A tropical approach to bilevel programming and an application to price incentives in telecom networks

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Motivation

- Bilevel programming :

$$\max_{y \in \mathcal{Y}} \quad F(x^*, y) \text{ s.t. } G(x^*, y) \leq 0$$

with x^* solution of:

$$\max_{x \in \mathcal{X}} \quad f(x, y) \text{ s.t. } g(x, y) \leq 0$$

- Game theory: Stackelberg equilibrium
- Player Y with strategies in $\mathcal{Y}$: "leader"
- Player X with strategies in $\mathcal{X}$: "follower"

Study of bilevel models

- A major class of models of pricing (Marcotte, Labbé, Brotcorne)
- Well-studied (Dempe)
- Generally *NP*-hard
- General approach based on replacing the low level program by its KKT conditions : non convex, non linear programs, sometimes mixed...

A special class of bilevel problems

We study the optimistic solution of :

$$\max_{y \in \mathbb{R}^n} \quad f(C^T x^*, y)$$

with x^* solution of:

$$\max_{x \in \mathcal{P}} \quad \langle \rho + Cy, x \rangle$$

where $\mathcal{P}$ integer polytope of $\mathbb{R}^k$, $C \in \mathcal{M}_{n,k}(\mathbb{Z})$ and $\rho \in \mathbb{R}^k$

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$$\max_{x \in \mathcal{E}(\mathcal{P})} \quad \langle \rho + Cy, x \rangle$$

where $\mathcal{E}(\mathcal{P})$: extreme points of $\mathcal{P}$.

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Low-level problem: Tropical polynomial

- In this talk: new approach based on tropical geometry for bilevel programming

How far is it possible to use the tropical structure to solve the bilevel problem?

- Tropical geometry applied to economy: introduced by Baldwin, Klemperer (2014), Yu, Tran (2015) for an auction problem
- Discrete convexity applied to economy: Danilov, Koshevoy, Murota (2001)

Tropical geometry

Tropical polynomials and hypersurfaces

- Tropical algebra: consider the max-plus semifield $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ defined by:

$$a \oplus b = \max(a, b) \quad \text{and} \quad a \odot b = a + b$$

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Example: $2 \oplus 3 = 3 \qquad 2 \odot 3 = 5$

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- "Tropical polynomial" : function P, continuous, piecewise-linear with integer slopes and convex:

$$P(x) = \max_{1 \leq k \leq p} (a_k + \langle c_k, x \rangle) = \text{"} \bigoplus_{1 \leq k \leq p} a_k x^{c_k} \text{"}$$

with $c_k \in \mathbb{Z}^n$ and $x \in \mathbb{R}^n$.

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  with $c_k \in \mathbb{Z}^n$ and $x \in \mathbb{R}^n$.

- "Tropical hypersurface" : set of points where P is not differentiable (= set of points where the maximum is attained at least "twice")

Example: tropical line

Ex (polynomial of degree 1): "$P(x, y) = \max(x, y, 0)$"

Example: tropical line

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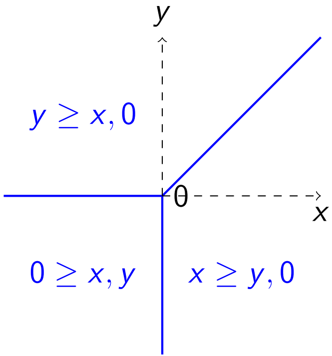


Subdivision

Subdivision $\mathcal{S}$ of a polyhedron Δ: collection of polyhedra (called *cells*) such that:

1. $\bigcup_{\mathcal{C}\in\mathcal{S}} \mathcal{C} = \Delta$
2. $\forall \mathcal{C} \neq \mathcal{C}' \in \mathcal{S},\ \mathrm{ri}(\mathcal{C}) \cap \mathrm{ri}(\mathcal{C}') = \emptyset$
3. $\forall \mathcal{C} \in \mathcal{S}$, $\forall F$ facet of $\mathcal{C}$, $F \in \mathcal{S}$.

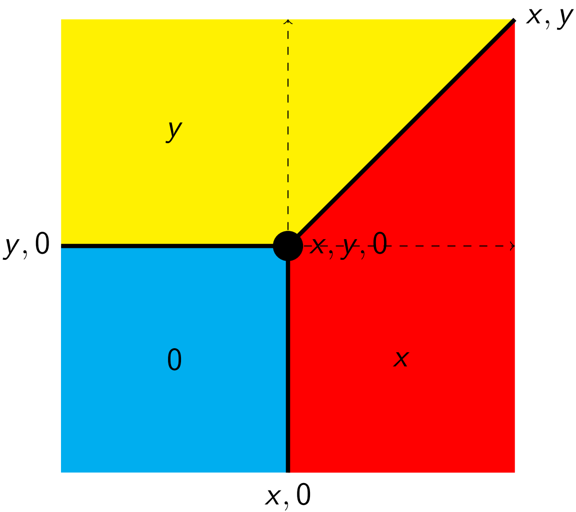
Remark: $\forall \mathcal{C} \neq \mathcal{C}' \in \mathcal{S},\ \mathcal{C} \cap \mathcal{C}' \in \mathcal{S}$ or $\mathcal{C} \cap \mathcal{C}' = \emptyset$.

Tropical polynomial : defines a subdivision $\mathcal{S}$ of $\mathbb{R}^n$!

Cells of $\mathcal{S}$: set of points corresponding to the same maximal monomial(s).

Subdivision

Ex : $P(x, y) = \max(x, y, 0)$



Subdivision $\mathcal{S}$:

- 3 two-dimensional polyhedra
- 3 one-dimensional polyhedra
- 1 zero-dimensional polyhedron

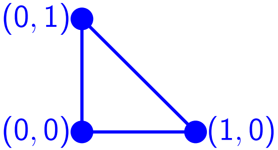
Newton polytope

Tropical polynomial $P(x) = \max_{1 \leq k \leq p} (a_k + \langle c_k, x \rangle)$.

Newton polytope $\mathrm{New}(P)$: convex hull of vectors c_k.

Example: $\max(x, y, 0) = \max(1x + 0y, 0x + 1y, 0x + 0y)$.

Newton polytope: convex hull of
$(1, 0)$, $(0, 1)$ and $(0, 0)$.

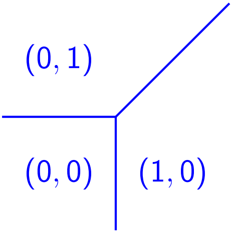


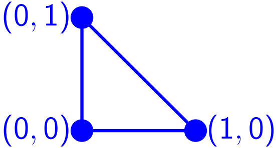
Dual subdivision

Theorem (Sturmfels 1994)

There exists a bijection ϕ between the subdivision $\mathcal{S}$ of $\mathbb{R}^n$ defined by a tropical polynomial P and a subdivision $\mathcal{S}'$ of the Newton polytope of P.

Δ: d-dimensional polyhedron in $\mathcal{S} \leftrightarrow \phi(\Delta)$: $(n-d)$-dimensional polyhedron in $\mathcal{S}'$.





Tropical representation of linear programming

Solving a linear program $\Leftrightarrow$ evaluate a tropical polynomial !

$$\max_{\alpha \in \mathcal{P}} \langle x, \alpha \rangle = \max_{\alpha \in \mathcal{E}(\mathcal{P})} \langle x, \alpha \rangle = \text{"} \bigoplus_{\alpha \in \mathcal{E}(\mathcal{P})} x^{\alpha} = P(x)$$

$\mathcal{E}(\mathcal{P}) \subset \mathbb{Z}^n$: set of vertices of $\mathcal{P}$.

$\mathcal{P}$: Newton polytope of P.

Low-level problem

Here: value of each low level problem is a tropical polynomial :

$$\max_{x \in \mathcal{P}} \langle \rho + Cy, x \rangle = \max_{x \in \mathcal{E}(\mathcal{P})} \langle y, C^T x \rangle + \langle \rho, x \rangle = \max_{z \in C^T \mathcal{E}(\mathcal{P})} \langle y, z \rangle + \varphi(z)$$

$$= \bigoplus_{z \in C^T \mathcal{E}(\mathcal{P})} \varphi(z) \odot y^{\odot z}$$

where $\varphi(z) = \max_{x \in \mathcal{P},\, C^T x = z} \langle \rho, x \rangle$ concave function in z.

Newton polytope: convex hull of $C^T \mathcal{E}(\mathcal{P}) = C^T \mathcal{P}$.

Low-level problem

$\mathcal{S}$: subdivision associated to this tropical polynomial.
ϕ: bijection between $\mathcal{S}$ and a subdivision of $C^T\mathcal{P}$.

Minimal cell containing $y \in \mathbb{R}^n$: $\mathcal{C}_y = \bigcap\{\mathcal{C} \in \mathcal{S} \mid y \in \mathcal{C}\}$.

Lemma

For $y \in \mathbb{R}^n$, let $\mathcal{C}_y$ be the minimal cell containing y. Then:

$$\arg\max_{z \in C^T\mathcal{P}} [\langle y, z\rangle + \varphi(z)] = \phi(\mathcal{C}_y)$$

Cell enumeration for the bilevel problem

Recall the continuous bilevel problem:

$$\max_{y \in \mathbb{R}^n} \quad f(C^T x^*, y)$$

with x^* solution of:

$$\max_{x \in \mathcal{P}} \quad \langle \rho + Cy, x \rangle$$

where $\mathcal{P}$ integer polytope of $\mathbb{R}^k$, $C \in \mathcal{M}_{n,k}(\mathbb{Z})$ and $\rho \in \mathbb{R}^k$, and the discrete one:

$$\max_{y \in \mathbb{R}^n} \quad f(C^T x^*, y)$$

with x^* solution of:

$$\max_{x \in \mathcal{E}(\mathcal{P})} \quad \langle \rho + Cy, x \rangle$$

Cell enumeration for the bilevel problem

We recall the continuous bilevel problem:

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with z^* solution of:

$$\max_{z \in C^T \mathcal{E}(\mathcal{P})} \quad \langle y, z \rangle + \varphi(z)$$

Cell enumeration for the bilevel problem

We recall the continuous bilevel problem:

$$\max_{y \in \mathbb{R}^n} \quad f(z^*, y)$$

subject to:

$$z^* \in \phi(\mathcal{C}_y)$$

and the discrete one:

$$\max_{y \in \mathbb{R}^n} \quad f(z^*, y)$$

subject to:

$$z^* \in \phi(\mathcal{C}_y) \cap C^T \mathcal{E}(\mathcal{P})$$

Cell enumeration for the bilevel problem

Continuous bilevel problem: $\max_{y \in \mathbb{R}^n} f(z^*, y)$ s.t. $z^* \in \phi(\mathcal{C}_y)$
Discrete: $\max_{y \in \mathbb{R}^n} f(z^*, y)$ s.t. $z^* \in \phi(\mathcal{C}_y) \cap C^T \mathcal{E}(\mathcal{P})$

Define $\mathcal{S}_n = \{\mathcal{C} \in \mathcal{S} \mid \mathcal{C}$ is a n-dimensional polyhedron$\}$.

Theorem (ABEGK 2018)

The continuous bilevel programming problem is equivalent to:

$$\max_{\mathcal{C} \in \mathcal{S}} \max_{y \in \mathcal{C},\, z \in \phi(\mathcal{C})} f(z, y)$$

The discrete bilevel programming problem is equivalent to:

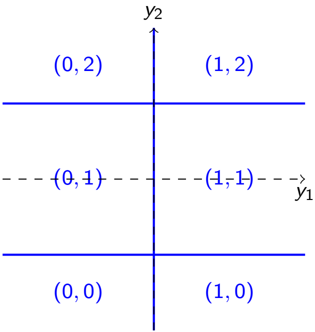
$$\max_{\mathcal{C} \in \mathcal{S}_n} \max_{y \in \mathcal{C},\, z \in \phi(\mathcal{C})} f(z, y)$$

Example

Consider $n = 2$ and $k = 4$.
Low-level : $\max_{x \in \mathcal{P}} \langle \rho + Cy, x \rangle$ with $\mathcal{P} = \{x \in [0,1]^4 \mid x_1 + x_3 \leq 1\}$ and

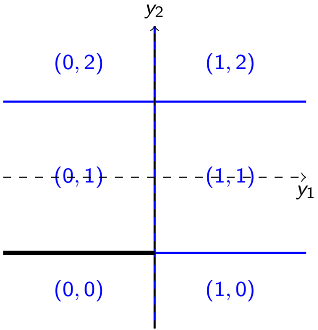
$$\rho = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{et} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Tropical polynomial : $\max(0, y_1, y_2 + 1, y_1 + y_2 + 1, 2y_2, y_1 + 2y_2)$



Example

Bilevel:
$\max_y f(z^*, y) = -(z_1^*)^2 - \langle y, z^* \rangle$
with $z^* = C^T x^*$ and x^* solution of the low-level problem.
Maximization over each cell
Optimal solution : 1 (black line)



Consequences

- Number of subproblems : number of cells in the subdivision
- Each subproblem : optimization over a separable domain in z and y
- f linear in y : only to consider the 0-dimensional cells of $\mathcal{S}$
- f linear in z : only to consider the 0-dimensional cells of $\phi(\mathcal{S})$ (i.e. the n-dimensional cells of $\mathcal{S} \Leftrightarrow$ the cells of $\mathcal{S}_n$).

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How many cells in $\mathcal{S}$?

Number of cells

We define $\Delta_d^n = \{x \in (\mathbb{R}_+)^n \mid \sum_{i=1}^n x_i \leq d\}$.

Theorem

Suppose $C^T\mathcal{P} \subset \Delta_d^n$. *Then:*

$$|\mathcal{S}_n| \leq \binom{n+d}{n} \qquad |\mathcal{S}| \leq \sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^i\binom{j}{i}\binom{n+(j+1-i)d}{n}.$$

$\Rightarrow$ Number of cells in $\mathcal{S}_n$ and in $\mathcal{S}$ in $\mathcal{O}(d^n)$: polynomial for fixed n.

Decomposition theorem

Important case: f does not depend on y.

Theorem (ABEG 2017)

The continuous bilevel problem is equivalent to:

1. *Find* $z^* \in \arg\max_{z \in C^T\mathcal{P}} f(z)$
2. *Find* x^* *and* y^* *such that* $z^* = C^T x^*$ *and* $x^* \in \arg\max_{x \in \mathcal{P}} \langle \rho + Cy^*, x \rangle$.

The discrete bilevel problem is equivalent to:

1. *Find* $z^* \in \arg\max_{z \in C^T\mathcal{E}(\mathcal{P})} f(z)$
2. *Find* x^* *and* y^* *such that* $z^* = C^T x^*$ *and* $x^* \in \arg\max_{x \in \mathcal{E}(\mathcal{P})} \langle \rho + Cy^*, x \rangle$.

Application: congestion problem in telecom networks

Motivation (Orange)

- Demand for using massive contents (video, downloads...)with mobile phones increases rapidly $\Rightarrow$ Spectrum crisis, congestion in different places at different hours
- Aim of providers: guarantee a sufficient quality of service (QoS)

One leverage: price incentives to shift the data consumption of the customers in time

Problem of Orange: How far is it possible to use price incentives to shift customers data consumption?

State of art

Smart data pricing problems (see Sen, Joe-Wong, Ha, Chiang 2014 for an overview)

Similar approaches:

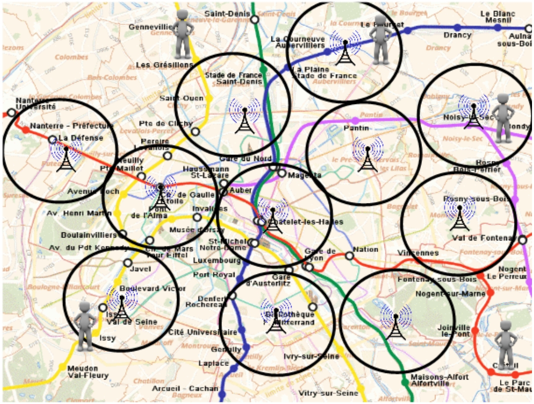
- Price incentives model depending on time (TUBE), implementation (Ha, Sen, Joe-Wong, Im, Chiang 2012)
- Model with anticipation of downloads (Tadrous, Eriylmaz, El Gamal 2013)
- Bilevel model taking the mobility into account (Ma, Liu, Huang 2014)

Congestion problem

Day divided in T time slots, network divided in L cells, K customers in the network.

Network at 3 AM.

No active customers.

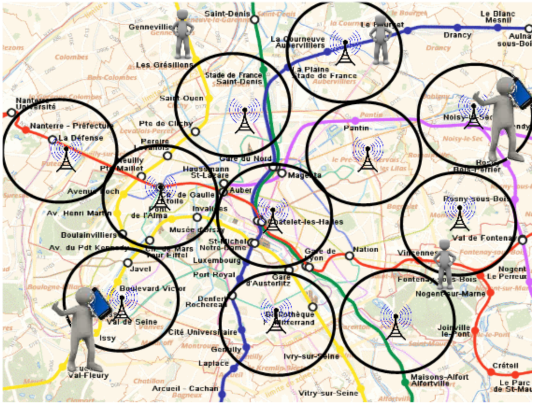


Congestion problem

Day divided in T time slots, network divided in L cells, K customers in the network.

Network at 7 AM.

- Issy : 1
- Noisy : 1

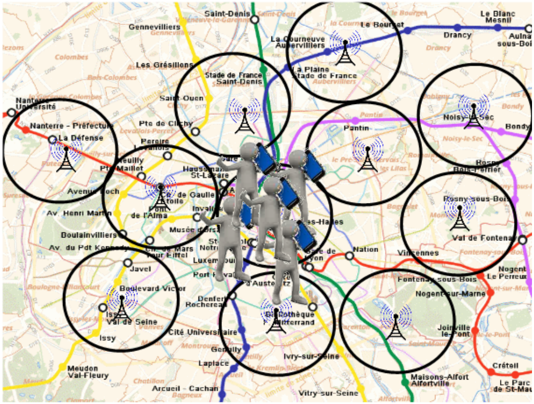


Congestion problem

Day divided in T time slots, network divided in L cells, K customers in the network.

Network at 9 AM.

- Chatelet : 5 !!!

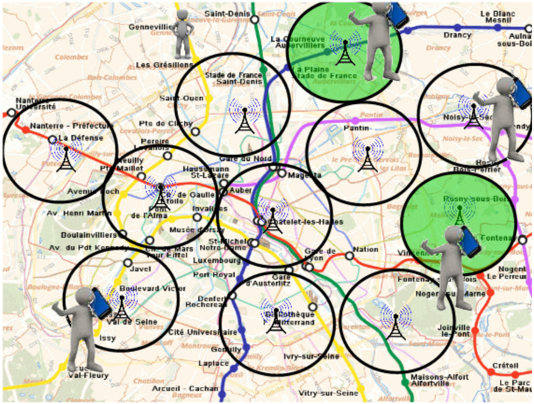


Congestion problem

Provider: proposes price incentive $y(t, \ell) \in \mathbb{R}_+$ at time t in the cell ℓ
Each customer: has a fixed total demand distributed on a day

Network at 7 AM.

- Issy : 1
- Noisy : 1
- La Courneuve : 1
- Vincennes : 1



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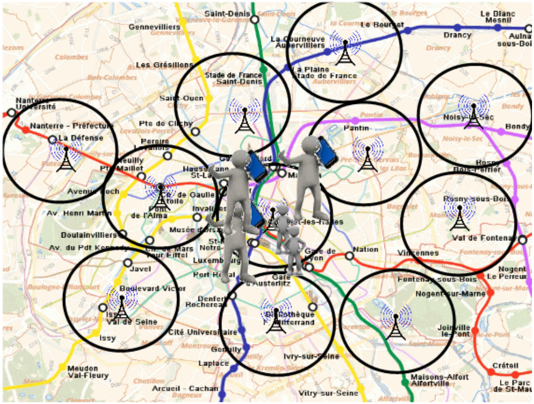
Network at 9 AM.

- Chatelet: only 3

  ...



A simplified customer model

Simple model: binary consumptions $u_k(t)$

A customer k wants to maximize his utility function:

$$\Rightarrow \max \sum_t \left[\rho_k(t) + y(t, L_k(t))\right] u_k(t)$$

subject to $u_k(t) \in \{0;1\}, \quad \sum_t u_k(t) = R_k$

- ρ_k: preferences of customer k
- L_k: trajectory of customer k
- R_k: number of requests made by k in one day
- Set of times during which the customer k does not want to consume : $\{t \mid \rho_k(t) = -\infty\}$

Example

Ex: $T = 5$, $L = 1$, $\rho_1 = [1, 2, -1, -\infty, -1]$, $R_1 = 2$.

Without incentives:

ρ_1	1	2	-1	$-\infty$	-1
u_1	1	1	0	0	0

With incentives $y = [0, 1, 3, 4, 0]$:

$\rho_1 + y$	1	3	2	$-\infty$	-1
u_1	0	1	1	0	0

The provider model

He wants to balance the traffic:

$$\Rightarrow \min s(N) = \sum_{t,\ell} s_{t,\ell}(N(t,\ell))$$

where:

- $N(t,\ell)$: total number of active customers at time t and cell ℓ:

$$N(t,\ell) = \sum_{k} u_k^*(t)\mathbb{1}(L_k(t) = \ell)$$

  and u_k^* optimal solution of the customer k.
- $s_{t,\ell}$: some convex functions

Example

Ex: $T = 5$, $L = 1$, $K = 2$.

- $\rho_1 = [1, 2, -1, -\infty, -1]$, $R_1 = 2$
- $\rho_2 = [3, 1, -\infty, 0, 3]$, $R_2 = 3$

Without incentives:

$$\left.\begin{array}{l} u_1 = [1, 1, 0, 0, 0] \\ u_2 = [1, 1, 0, 0, 1] \end{array}\right\} N = [2, 2, 0, 0, 1]$$

With incentives $y = [0, 1, 3, 4, 0]$:

$$\left.\begin{array}{l} u_1 = [0, 1, 1, 0, 0] \\ u_2 = [1, 0, 0, 1, 1] \end{array}\right\} N = [1, 1, 1, 1, 1]$$

Bilevel model

It leads to a bilevel model.
Provider : proposes discounts y.

- Low-level problem (each customer k)

$$\max_{u_k \in \mathcal{F}_k} \langle \rho_k + y, u_k \rangle \tag{1}$$

  Extreme points of a hypersimplex $\mathcal{F}_k = \{u_k \in \{0;1\}^n \mid \sum_i u_k(i) = R_k,\ (\rho_k(i) = -\infty \Rightarrow u_k(i) = 0)\}$
- High-level problem (provider)

$$\min_{y \in \mathbb{R}_+^n} s(N) = \sum_i s_i(N_i) \tag{2}$$

  with $N_i = \sum_k u_k^*(i)$ and $\forall k,\ u_k^*$ solution of (1).

Bilevel model

We study the following model:

$$\min_{y \in \mathbb{R}^n_+} \sum_{i=1}^{n} s_i(N_i)$$

$$s.t. \begin{cases} N_i = \sum_k u_k^*(i) \\ \forall k,\ u_k^* \in \arg\max_{u_k \in \mathcal{F}_k} \langle \rho_k + y, u_k \rangle \end{cases}$$

Bilevel model

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$\forall k, \forall u_k \in \mathcal{F}_k$, $\sum_i u_k(i)$ constant $\Rightarrow$ same solution for the low-level problems by replacing y by $y + \alpha(1, \dots, 1)$ for all $\alpha \in \mathbb{R}$.

Bilevel model

Bilevel model:

$$\min_{y \in \mathbb{R}^n} s(N^*)$$
$$s.t. \begin{cases} N^* = C^T u^* \\ u^* \in \arg\max_{u \in \mathcal{E}(\mathcal{P})} \langle \rho + Cy, u \rangle \end{cases}$$

with $C^T = [I_n \dots I_n] \in \mathcal{M}_{n,Kn}(\mathbb{Z})$, $\mathcal{E}(\mathcal{P}) = \mathcal{F}_1 \times \dots \mathcal{F}_K$, $\rho^T = \left[\rho_1^T \dots \rho_K^T\right] \in \mathbb{R}^{Kn}$ and $s(N^*) = \sum_i s_i(N_i^*)$.

Bilevel model

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$$\min_{y \in \mathbb{R}^n} s(N^*)$$
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Discrete bilevel problem

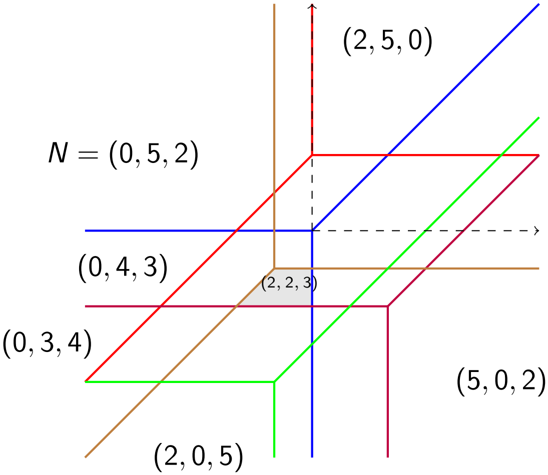
Tropical representation of customers' responses

- Value of the low-level problem for each customer : tropical polynomial
- Arrangement of tropical hypersurfaces $\Rightarrow$ Hypersurface corresponding to the product of different tropical polynomials.

Global example with 5 customers:

- $\rho_1 = [0, 0, 0]$, $R_1 = 1$
- $\rho_2 = [0, -1, 0]$, $R_2 = 2$
- $\rho_3 = [-1, 1, 0]$, $R_3 = 1$
- $\rho_4 = [1/2, 1/2, 0]$, $R_4 = 2$
- $\rho_5 = [1/2, 2, 0]$, $R_5 = 1$.

Tropical representation of customers' responses



Bilevel model

Bilevel model:

$$\min_{y \in \mathbb{R}^n} s(N^*)$$
$$s.t. \begin{cases} N^* = C^T u^* \\ u^* \in \arg\max_{u \in \mathcal{E}(\mathcal{P})} \langle \rho + Cy, u \rangle \end{cases}$$

with $C^T = [I_n \dots I_n] \in \mathcal{M}_{n,Kn}(\mathbb{Z})$, $\mathcal{E}(\mathcal{P}) = \mathcal{F}_1 \times \dots \mathcal{F}_K$, $\rho^T = \left[\rho_1^T \dots \rho_K^T\right] \in \mathbb{R}^{Kn}$ and $s(N^*) = \sum_i s_i(N_i^*)$.

- Discrete bilevel problem
- High-level problem does not depend on y.

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- Discrete bilevel problem
- High-level problem does not depend on y.

$\Rightarrow$ Decomposition theorem

Decomposition theorem

Theorem (Akian, Bouhtou, E., Gaubert, 2017)

The optimal value y^* of the bilevel program can be obtained by:

1. *Computing N^* optimal solution of* $\min_{N \in \sum_k \mathcal{F}_k} s(N)$
2. *Finding y^* and $u_k^* \in \mathcal{F}_k$ such that:*

$$N^* = \sum_k u_k^*$$

$$\forall k, u_k^* \in \arg\max_{u_k \in \mathcal{F}_k} \langle \rho_k + y^*, u_k \rangle$$

Decomposition theorem

Theorem (Akian, Bouhtou, E., Gaubert, 2017)

The optimal value y^* of the bilevel program can be obtained by:

1. *Computing N^* optimal solution of $\min_{N \in \sum_k \mathcal{F}_k} s(N)$* *POLYNOMIAL ???*
2. *Finding y^* and $u_k^* \in \mathcal{F}_k$ such that:* *POLYNOMIAL*

$$N^* = \sum_k u_k^*$$

$$\forall k, u_k^* \in \arg\max_{u_k \in \mathcal{F}_k} \langle \rho_k + y^*, u_k \rangle$$

High-level problem

Minimizing a convex function over $\sum_k \mathcal{F}_k$

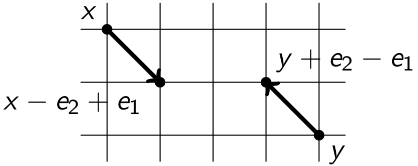
Tool: discrete convexity ! (developed by Danilov, Koshevoy and Murota)

M-convex set

Consider $(e_1, \dots, e_n)$ the canonical basis of $\mathbb{R}^n$.

Definition

A set $E \subset \mathbb{Z}^n$ is M-convex if $\forall x, y \in E, \forall i$ such that $x_i > y_i$, $\exists j$ such that $x_j < y_j$ with $x - e_i + e_j \in E$ and $y + e_i - e_j \in E$



M-convex set

Example: $\mathcal{F}_k$ is a M-convex set for all k

$$
\begin{aligned}
x &= (0,0,1,1,0,0,1) \\
y &= (0,1,1,0,1,0,0) \\
x - e_4 + e_5 &= (0,0,1,0,1,0,1) \\
y + e_4 - e_5 &= (0,1,0,1,0,0,1)
\end{aligned}
$$

Theorem (Murota, 1996)

The Minkowski sum of M-convex sets is a M-convex set.

Corollary

$\sum_k \mathcal{F}_k$ *is a M-convex set.*

M-convex function

Definition

Function $f : \mathbb{Z}^n \mapsto \mathbb{R} \cup \{+\infty\}$ M-convex iff $\forall x, y \in \text{dom}(f), \forall i$ such that $x_i > y_i$, $\exists j$ such that $x_j < y_j$ verifying:

$$f(x) + f(y) \geq f(x - e_i + e_j) + f(y + e_i - e_j)$$

Theorem (Murota, 1996)

A separable convex function defined on a M-convex set is a M-convex function

$\Rightarrow$: High-level problem : minimization of a M-convex function $s + \chi_{\sum_k \mathcal{F}_k}$.

Minimization of a M-convex function

Theorem (Murota, 1996)

For a M-convex function, local optimality guarantees global optimality in sense that:

$$\forall y \in dom\ f, f(x) \leq f(y) \Leftrightarrow \forall i, j, f(x) \leq f(x - e_i + e_j)$$

Theorem (Shioura, 1998)

The minimization of a M-convex function over $\mathbb{Z}^n$ can be achieved in polynomial time in the dimension n.

$\Rightarrow$ Bilevel problem can be solved in POLYNOMIAL TIME !

Greedy algorithm for M-convex minimization

Simple greedy algorithm, generally pseudo-polynomial, polynomial in our case, for solving the high-level problem:

1. Take $N \in \sum_k \mathcal{F}_k$
2. Compute i, j such that:

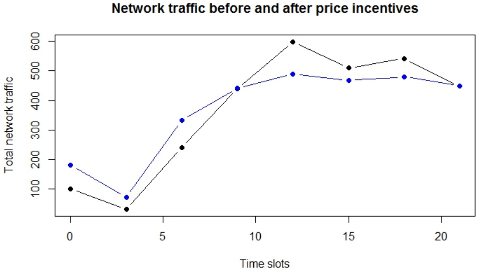
$$s(N - e_i + e_j) = \min_{u,v \text{ with } N - e_u + e_v \in \sum_k \mathcal{F}_k} s(N - e_u + e_v)$$

3. If $s(N - e_i + e_j) \geq s(N)$ then $N^* := N$
4. Else $N := N - e_i + e_j$ and go back to 1

Numerical results

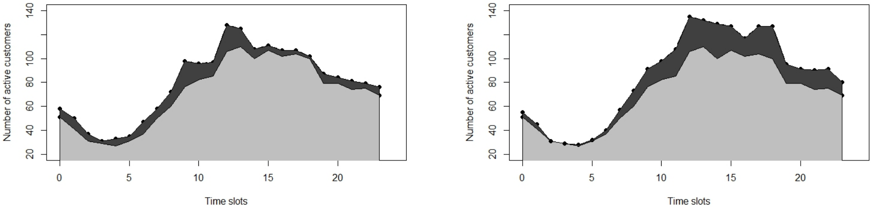
Example on real data with 8 time slots, 43 cells: $n = 344$. More than 2000 customers ($K > 2000$). $s(N) = \sum_i N_i^2$

Case	Optimal value	Most loaded cell
Without incentives	47 189	60
With incentives	35 499	31



Other numerical results

More developed and realistic telecom model: take into account different kind of customers, different applications . . .
Discounts only for download. Network with more than 2000 customers in 43 cells. Day divided in 24 hours.

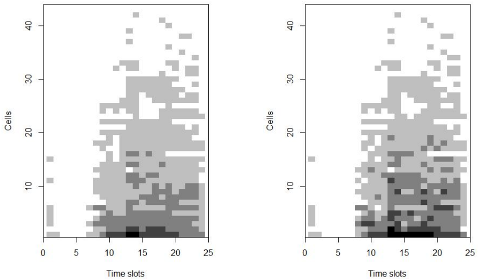


With incentives Without incentives

Figure: Active customers in the most loaded cell

Other numerical results

With incentives



Without incentives

Satisfaction of customers. Gray levels characterize the quality of service from white (very good quality) to black (very bad)

Conclusion

- Decomposition approach for solving a class of bilevel problems thanks to tropical geometry
- Complexity bounds of the method
- Application to a concrete problem

Next step:

- Improve the bounds
- Obtain more precise results in the case of separable low-levels
- Try to develop a "pivoting" algorithm

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