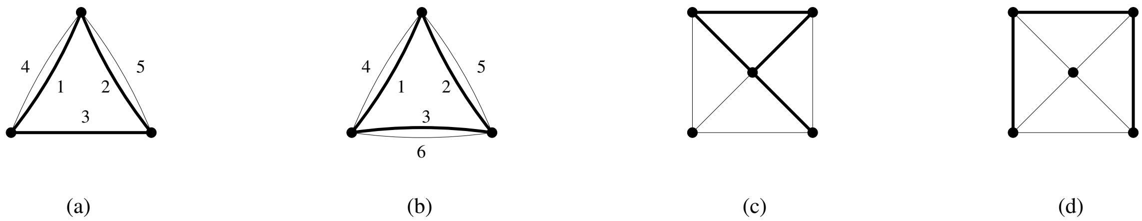


Figure 4.6: Packing/covering of odd circuits and flows

Let us define:

$$
\begin{aligned}
\nu_{\text{odd}}(G,R) &:= \text{ maximum number of edge-disjoint odd circuits of } (G,R)\\
\nu_{\text{flow}}(G,R) &:= \text{ maximum number of edge-disjoint flows of } (G,R)\\
\tau_{\text{odd}}(G,R) &:= \text{ minimum cardinality of a } T \subseteq E \text{ intersecting every odd circuit of } (G,R)\\
\tau_{\text{flow}}(G,R) &:= \text{ minimum cardinality of a } T \subseteq E \text{ intersecting every flow of } (G,R)
\end{aligned}
$$

For instances, in the signed graph of Figure 4.6(c), one has $\nu_{\text{odd}}(G,R) = 2 = \tau_{\text{odd}}(G,R)$ and one has $\nu_{\text{flow}}(G,R) = 1 \neq 2 = \tau_{\text{flow}}(G,R)$. In the signed graph of Figure 4.6(d), one has $\nu_{\text{odd}}(G,R) = 2 \neq 3 = \tau_{\text{odd}}(G,R)$ and one has $\nu_{\text{flow}}(G,R) = 2 = \tau_{\text{flow}}(G,R)$. Obviously, one has the four inequalities

$$
\begin{array}{ccc}
\nu_{\text{odd}}(G,R) & \leq & \tau_{\text{odd}}(G,R)\\
\text{IV} & & \text{IV}\\
\nu_{\text{flow}}(G,R) & \leq & \tau_{\text{flow}}(G,R)
\end{array}
\qquad \text{for all } (G,R)
$$

For all signed graphs (G,R) in which the cut-condition holds, then

$$\nu_{\text{flow}}(G,R) = \tau_{\text{flow}}(G,R) \quad \text{if and only if} \quad \nu_{\text{odd}}(G,R) = \tau_{\text{odd}}(G,R)$$

moreover, when the two equalities hold, then

$$\nu_{\text{flow}}(G,R) = \nu_{\text{odd}}(G,R) = |R| = \tau_{\text{flow}}(G,R) = \tau_{\text{odd}}(G,R).$$

Known classes of signed graphs (G,R) for which the odd-circuit clutter min-max relation holds are

$$\nu_{\text{odd}}(G,R) = \tau_{\text{odd}}(G,R) \quad \text{if} \quad \begin{array}{l} (G,R) \text{ has no odd-}K_4 \text{ minor [39], or} \\ (G,R) \text{ is Eulerian and has no odd-}K_5 \text{ minor [28].} \end{array}$$

Chapter 5

Further work

The concept of minimal forbidden structure in hypergraph, e.g. forbidden minor or forbidden subgraph, can be applied to another domain.

We explain our contribution in applying this concept to social choice theory, which deals with profile matrices instead of 0-1 matrices, in the next section. In the second section we list some problems.

5.1 Structure and algorithm in elections

Let C be a finite set of *candidate*. A vector is called a *voter* if it is a permutation of C.

A matrix $\mathcal{P}$ is called a *profile* if each column is a voter. A *subprofile* $\mathcal{P}'$ *of* $\mathcal{P}$ is a profile obtained from $\mathcal{P}$ by deleting candidates and voters.

For instance

$$\mathcal{P}' = \begin{pmatrix} a & c & b \\ b & b & a \\ c & a & c \end{pmatrix} \quad \text{is a subprofile of} \quad \mathcal{P} = \begin{pmatrix} a & c & b & b & a \\ b & d & a & a & c \\ c & b & c & d & b \\ d & a & d & c & d \end{pmatrix}$$

obtained by removing candidate d and the two last voters.

Throughout we let $\mathcal{P}$ be a profile with candidate set C and voter set V.

Let c be a candidate and v a voter. Denote the position of c in v by $\chi_v^c \in I := \{1, \dots, |C|\}$. For instance, if v is the second column in the above profile $\mathcal{P}$, then $\chi_v^a = 4$, $\chi_v^b = 3$, $\chi_v^c = 1$ and $\chi_v^d = 2$.

A subset $C' \subseteq C$ of k candidates is a *representative k-set* if it minimizes

$$\sum_{v \in V} \min_{c \in C'} \chi_v^c$$

The problem of determining a representative k-set is NP-hard in general. It becomes polynomial if $\mathcal{P}$ is *single-peaked*, that is, if there exists a path P with vertex-set C such that, for all voter $v \in V$, the subgraph of P induced by $\{c \in C : \chi_v^c \leq i\}$ is connected, for all $i \in I$.

For instance the above subprofile $\mathcal{P}'$ is single-peaked: take a path P with extremities a and c. The above profile $\mathcal{P}$ is not single-peaked because it has three distinct candidates in its last row.

An *interval of* $\mathcal{P}$ is a (non-trivial) subset of candidates which are consecutive on every voter of $\mathcal{P}$. For instance, the interval incidence matrix of

$$\mathcal{P} = \begin{pmatrix} a & a & f \\ b & c & g \\ c & b & h \\ d & d & e \\ e & e & d \\ f & h & c \\ g & g & a \\ h & f & b \end{pmatrix} \quad \text{is} \quad I(\mathcal{P}) = \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & & & & & \\ 1 & 1 & 1 & 1 & & & & \\ 1 & 1 & 1 & 1 & 1 & & & \\ & & & 1 & 1 & & & \\ & & & 1 & 1 & 1 & 1 & 1 \\ & & & & 1 & 1 & 1 & 1 \\ & & & & & 1 & 1 & \\ & & & & & 1 & 1 & 1 \\ & & & & & & 1 & 1 \end{pmatrix} \end{array}$$

Given a partition $\mathcal{C}$ of C into intervals, let $\mathcal{P}/\mathcal{C}$ be the subprofile obtained by removing all candidate but one in each interval of the partition.

Definition 14 *The* single-peaked width of $\mathcal{P}$ *is the maximum size of an interval in a partition* $\mathcal{C}$ *such that* $\mathcal{P}/\mathcal{C}$ *is single-peaked.*

Theorem 11 (C., Galand and Spanjaard [17, 18]) *The single-peaked width of any profile can be determined in a polynomial time. Moreover, a representative* k*-set can be found in a polynomial time in bounded single-peaked width profiles.*

We used a minimal forbidden structure characterization for the proof:

- A profile is single-peaked if and only if it has no subprofile isomorphic to:

$$\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}, \quad \begin{pmatrix} a & a & b \\ b & c & c \\ c & b & a \end{pmatrix}, \quad \begin{pmatrix} d & d \\ a & c \\ b & b \\ c & a \end{pmatrix}, \quad \begin{pmatrix} a & d \\ d & c \\ b & b \\ c & a \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} a & c \\ d & d \\ b & b \\ c & a \end{pmatrix}$$

This approach turns out to be successful to generalize positive results for another hard problem. Namely, finding a *Kemeny* voter, that is, a voter u (not necessarily belonging to V) minimizing the sum $\sum_{v \in V} d(u,v)$ of its distances with all voters of the profile $\mathcal{P}$. Here the *distance* $d(u,v)$ *of* u *with* v is

$$d(u,v) := \sum_{ab \text{ pair of } C} (\mathbf{1}^u_{ab} - \mathbf{1}^v_{ab})^2 \quad \text{where} \quad \mathbf{1}^u_{ab} := \begin{cases} 1 & \text{if } \chi^a_u < \chi^b_u \\ 0 & \text{otherwise} \end{cases}$$

The problem of determining a Kemeny voter is NP-hard in general. It becomes polynomial if $\mathcal{P}$ is single-peaked or *single-crossing*, that is, if there exists a path P with vertex-set V such that, for all ordered pair ab of candidates, the subgraph of P induced by $\{v \in V : \chi^a_v < \chi^b_v\}$ is connected.

It is NP-hard to determine the minimum number of intervals in a partition $\mathcal{C}$ such that $\mathcal{P}/\mathcal{C}$ is single-crossing.

Definition 15 *The* single-crossing width of $\mathcal{P}$ *is the maximum size of an interval in a partition* $\mathcal{C}$ *such that* $\mathcal{P}/\mathcal{C}$ *is single-crossing.*

Theorem 12 (C., Galand and Spanjaard [18]) *The single-crossing width of any profile can be determined in a polynomial time. Moreover, a Kemeny voter can be found in a polynomial time in bounded single-crossing (-peaked) width profiles.*

5.2 Open questions

Here is a list of problems with a various level of difficulty.

5.2.1 Graph classes

Sandwich line-graphs

A graph is a sandwich line-graph is if it is the graph with vertex-set the arc-set of an orientated graph $\vec{G}$ where two vertices are linked if they correspond to adjacent arcs which are not simplicial.

A sandwich line-graph can contain any graph H as induced subgraph. To see this, let $G = \overline{H}$ and add a universal tail u, so $\delta(u)$ induce H in $S(G)$. If the underlaying graph G is complete, the sandwich line-graph $S(\vec{G})$ coincide with the facility location graph (up to reversing the orientation) which is hard to recognize in general [2].

Problem 1 *Is it hard to recognize a sandwich line-graph?*

Complex threshold graphs

We can generalize the class of i-threshold graphs with weights taking values in the whole complex set.

A graph $G = (V, E)$ is *complex threshold* if there is a $t \in \mathbb{C}^V$ such that

$$uv \in E \;\Leftrightarrow\; \Re(t(u) + t(v)) - \Im(t(u))\Im(t(v)) > 0$$

Figure 5.1 shows two complex threshold graphs.

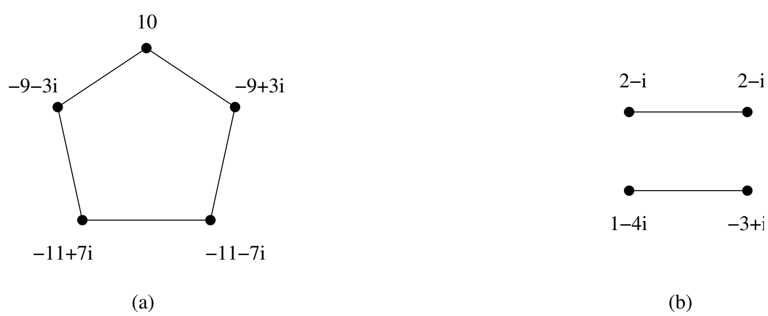


Figure 5.1: Complex threshold graphs

Notice that this graph class is closed under taking induced subgraphs and that it is not contained in the class of perfect graphs.

Problem 2 *What are the forbidden induced subgraphs for the class of complex threshold graphs?*

5.2.2 Polytopes

Degenerated facets of the stable set polytope

A facet $a^\top x \leq \beta$ of the stable set polytope is called *trivial* if the vector a has only one non-zero entries. A non-trivial facet is called *degenerate* if there exist two vertices u and v, non-adjacent, such that:

$$a(S) = \beta \quad \Rightarrow \quad \text{either } u \in S \text{ or } v \in S \qquad \text{(for every stable set } S\text{)}$$

Clearly clique, odd-circuit, and odd-wheel inequalities are non-trivial and non-degenerate facets.

Problem 3 *Is there a graph with a stable set polytope having a degenerate facet ?*

Minimally 2-edge-connected

A 2-edge-connected graph is *minimally 2-edge-connected* if and only if it contains no *Theta-graph*, that is, a cycle (not necessarily elementary) with one chord.

In K_4 the 2-edge-connected spanning subgraph polytope has dimension six. The dimension falls down to only two if one requires minimality. The Theta-graph inequalities can be separated using basic flow technique.

Problem 4 *Given a 2-edge-connected graph G and an edge e, is it possible to determine whether or not e belongs to at least one minimality 2-edge-connected spanning subgraph of G?*

Series-parallel graphs

Chen, Ding and Zang [6] proved that G is series-parallel if and only if (2CON) is box-TDI.

$$\text{(2CON)}\left\{\begin{array}{rl} x_e \geq 0 & \text{for all edge } e \text{ of } G, \\ \frac{1}{2}x(D) \geq 1 & \text{for all cut } D \text{ of } G \end{array}\right.$$

Such a characterization of series-parallel graphs could exists with a generalized max-multiflow/min-multicut relation.

Problem 5 *Is it true that (CONE) is box-TDI if and only if G is series-parallel?*

5.2.3 Grids and bicliques

We denote by $bc(G)$ the minimum number of bicliques covering the edges of G. Let $G_{p\times q}$ be the complete grid-graph on pq vertices with $p \leq q$.

Maximal bicliques of a grid-graph are 3- and 4-stars and 4-cycles. We proved that

$$\tau(G_{p\times q}) - 1 \leq bc(G_{p\times q}) \leq \tau(G_{p\times q}) = \left\lfloor \frac{pq}{2} \right\rfloor$$

and that, moreover, if p is odd, the right inequality holds with equality. A covering using only stars shows that the right inequality holds. The left inequality follows from a construction of half-integer dual solutions (with the ILP formulation of Chapter 2).

If p is even and if the Frobenius equation $(p-1)x_1 + (p+1)x_2 = q - p$ has a solution (that is, it is satisfied by some $x \in \mathbb{Z}^2_+$), one can construct a biclique covering showing that the left inequality holds with equality in this case.

Using Sylvester's formula (1884), the Frobenius equation has a solution if $q \geq p^2 - p$. However for p even and $q < p^2 - p$, we could not determine when the left or the right inequality holds with equality.

Problem 6 *Is it true that, if p is even and if $(p-1)x_1 + (p+1)x_2 \neq q - p$ for all $x \in \mathbb{Z}^2_+$, then $bc(G_{p\times q}) = \frac{pq}{2}$?*

Computational experiments show that for $G_{6\times 17}$ the gap between $\frac{pq}{2}$ and the optimum of the relaxation is greater than one.

5.2.4 Strong minors of the flow clutter

In a signed graph (G, R) the collection of odd circuits of (G, R) is closed under *resigning*, that is, resetting $R := R \triangle D$ for some cut D of G. A *minor of* (G, R) is a signed graph obtained from (G, R) by a series of deletion of edges, resigning, and contraction of unsigned edges (one can always resign so an edge becomes unsigned). So the minors of (G, R) are in 1-to-1 correspondence with the minors of its odd-circuit clutter.

An *odd-K_4* is a signed K_4 where each triangle is odd.

Problem 7 *Does $\nu_{flow}(G, R) = \tau_{flow}(G, R)$ holds if (G, R) has no minor odd-K_4?*

For an odd-K_4, the incidence matrices of the odd-circuit clutter and of the flow clutter are respectively

$$Q_6 = \begin{pmatrix} 1 & 1 & 1 & & & \\ 1 & & & 1 & 1 & \\ & 1 & & 1 & & 1 \\ & & 1 & & 1 & 1 \end{pmatrix} \quad \text{and} \quad Q_6^- = \begin{pmatrix} 1 & & & 1 & 1 & \\ & 1 & & 1 & & 1 \\ & & 1 & & 1 & 1 \end{pmatrix}$$

It is well known that Q_6 is minimally non-Mengerian. On the contrary, Q_6^- is non-Mengerian but not minimally as it has a minor $\mathcal{C}_3^2$. Thus no minor operation in (G, R) can correspond to minor operation in its flow clutter.

Call *strong minor of* (G, R) a signed graph that arises from (G, R) by a series of deletion of edges in E and contraction of edges in $E \setminus R$.

Call *flow odd-wheel* a signed odd wheel with signed edges the odd cycle (so the odd-K_4 with three signed edges is a flow odd wheel). Call *flow-K_4* the odd-K_4 with two signed edges.

Problem 8 *Does $\nu_{flow}(G, R) = \tau_{flow}(G, R)$ holds if (G, R) has no strong minor flow odd-wheel nor flow-K_4?*

Chapter 6

Conclusion

Let us summarize our results.

1. Let $H = (E, \mathcal{C})$ be the minimal non-biclique clutter of a graph $G = (V, E)$, then

$$\omega(G) - 1 \leq \max_{C \in \mathcal{C}} |C| \leq \omega(G) \quad \text{(with equality at the right if } \omega(G) \text{ is odd)}$$

2. Let $H = (E, \mathcal{C})$ be the minimal non-multiclique clutter of a graph $G = (V, E)$, then

$$|C| = 2 \quad (\forall C \in \mathcal{C}) \qquad \Longleftrightarrow \qquad G \text{ is fan- and prism-free}$$

3. Several difficult problems over a graph $G = (V, E)$, including finding a maximum edge-weight vertex-induced bipartite subgraph, can be formulate as finding $\tau_w(H)$ where $H = (E, \mathcal{C})$ is a clutter such that

$$\text{Finding a } C \in \mathcal{C} \text{ minimizing } w(C) \text{ is polynomial in the size of } G \qquad (\forall w \in \mathbb{Q}_+^E)$$

4. For any sandwich line-graph $S(\vec{G})$ of G, where $\vec{G}$ has no 3-dicycle, then

$$\alpha(G) + \overline{\chi}(S(\vec{G})) = |V(G)| = \alpha(S(\vec{G})) + \overline{\chi}(G)$$

5. The operator Φ_β for graph parameters β satisfies

$$\alpha(G) \leq \Phi_{\overline{\chi}_f}(G) \leq \overline{\chi}_f(G) \quad \text{and} \quad \vartheta(G) \leq \Phi_\vartheta(G) \leq \overline{\chi}(G) \qquad \text{(for all graph } G\text{)}$$

6. Let $G = (V, E)$ be a graph with $c \in \mathbb{Z}_+^V$. There is a sandwich line-graph $S(\vec{G})$ with vertex-weight $w \in \mathbb{Z}_+^E$ such that

$$\alpha_w(S(\vec{G})) + \overline{\chi}_{\max}(G, c) = \mathbf{1}^\top c$$

7. Let $G = (V, E)$ be a graph with a partition $\mathcal{V} = \{V_i,\ i \in I\}$ of V and let $M(G, \mathcal{V})$ be the incidence matrix of the clutter $H = (I, \mathcal{J})$ where $J \in \mathcal{J}$ if G has a stable set intersecting all V_j, $j \in J$. Then

$$\begin{array}{rrcl} & M(G,\mathcal{V}) \text{ is minimally non-conformal} & \Longrightarrow & \mathcal{V} = \{V_1, V_2, V_3\} \text{ with } |V_i| \leq 2 \\ \text{and,} & M(G,\mathcal{V}) \text{ is perfect for all } \mathcal{V} & \Longleftrightarrow & G \text{ is } i\text{-threshold} \end{array}$$

8. The complete sets inequalities and the inequalities induced by non-trivial and non-degenerate facets of the clique polytope are facets of the clique-connecting forest polytope.

9. Let $\chi''(G)$ be the minimum number of ocm sets covering G (so $\chi''(G) = \chi'(G)$ if G bipartite). Then

$$\Delta(G) = \chi''(G), \quad \text{for all graph } G$$

In other words, the following system is minimal-TDI and describes the star polytope of all graphs

$$(\text{OCM}) \begin{cases} x_e \geq 0 & \text{for all } e \in E, \\ \frac{1}{2}x(C) + x(M) \leq 1 & \text{for all maximal ocm set } (C, M) \text{ of } G \end{cases}$$

10. If $G + H$ is series-parallel, then the maximum integer multiflow is equal to the minimum multicut. More generally, a graph G is series-parallel if and only if the following system is TDI

$$(\text{MCUT}) \begin{cases} 0 \leq x_e \leq 1 & \text{for all edge } e \text{ of } G, \\ x(C \setminus \{e\}) \leq x_e & \text{for all circuit } C \text{ of } G \text{ and for all edge } e \text{ of } C \end{cases}$$

11. The single-peaked width of any profile can be determined in a polynomial time. Moreover, a representative k-set can be found in a polynomial time in bounded single-peaked width profiles.

12. The single-crossing width of any profile can be determined in a polynomial time. Moreover, a Kemeny voter can be found in a polynomial time in bounded single-crossing or -peaked width profiles.

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Résumé

Nous étudions des structures interdites minimales que l'on peut voir comme des homologues sophistiqués des cycles impairs dans les graphes signés. Initiée avec les structures associées aux bicliques, cette approche s'avère être plus générale. Elle relie plusieurs problèmes complexes de graphes.

De plus, elle nous conduit à une version chromatique de deux identités bien connues de Gallai. Ces identités chromatiques permettent d'opérer sur les paramètres classiques liés à la coloration de graphes, on déplace ainsi la relaxation semi-définie positive de Lovász vers le nombre chromatique et la relaxation linéaire vers le nombre clique. Une application concerne un problème de coloration spécial, où le coût d'une couleur est le poids maximum d'un sommet de cette couleur. Par ailleurs, nous montrons comment sont liés le polytope des stables et celui des forêts clique-connectantes, qui sont à la base des identités de Gallai chromatiques.

Pour un autre problème de coloration, où l'on colore un sous-graphe induit par exactement un sommet par cluster, il est difficile de reconnaitre si la matrice 0-1 associée est conforme, c'est-à-dire, si c'est une matrice clique-sommet. Après avoir prouvé que, pour toutes les matrices provenant des graphes clusterisés, il suffit d'une conformalité locale pour avoir la conformalité totale, nous caractérisons les graphes avec une propriété de perfection définie non seulement pour les graphes, mais pour les graphes clusterisés.

Concernant le problème de coloration d'arêtes, nous donnons une relation min-max décrivant le polytope des étoiles, cette relation est optimale puisqu'elle correspond à un système linéaire minimal. Un autre relation min-max, de type max-multiflot/min-multicoupe, est donnée pour caractériser les graphes série-parallèles, et elle améliore une caractérisation similaire antérieure.

Enfin, nous étudions l'intérêt d'une approche par structures interdites minimales dans un autre domaine que la théorie des graphes: les problèmes électoraux en choix social. Puis nous énumérons quelques questions que nous avons laissées ouvertes.