Spatial Decomposition/Coordination Methods for Stochastic Optimal Control Problems

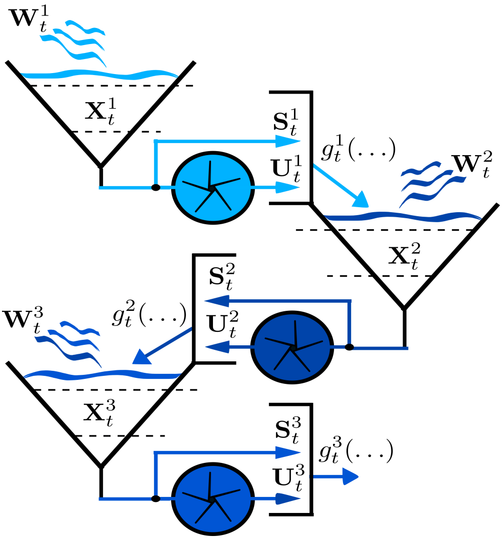
Practical aspects and theoretical questions

J.-C. Alais, P. Carpentier, J-Ph. Chancelier,
M. De Lara, V. Leclère

École des Ponts ParisTech

3 November 2014

Large scale storage systems stand as powerful motivation



To make a long story short

We look after strategies as solutions of large scale stochastic optimal control problems,
for example, the optimal management over a given time horizon
of a large amount of dynamical production units

- To obtain decision strategies (closed-loop controls),
  we use Dynamic Programming or related methods
  - **Assumption**: Markovian case
  - **Difficulty**: curse of dimensionality
- To use decomposition/coordination techniques, we have to deal with the information pattern of the stochastic optimization problem

A long-term effort in our group

1976 A. Benveniste, P. Bernhard, G. Cohen, "On the decomposition of stochastic control problems", *IRIA-Laboria research report*, No. 187, 1976.

1996 P. Carpentier, G. Cohen, J.-C. Culioli, A. Renaud, "Stochastic optimization of unit commitment: a new decomposition framework", *IEEE Transactions on Power Systems*, Vol. 11, No. 2, 1996.

2006 C. Strugarek, "Approches variationnelles et autres contributions en optimisation stochastique", *Thèse de l'ENPC*, mai 2006.

2010 K. Barty, P. Carpentier, P. Girardeau, "Decomposition of large-scale stochastic optimal control problems", *RAIRO Operations Research*, Vol. 44, No. 3, 2010.

2014 V. Leclère, "Contributions to decomposition methods in stochastic optimization", *Thèse de l'Université Paris-Est*, juin 2014.

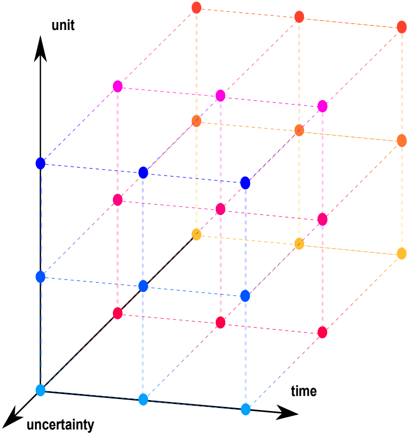
Lecture outline

1. Decomposition and coordination
   - A bird's eye view of decomposition methods
   - (A brief insight into Progressive Hedging)
   - Spatial decomposition methods in the deterministic case
   - The stochastic case raises specific obstacles
2. Dual approximate dynamic programming (DADP)
   - Problem statement
   - DADP principle and implementation
   - Numerical results on a small size problem
3. Theoretical questions
   - Existence of a saddle point
   - Convergence of the Uzawa algorithm
   - Convergence w.r.t. information
4. Conclusion

Decomposition-coordination: divide and conquer

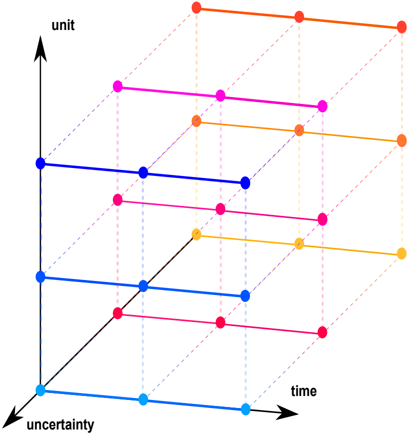
- Spatial decomposition
  - Multiple players with their local information
  - Scales: local / regional / national /supranational
- Temporal decomposition
  - A state is an information summary
  - Time coordination realized through Dynamic Programming, by value functions
  - Hard nonanticipativity constraints
- Scenario decomposition
  - Along each scenario, sub-problems are deterministic (powerful algorithms)
  - Scenario coordination realized through Progressive Hedging, by updating nonanticipativity multipliers
  - Soft nonanticipativity constraints

Couplings for stochastic problems



$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_{\omega} L_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1})$$

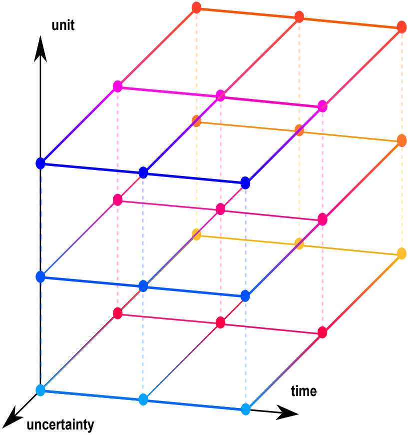
Couplings for stochastic problems: in time



$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_\omega L_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1})$$

$$\text{s.t.} \quad \mathbf{x}_{t+1}^i = f_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1})$$

Couplings for stochastic problems: in uncertainty

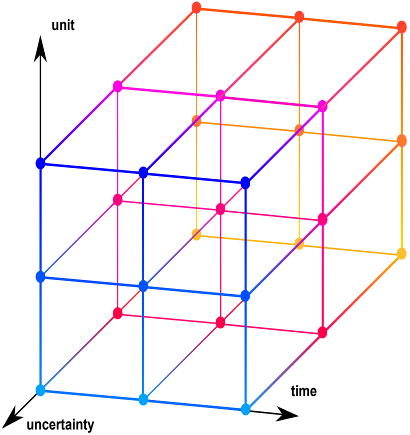


$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_{\omega} L_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1})$$

$$\text{s.t.} \quad \mathbf{x}_{t+1}^i = f_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1})$$

$$\mathbf{u}_t^i = \mathbb{E}\left(\mathbf{u}_t^i \;\middle|\; \mathbf{w}_1, \ldots, \mathbf{w}_t\right)$$

Couplings for stochastic problems: in space



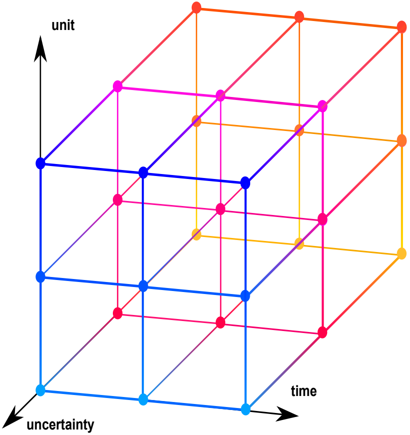
$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_{\omega} L_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1})$$

s.t. $\mathbf{x}_{t+1}^i = f_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1})$

$$\mathbf{u}_t^i = \mathbb{E}\left(\mathbf{u}_t^i \;\middle|\; \mathbf{w}_1, \ldots, \mathbf{w}_t\right)$$

$$\sum_i \theta_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i) = 0$$

Can we decouple stochastic problems?



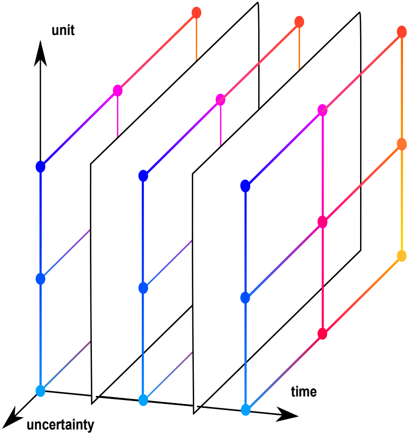
$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_{\omega} L_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1})$$

$$\text{s.t.} \quad \mathbf{x}_{t+1}^i = f_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1})$$

$$\mathbf{u}_t^i = \mathbb{E}\left(\mathbf{u}_t^i \;\middle|\; \mathbf{w}_1, \ldots, \mathbf{w}_t\right)$$

$$\sum_{i} \theta_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i) = 0$$

Decompositions for stochastic problems: in time



$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_{\omega} L_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1})$$

s.t. $$\mathbf{x}_{t+1}^i = f_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1})$$

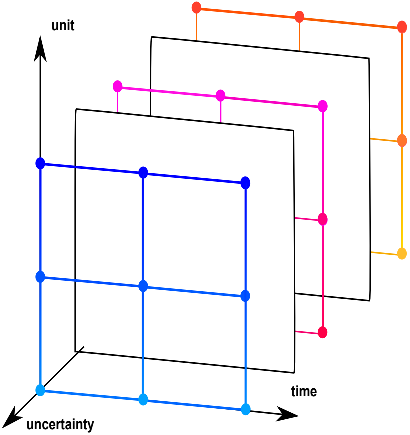
$$\mathbf{u}_t^i = \mathbb{E}\left(\mathbf{u}_t^i \;\middle|\; \mathbf{w}_1, \ldots, \mathbf{w}_t\right)$$

$$\sum_{i} \theta_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i) = 0$$

Dynamic Programming

Bellman (56)

Decompositions for stochastic problems: in uncertainty



$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_{\omega} L_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1})$$

$$\text{s.t. } \mathbf{x}_{t+1}^i = f_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1})$$

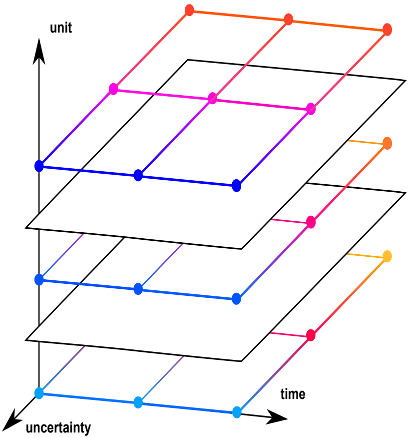
$$\mathbf{u}_t^i = \mathbb{E}\Big(\mathbf{u}_t^i \;\Big|\; \mathbf{w}_1, \ldots, \mathbf{w}_t\Big)$$

$$\sum_{i} \theta_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i) = 0$$

Progressive Hedging

Rockafellar - Wets (91)

Decompositions for stochastic problems: in space



$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_{\omega} L_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1})$$

$$\text{s.t.} \quad \mathbf{x}_{t+1}^i = f_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1})$$

$$\mathbf{u}_t^i = \mathbb{E}\left(\mathbf{u}_t^i \;\middle|\; \mathbf{w}_1, \ldots, \mathbf{w}_t\right)$$

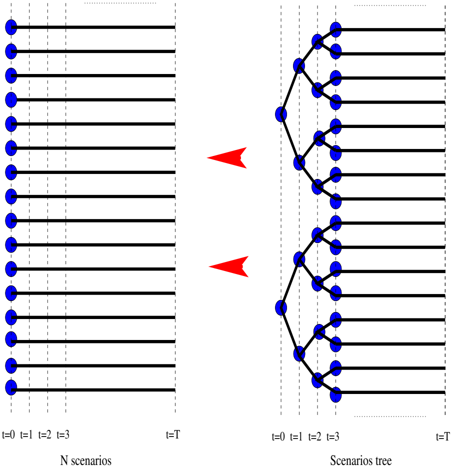
$$\sum_{i} \theta_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i) = 0$$

Dual Approximate
Dynamic Programming

Outline of the presentation

1. Decomposition and coordination
   - A bird's eye view of decomposition methods
   - (A brief insight into Progressive Hedging)
   - Spatial decomposition methods in the deterministic case
   - The stochastic case raises specific obstacles
2. Dual approximate dynamic programming (DADP)
   - Problem statement
   - DADP principle and implementation
   - Numerical results on a small size problem
3. Theoretical questions
   - Existence of a saddle point
   - Convergence of the Uzawa algorithm
   - Convergence w.r.t. information
4. Conclusion

Non-anticipativity constraints are linear



- From tree to scenarios (comb)
- Equivalent formulations of the non-anticipativity constraints
  - pairwise equalities
  - all equal to their mathematical expectation
- Linear structure

$$\mathbf{u}_t = \mathbb{E}\Big(\mathbf{u}_t \;\Big|\; \mathbf{w}_1, \ldots, \mathbf{w}_t\Big)$$

Progressive Hedging stands as a scenario decomposition method by dualizing the non-anticipativity constraints

- When the criterion is strongly convex,
  we use an algorithm "à la Uzawa"
  to obtain a scenario decomposition
- When the criterion is linear,
  Rockafellar - Wets (91) propose to use an augmented Lagrangian,
  and obtain the Progressive Hedging algorithm

Data: Initial multipliers $\{\{\lambda_t^{(0)}(\omega)\}_{t=0}^{T-1}\}_{\omega\in\Omega}$ and mean control $\{\overline{U}_n^{(0)}\}_{n\in\mathcal{T}}$;

Result: optimal feedback;

repeat

forall the *scenario* $\omega \in \Omega$ **do**

Solves the deterministic minimization problem for scenario ω with a measurability penalization, and obtain optimal control $\mathbf{u}^{(k+1)}$;

Update the mean controls

$$\overline{u}_n^{(k+1)} = \frac{\sum_{\omega\in n} \mathbf{u}_t^{(k+1)}(\omega)}{|n|}$$

Update the measurability penalization with

$$\lambda_t^{(k+1)}(\omega) = \lambda_t^{(k)}(\omega) + \rho\big(U_t(\omega)^{(k+1)} - \overline{u}_{n_t(\omega)}^{(k+1)}\big)$$

until $\mathbf{u}_t - \mathbb{E}\big(u_t^i \bigm| \mathbf{w}_1, \ldots, \mathbf{w}_t\big) = 0$;

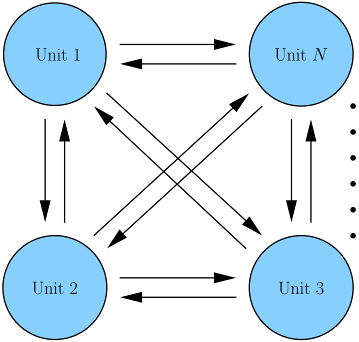
1. Decomposition and coordination
   - A bird's eye view of decomposition methods
   - (A brief insight into Progressive Hedging)
   - Spatial decomposition methods in the deterministic case
   - The stochastic case raises specific obstacles

2. Dual approximate dynamic programming (DADP)
   - Problem statement
   - DADP principle and implementation
   - Numerical results on a small size problem

3. Theoretical questions
   - Existence of a saddle point
   - Convergence of the Uzawa algorithm
   - Convergence w.r.t. information

4. Conclusion

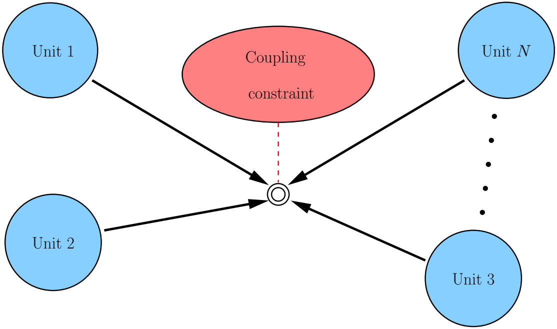
Decomposition and coordination



Interconnected units

- The system to be optimized consists of interconnected subsystems
- We want to use this structure to formulate optimization subproblems of reasonable complexity
- But the presence of interactions requires a level of coordination
- Coordination iteratively provides a local model of the interactions for each subproblem
- We expect to obtain the solution of the overall problem by concatenation of the solutions of the subproblems

Example: the "flower model"

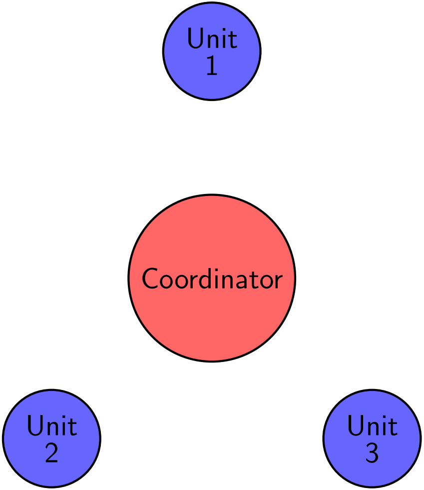


$$\min_u \sum_{i=1}^{N} J_i(u_i)$$

$$\text{s.t.} \quad \sum_{i=1}^{N} \theta_i(u_i) = \theta$$

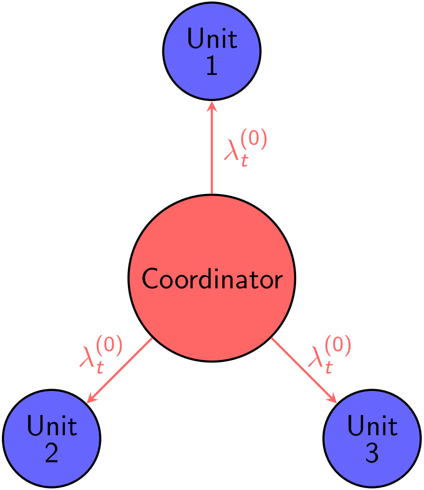
Unit Commitment Problem

Intuition of spatial decomposition



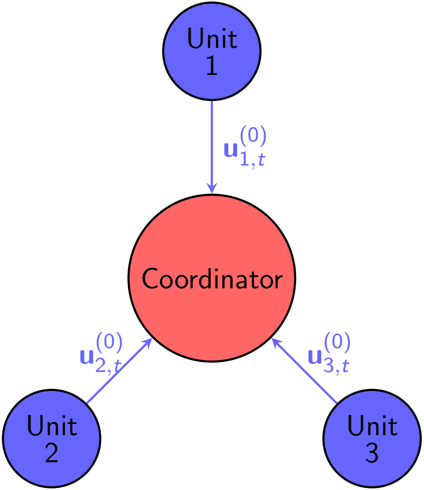
- Purpose: satisfy a demand with N production units, at minimal cost
- Price decomposition
  - the coordinator sets a price λ_t
  - the units send their production $\mathbf{u}_t^{(i)}$
  - the coordinator compares total production and demand, and then updates the price
  - and so on...

Intuition of spatial decomposition



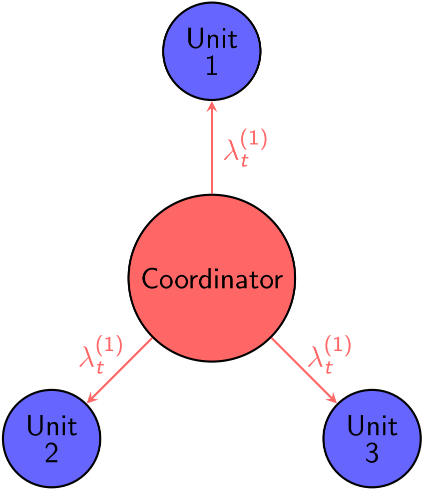
- Purpose: satisfy a demand with N production units, at minimal cost
- Price decomposition
  - the coordinator sets a price λ_t
  - the units send their production $\mathbf{u}_t^{(i)}$
  - the coordinator compares total production and demand, and then updates the price
  - and so on...

Intuition of spatial decomposition



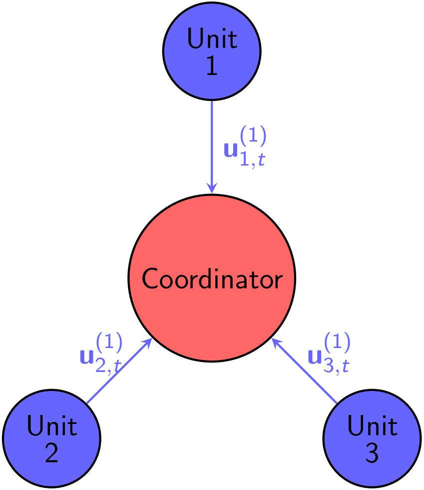
- Purpose: satisfy a demand with N production units, at minimal cost
- Price decomposition
  - the coordinator sets a price λ_t
  - the units send their production $\mathbf{u}_t^{(i)}$
  - the coordinator compares total production and demand, and then updates the price
  - and so on...

Intuition of spatial decomposition



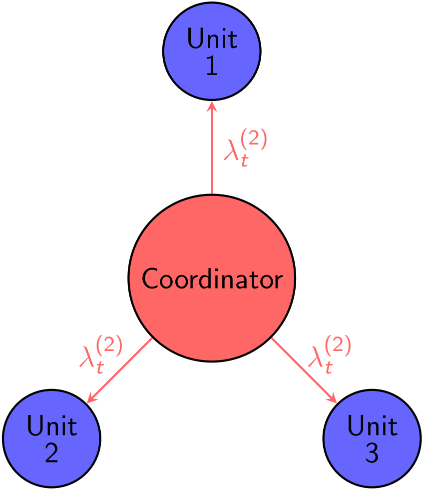
- Purpose: satisfy a demand with N production units, at minimal cost
- Price decomposition
  - the coordinator sets a price λ_t
  - the units send their production $\mathbf{u}_t^{(i)}$
  - the coordinator compares total production and demand, and then updates the price
  - and so on...

Intuition of spatial decomposition



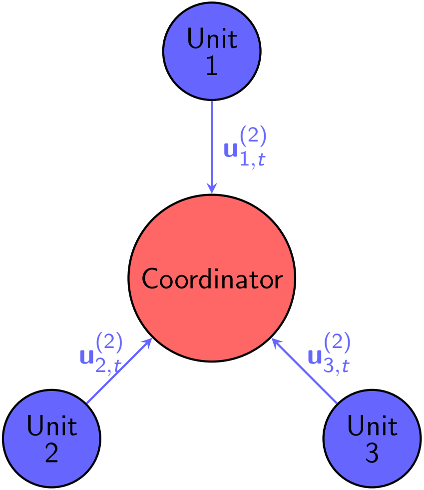
- Purpose: satisfy a demand with N production units, at minimal cost
- Price decomposition
  - the coordinator sets a price λ_t
  - the units send their production $\mathbf{u}_t^{(i)}$
  - the coordinator compares total production and demand, and then updates the price
  - and so on...

Intuition of spatial decomposition



- Purpose: satisfy a demand with N production units, at minimal cost
- Price decomposition
  - the coordinator sets a price λ_t
  - the units send their production $\mathbf{u}_t^{(i)}$
  - the coordinator compares total production and demand, and then updates the price
  - and so on...

Intuition of spatial decomposition



- Purpose: satisfy a demand with N production units, at minimal cost
- Price decomposition
  - the coordinator sets a price λ_t
  - the units send their production $\mathbf{u}^{(i)}_t$
  - the coordinator compares total production and demand, and then updates the price
  - and so on...

Price decomposition relies on dualization

$$\min_{u_i \in \mathcal{U}_i, i=1\dots N} \sum_{i=1}^{N} J_i(u_i) \quad \text{subject to} \quad \sum_{i=1}^{N} \theta_i(u_i) - \theta = 0$$

1. Form the Lagrangian and assume that a saddle point exists

$$\max_{\lambda \in \mathcal{V}} \min_{u_i \in \mathcal{U}_i, i=1\dots N} \sum_{i=1}^{N} \Big(J_i(u_i) + \langle \lambda , \theta_i(u_i) \rangle \Big) - \langle \lambda , \theta \rangle$$

2. Solve this problem by the dual gradient algorithm "à la Uzawa"

$$u_i^{(k+1)} \in \arg\min_{u_i \in \mathcal{U}_i} J_i(u_i) + \left\langle \lambda^{(k)} , \theta_i(u_i) \right\rangle , \quad i = 1 \dots , N$$

$$\lambda^{(k+1)} = \lambda^{(k)} + \rho \Big(\sum_{i=1}^{N} \theta_i \Big(u_i^{(k+1)} \Big) - \theta \Big)$$

Price decomposition relies on dualization

$$\min_{u_i \in \mathcal{U}_i, i=1\dots N} \sum_{i=1}^{N} J_i(u_i) \quad \text{subject to} \quad \sum_{i=1}^{N} \theta_i(u_i) - \theta = 0$$

1. Form the Lagrangian and assume that a saddle point exists

$$\max_{\lambda \in \mathcal{V}} \min_{u_i \in \mathcal{U}_i, i=1\dots N} \sum_{i=1}^{N} \Big(J_i(u_i) + \left\langle \lambda \,, \theta_i(u_i) \right\rangle \Big) - \left\langle \lambda \,, \theta \right\rangle$$

2. Solve this problem by the dual gradient algorithm "à la Uzawa"

$$u_i^{(k+1)} \in \arg\min_{u_i \in \mathcal{U}_i} J_i(u_i) + \left\langle \lambda^{(k)} \,, \theta_i(u_i) \right\rangle \,, \quad i = 1 \dots, N$$

$$\lambda^{(k+1)} = \lambda^{(k)} + \rho \Big(\sum_{i=1}^{N} \theta_i \Big(u_i^{(k+1)} \Big) - \theta \Big)$$

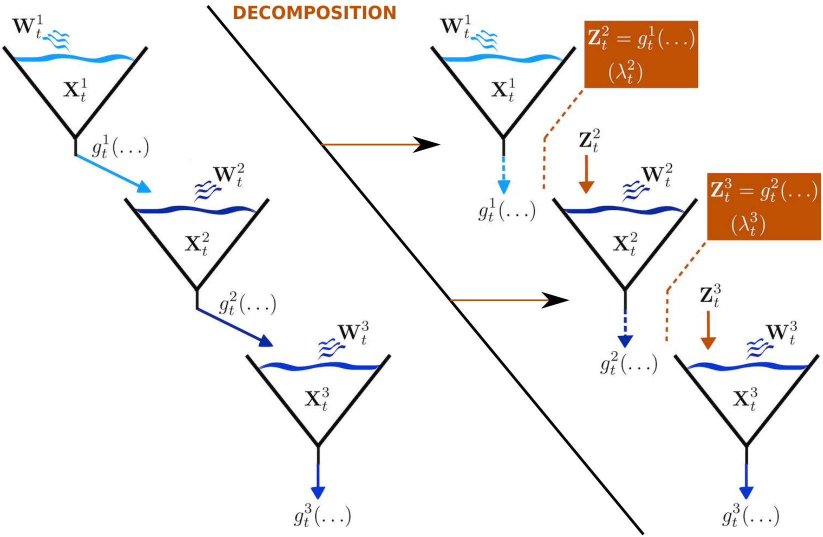
Remarks on decomposition methods

- ▷ The theory is available for infinite dimensional Hilbert spaces, and thus applies in the stochastic framework, that is, when the $\mathcal{U}_i$ are spaces of random variables
- ▷ The minimization algorithm used for solving the subproblems is not specified in the decomposition process
- ▷ New variables $\lambda^{(k)}$ appear in the subproblems arising at iteration k of the optimization process

$$\min_{u_i \in \mathcal{U}_i} J_i(u_i) + \left\langle \lambda^{(k)} , \theta_i(u_i) \right\rangle$$

- ▷ These variables are fixed when solving the subproblems, and do not cause any difficulty, at least in the deterministic case

Price decomposition applies to various couplings



1. Decomposition and coordination
   - A bird's eye view of decomposition methods
   - (A brief insight into Progressive Hedging)
   - Spatial decomposition methods in the deterministic case
   - The stochastic case raises specific obstacles

2. Dual approximate dynamic programming (DADP)
   - Problem statement
   - DADP principle and implementation
   - Numerical results on a small size problem

3. Theoretical questions
   - Existence of a saddle point
   - Convergence of the Uzawa algorithm
   - Convergence w.r.t. information

4. Conclusion

Stochastic optimal control (SOC) problem formulation

Consider the following SOC problem

$$\min_{\mathbf{u},\mathbf{x}} \mathbb{E}\Big(\sum_{i=1}^{N} \Big(\sum_{t=0}^{T-1} L_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1}) + K^i(\mathbf{x}_T^i) \Big) \Big)$$

subject to the constraints

$$\mathbf{x}_0^i = f_{-1}^i(\mathbf{w}_0) \; , \qquad i = 1 \ldots N$$

$$\mathbf{x}_{t+1}^i = f_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1}) \; , \qquad t = 0 \ldots T-1 \; , \; i = 1 \ldots N$$

$$\mathbf{u}_t^i \preceq \mathcal{F}_t = \sigma(\mathbf{w}_0, \ldots, \mathbf{w}_t) \; , \; t = 0 \ldots T-1 \; , \; i = 1 \ldots N$$

$$\sum_{i=1}^{N} \theta_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i) = 0 \; , \qquad t = 0 \ldots T-1$$

Stochastic optimal control (SOC) problem formulation

Consider the following SOC problem

$$\min_{\mathbf{u},\mathbf{x}} \sum_{i=1}^{N} \Bigg(\mathbb{E}\Big(\sum_{t=0}^{T-1} L_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1}) + K^i(\mathbf{x}_T^i) \Big) \Bigg)$$

subject to the constraints

$$\mathbf{x}_0^i = f_{\text{-}1}^i(\mathbf{w}_0) \;, \qquad i = 1 \dots N$$

$$\mathbf{x}_{t+1}^i = f_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1}) \;, \qquad t = 0 \dots T-1 \;, \quad i = 1 \dots N$$

$$\mathbf{u}_t^i \preceq \mathcal{F}_t = \sigma(\mathbf{w}_0, \dots, \mathbf{w}_t) \;, \quad t = 0 \dots T-1 \;, \quad i = 1 \dots N$$

$$\sum_{i=1}^{N} \theta_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i) = 0 \;, \qquad t = 0 \dots T-1$$

Dynamic Programming yields centralized controls

- As we want to solve this SOC problem using Dynamic Programming (**DP**), we suppose to be in the Markovian setting, that is, $\mathbf{w}_0, \ldots, \mathbf{w}_T$ are a white noise
- The system is made of N interconnected subsystems, with the control $\mathbf{u}_t^i$ and the state $\mathbf{x}_t^i$ of subsystem i at time t
- The optimal control $\mathbf{u}_t^i$ of subsystem i is a function of the whole system state $\left(\mathbf{x}_t^1, \ldots, \mathbf{x}_t^N\right)$

$$\mathbf{u}_t^i = \gamma_t^i\left(\mathbf{x}_t^1, \ldots, \mathbf{x}_t^N\right)$$

Naive decomposition should lead to decentralized feedbacks

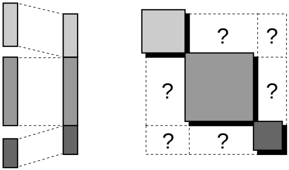
$$\mathbf{u}_t^i = \widehat{\gamma}_t^i(\mathbf{x}_t^i)$$

which are, in most cases, far from being optimal. . .

Straightforward decomposition of Dynamic Programming?

The crucial point is that the optimal feedback of a subsystem a priori depends on the state of all other subsystems, so that using a decomposition scheme by subsystems is not obvious. . .

As far as we have to deal with Dynamic Programming, the central concern for decomposition/coordination purpose boils down to



- ▷ how to decompose a feedback γ_t w.r.t. its domain $\mathbb{X}_t$ rather than its range $\mathbb{U}_t$?

  And the answer is
- ▷ impossible in the general case!

Price decomposition and Dynamic Programming

When applying price decomposition to the problem by dualizing the (almost sure) coupling constraint $\sum_i \theta_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i) = 0$,
multipliers $\mathbf{\Lambda}_t^{(k)}$ appear in the subproblems arising at iteration k

$$\min_{\mathbf{u}^i, \mathbf{x}^i} \mathbb{E}\Big(\sum_t L_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1}) + \mathbf{\Lambda}_t^{(k)} \cdot \theta_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i) \Big)$$

- ▷ The variables $\mathbf{\Lambda}_t^{(k)}$ are fixed random variables, so that the random process $\mathbf{\Lambda}^{(k)}$ acts as an additional input noise in the subproblems
- ▷ But this process may be correlated in time, so that the white noise assumption has no reason to be fulfilled
- ▷ DP cannot be applied in a straightforward manner!

Question: how to handle the coordination instruments $\mathbf{\Lambda}_t^{(k)}$ to obtain (an approximation of) the overall optimum?

Price decomposition and Dynamic Programming

When applying price decomposition to the problem by dualizing the (almost sure) coupling constraint $\sum_i \theta_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i) = 0$,
multipliers $\boldsymbol{\Lambda}_t^{(k)}$ appear in the subproblems arising at iteration k

$$\min_{\mathbf{u}^i, \mathbf{x}^i} \mathbb{E}\Big(\sum_t L_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1}) + \boldsymbol{\Lambda}_t^{(k)} \cdot \theta_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i) \Big)$$

- ▷ The variables $\boldsymbol{\Lambda}_t^{(k)}$ are fixed random variables, so that the random process $\boldsymbol{\Lambda}^{(k)}$ acts as an additional input noise in the subproblems
- ▷ But this process may be correlated in time, so that the white noise assumption has no reason to be fulfilled
- ▷ DP cannot be applied in a straightforward manner!

Question: how to handle the coordination instruments $\boldsymbol{\Lambda}_t^{(k)}$ to obtain (an approximation of) the overall optimum?

1 Decomposition and coordination

2 Dual approximate dynamic programming (DADP)

3 Theoretical questions

4 Conclusion

1. Decomposition and coordination
   - A bird's eye view of decomposition methods
   - (A brief insight into Progressive Hedging)
   - Spatial decomposition methods in the deterministic case
   - The stochastic case raises specific obstacles
2. Dual approximate dynamic programming (DADP)
   - Problem statement
   - DADP principle and implementation
   - Numerical results on a small size problem
3. Theoretical questions
   - Existence of a saddle point
   - Convergence of the Uzawa algorithm
   - Convergence w.r.t. information
4. Conclusion

Optimization problem

The SOC problem under consideration reads

$$\min_{\mathbf{u},\mathbf{x}} \; \mathbb{E}\bigg(\sum_{i=1}^{N} \Big(\sum_{t=0}^{T-1} L_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1}) + K^i(\mathbf{x}_T^i) \Big) \bigg) \tag{1a}$$

subject to dynamics constraints

$$\mathbf{x}_0^i = f_{-1}^i(\mathbf{w}_0) \tag{1b}$$

$$\mathbf{x}_{t+1}^i = f_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1}) \tag{1c}$$

to measurability constraints:

$$\mathbf{u}_t^i \preceq \sigma(\mathbf{w}_0, \dots, \mathbf{w}_t) \tag{1d}$$

and to instantaneous coupling constraints

$$\sum_{i=1}^{N} \theta_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i) = 0 \qquad \text{Constraints to be } \textbf{dualized} \tag{1e}$$

Assumptions

Assumption 1 (White noise)

Noises $\mathbf{w}_0, \dots, \mathbf{w}_T$ are independent over time

Hence Dynamic Programming applies: there is no optimality loss to look after the controls $\mathbf{u}_t^i$ as functions of the state at time t

Assumption 2 (Constraint qualification)

A saddle point of the Lagrangian $\mathcal{L}$ exists

$$\mathcal{L}(\mathbf{x}, \mathbf{u}, \mathbf{\Lambda}) = \mathbb{E}\Bigg(\sum_{i=1}^{N} \Big(\sum_{t=0}^{T-1} L_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1}) + K^i(\mathbf{x}_T^i) + \sum_{t=0}^{T-1} \mathbf{\Lambda}_t \cdot \theta_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i) \Big) \Bigg)$$

where the $\mathbf{\Lambda}_t$ are $\sigma(\mathbf{w}_0, \dots, \mathbf{w}_t)$-measurable random variables

Assumption 3 (Dual gradient algorithm)

Uzawa algorithm applies...

Uzawa algorithm

At iteration k of the algorithm,

1. Solve Subproblem i, $i = 1, \ldots, N$, with $\mathbf{\Lambda}^{(k)}$ fixed

$$\min_{\mathbf{u}^i, \mathbf{x}^i} \mathbb{E}\left(\sum_{t=0}^{T-1} \left(L_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1}) + \mathbf{\Lambda}_t^{(k)} \cdot \theta_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i) \right) + K^i(\mathbf{x}_T^i) \right)$$

subject to

$$\begin{aligned} \mathbf{x}_{t+1}^i &= f_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1}) \\ \mathbf{u}_t^i &\preceq \sigma(\mathbf{w}_0, \ldots, \mathbf{w}_t) \end{aligned}$$

whose solution is denoted $\left(\mathbf{u}^{i,(k+1)}, \mathbf{x}^{i,(k+1)}\right)$

2. Update the multipliers $\mathbf{\Lambda}_t$

$$\mathbf{\Lambda}_t^{(k+1)} = \mathbf{\Lambda}_t^{(k)} + \rho_t \left(\sum_{i=1}^{N} \theta_t^i\left(\mathbf{x}_t^{i,(k+1)}, \mathbf{u}_t^{i,(k+1)}\right) \right)$$

Structure of a subproblem

- Subproblem i reads

$$\min_{\mathbf{u}^i,\mathbf{x}^i} \mathbb{E}\left(\sum_{t=0}^{T-1} \Big(L_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1}) + \mathbf{\Lambda}_t^{(k)} \cdot \theta_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i) \Big) \right)$$

subject to

$$\mathbf{x}_{t+1}^i = f_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1})$$
$$\mathbf{u}_t^i \preceq \sigma(\mathbf{w}_0, \dots, \mathbf{w}_t)$$

- Without some knowledge of the process $\mathbf{\Lambda}^{(k)}$
(we just know that $\mathbf{\Lambda}_t^{(k)} \preceq (\mathbf{w}_0, \dots, \mathbf{w}_t)$),
the informational state of this subproblem i at time t
cannot be summarized by the physical state $\mathbf{x}_t^i$

1. Decomposition and coordination
   - A bird's eye view of decomposition methods
   - (A brief insight into Progressive Hedging)
   - Spatial decomposition methods in the deterministic case
   - The stochastic case raises specific obstacles
2. Dual approximate dynamic programming (DADP)
   - Problem statement
   - DADP principle and implementation
   - Numerical results on a small size problem
3. Theoretical questions
   - Existence of a saddle point
   - Convergence of the Uzawa algorithm
   - Convergence w.r.t. information
4. Conclusion

We outline the main idea in DADP

- To overcome the difficulty induced by the term $\boldsymbol{\Lambda}_t^{(k)}$, we introduce a new adapted information process $\mathbf{y}^i = \big(\mathbf{y}_0^i, \ldots, \mathbf{y}_{T-1}^i\big)$ for Subsystem i
- at each time t, the random variable $\mathbf{y}_t^i$ is measurable w.r.t. the past noises $\big(\mathbf{w}_0, \ldots, \mathbf{w}_t\big)$
- The core idea is to replace the multiplier $\boldsymbol{\Lambda}_t^{(k)}$ at iteration k by its conditional expectation $\mathbb{E}(\boldsymbol{\Lambda}_t^{(k)} \mid \mathbf{y}_t^i)$
- (More on the interpretation later)

Note that we require that the information process is not influenced by controls

We can now approximate Subproblem i

- Using this idea, we replace Subproblem i by

$$\min_{\mathbf{u}^i,\mathbf{x}^i} \mathbb{E}\Big(\sum_{t=0}^{T-1} \Big(L_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1}) + \mathbb{E}(\mathbf{\Lambda}_t^{(k)} \mid \mathbf{y}_t^i) \cdot \theta_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i) \Big) + K^i(\mathbf{x}_T^i) \Big)$$

subject to

$$\begin{aligned} \mathbf{x}_{t+1}^i &= f_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1}) \\ \mathbf{u}_t^i &\preceq \sigma(\mathbf{w}_0, \ldots, \mathbf{w}_t) \end{aligned}$$

- The conditional expectation $\mathbb{E}(\mathbf{\Lambda}_t^{(k)} \mid \mathbf{y}_t^i)$ is an (updated) function of the variable $\mathbf{y}_t^i$,
- so that Subproblem i involves the two noises processes $\mathbf{w}$ and $\mathbf{y}^i$

If $\mathbf{y}^i$ follows a dynamical equation, DP applies

We obtain a Dynamic Programming equation by subsystem

Assuming a non-controlled dynamics $\mathbf{y}^i_{t+1} = h^i_t\big(\mathbf{y}^i_t, \mathbf{w}_{t+1}\big)$ for the information process $\mathbf{y}^i$, the DP equation writes

$$
\begin{aligned}
V^i_T(x, y) &= K^i(x) \\
V^i_t(x, y) &= \min_u \mathbb{E}\Big(L^i_t(x, u, \mathbf{w}_{t+1}) \\
&\qquad\qquad + \mathbb{E}\big(\mathbf{\Lambda}^{(k)}_t \bigm| \mathbf{y}^i_t = y\big) \cdot \theta^i_t(x, u) \\
&\qquad\qquad\qquad + V^i_{t+1}\big(\mathbf{x}^i_{t+1}, \mathbf{y}^i_{t+1}\big)\Big)
\end{aligned}
$$

subject to the dynamics

$$
\begin{aligned}
\mathbf{x}^i_{t+1} &= f^i_t(x, u, \mathbf{w}_{t+1}) \\
\mathbf{y}^i_{t+1} &= h^i_t(y, \mathbf{w}_{t+1})
\end{aligned}
$$

DADP displays three interpretations

- DADP as an approximation of the optimal multiplier

$$\lambda_t \quad \rightsquigarrow \quad \mathbb{E}\big(\lambda_t \bigm| \mathbf{y}_t\big)$$

- DADP as a decision-rule approach in the dual

$$\max_{\lambda} \min_{\mathbf{u}} L\big(\lambda, \mathbf{u}\big) \quad \rightsquigarrow \quad \max_{\lambda_t \preceq \mathbf{y}_t} \min_{\mathbf{u}} L\big(\lambda, \mathbf{u}\big)$$

- DADP as a constraint relaxation

$$\sum_{i=1}^{n} \theta_t^i\big(\mathbf{u}_t^i\big) = 0 \quad \rightsquigarrow \quad \mathbb{E}\bigg(\sum_{i=1}^{n} \theta_t^i\big(\mathbf{u}_t^i\big) \biggm| \mathbf{y}_t\bigg) = 0$$

A bunch of practical questions remains open

- ⋆ How to choose the information variables $\mathbf{y}_t^i$?
  - ▷ Perfect memory: $\mathbf{y}_t^i = \big(\mathbf{w}_0, \dots, \mathbf{w}_t\big)$
  - ▷ Minimal information: $\mathbf{y}_t^i \equiv \mathrm{cste}$
  - ▷ Static information: $\mathbf{y}_t^i = h_t^i\big(\mathbf{w}_t\big)$
  - ▷ Dynamic information: $\mathbf{y}_{t+1}^i = h_t^i\big(\mathbf{y}_t^i, \mathbf{w}_{t+1}\big)$
- ⋆ How to obtain a feasible solution from the relaxed problem?
  - ▷ Use an appropriate heuristic!
- ⋆ How to accelerate the gradient algorithm?
  - ▷ Augmented Lagrangian
  - ▷ More sophisticated gradient methods
- ⋆ How to handle more complex structures than the flower model?

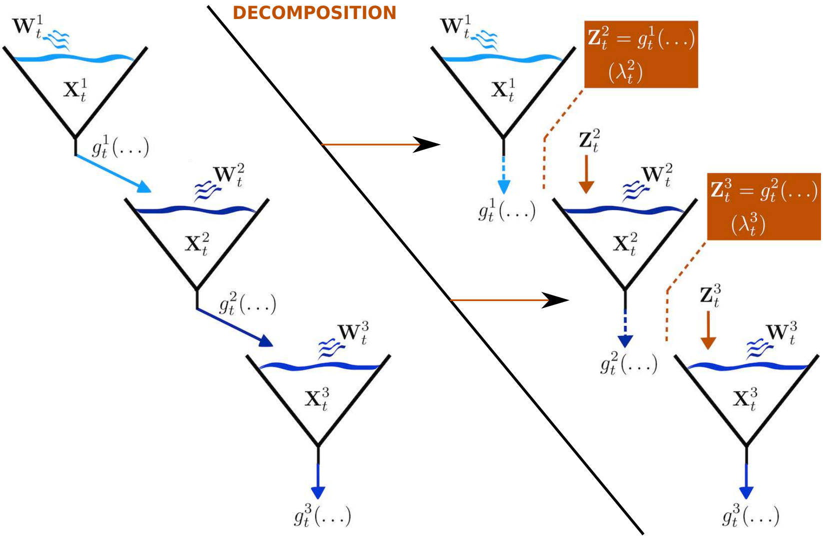
1. Decomposition and coordination
   - A bird's eye view of decomposition methods
   - (A brief insight into Progressive Hedging)
   - Spatial decomposition methods in the deterministic case
   - The stochastic case raises specific obstacles

2. Dual approximate dynamic programming (DADP)
   - Problem statement
   - DADP principle and implementation
   - Numerical results on a small size problem

3. Theoretical questions
   - Existence of a saddle point
   - Convergence of the Uzawa algorithm
   - Convergence w.r.t. information

4. Conclusion

We consider 3 dams in a row, amenable to DP



Problem specification

- We consider a 3 dam problem, over 12 time steps
- We relax each constraint with a given information process $\mathbf{y}^i$
- All random variable are discrete (noise, control, state)

- We test the following information processes

  Constant information: equivalent to replace the a.s. constraint by an expected constraint

  Part of noise: the information process is the inflow of the above dam $\mathbf{Y}_t^i = \mathbf{w}_t^{i-1}$

  Phantom state: the information process mimicks the optimal trajectory of the state of the first dam (by statistical regression over the known optimal trajectory in this case)

Numerical results are encouraging

	DADP - $\mathbb{E}$	DADP - $\mathbf{w}^{i-1}$	DADP - dyn.	DP
Nb of it.	165	170	25	1
Time (min)	2	3	67	41
Lower Bound	-1.386×10^6	-1.379×10^6	-1.373×10^6	
Final Value	-1.335×10^6	-1.321×10^6	-1.344×10^6	-1.366×10^6
Loss	-2.3%	-3.3%	-1.6%	ref.

$\rightsquigarrow$ *PhD thesis of J.-C. Alais*

Summing up DADP

- **Choose** an information process $\mathbf{y}$ following $\mathbf{y}_{t+1} = \widetilde{f}_t(\mathbf{y}_t, \mathbf{w}_{t+1})$
- Relax the almost sure coupling constraint into a conditional expectation
- Then apply a price decomposition scheme to the relaxed problem
- The subproblems can be solved by dynamic programming with the modest state $(\mathbf{x}_t^i, \mathbf{y}_t)$
- In the theoretical part, we give
  - a consistency result (family of information process)
  - a convergence result (fixed information process)
  - conditions for the existence of multiplier

1 Decomposition and coordination

2 Dual approximate dynamic programming (DADP)

3 Theoretical questions

4 Conclusion

What are the issues to consider?

- ▷ We treat the coupling constraints in a stochastic optimization problem by duality methods
- ▷ Uzawa algorithm is a dual method which is naturally described in an Hilbert space, but we cannot guarantee the existence of an optimal multiplier in the space $\mathrm{L}^2\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big)$!
- ▷ Consequently, we extend the algorithm to the non-reflexive Banach space $\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big)$, by giving a set of conditions ensuring the existence of a $\mathrm{L}^1\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big)$ optimal multiplier, and by providing a convergence result of the algorithm

- ▷ We also have to deal with the approximation induced by the information variable: we give an epi-convergence result related to such an approximation

⇝ *PhD thesis of V. Leclère*

Abstract formulation of the problem

We consider the following abstract optimization problem

$$(\mathcal{P}) \qquad \min_{\mathbf{u} \in \mathcal{U}^{\mathrm{ad}}} J(\mathbf{u}) \quad \text{s.t.} \quad \Theta(\mathbf{u}) \in -C$$

where $\mathcal{U}$ and $\mathcal{V}$ are two Banach spaces, and

- ▷ $J : \mathcal{U} \to \overline{\mathbb{R}}$ is the objective function
- ▷ $\mathcal{U}^{\mathrm{ad}}$ is the admissible set
- ▷ $\Theta : \mathcal{U} \to \mathcal{V}$ is the constraint function to be dualized
- ▷ $C \subset \mathcal{V}$ is the cone of constraint

Let $\mathcal{U}^{\Theta} = \left\{ \mathbf{u} \in \mathcal{U},\ \Theta(\mathbf{u}) \in -C \right\}$ be the associated constraint set

Here, $\mathcal{U}$ is a space of random variables, and J is defined by

$$J(\mathbf{u}) = \mathbb{E}\big(j(\mathbf{u}, \mathbf{w})\big)$$

The relationship with Problem (1) is almost straightforward. . .

1. Decomposition and coordination
   - A bird's eye view of decomposition methods
   - (A brief insight into Progressive Hedging)
   - Spatial decomposition methods in the deterministic case
   - The stochastic case raises specific obstacles
2. Dual approximate dynamic programming (DADP)
   - Problem statement
   - DADP principle and implementation
   - Numerical results on a small size problem
3. Theoretical questions
   - Existence of a saddle point
   - Convergence of the Uzawa algorithm
   - Convergence w.r.t. information
4. Conclusion

Standard duality in L^2 spaces (I)

Assume that $\mathcal{U} = \mathrm{L}^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ and $\mathcal{V} = \mathrm{L}^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m)$

The standard sufficient constraint qualification condition

$$0 \in \mathrm{ri}\Big(\Theta\big(\mathcal{U}^{\mathrm{ad}} \cap \mathrm{dom}(J)\big) + C\Big)$$

is scarcely satisfied in such a stochastic setting

Proposition 1

If the σ-algebra $\mathcal{F}$ is not finite modulo $\mathbb{P}$,[a]
then for any subset $U^{\mathrm{ad}} \subset \mathbb{R}^n$ that is not an affine subspace, the set

$$\mathcal{U}^{\mathrm{ad}} = \Big\{\mathbf{u} \in \mathrm{L}^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n) \mid \mathbf{u} \in U^{\mathrm{ad}} \quad \mathbb{P} - a.s.\Big\}$$

has an empty relative interior in L^p, for any $p < +\infty$

[a] If the σ-algebra is finite modulo $\mathbb{P}$, $\mathcal{U}$ and $\mathcal{V}$ are finite dimensional spaces

Standard duality in L^2 spaces (II)

Consider the following optimization problem:

$$\begin{aligned} \inf_{u_0, \mathbf{u}_1} \quad & u_0^2 + \mathbb{E}\big((\mathbf{u}_1 + \alpha)^2\big) \\ \text{s.t.} \quad & u_0 \geq a \\ & \mathbf{u}_1 \geq 0 \\ & u_0 - \mathbf{u}_1 \geq \mathbf{w} \qquad \text{to be dualized} \end{aligned}$$

where $\mathbf{w}$ is a random variable uniform on $[1, 2]$

For $a < 2$, we can construct a maximizing sequence of multipliers for the dual problem that does not converge in L^2.
(We are in the so-called *non relatively complete recourse* case, that is, the case where the constraints on $\mathbf{u}_1$ induce a stronger constraint on u_0)

An optimal multiplier is available in $(\mathrm{L}^\infty)^\star$...

Standard duality in L^2 spaces (II)

Consider the following optimization problem:

$$\begin{aligned} \inf_{u_0, \mathbf{u}_1} \quad & u_0^2 + \mathbb{E}\big((\mathbf{u}_1 + \alpha)^2\big) \\ \text{s.t.} \quad & u_0 \geq a \\ & \mathbf{u}_1 \geq 0 \\ & u_0 - \mathbf{u}_1 \geq \mathbf{w} \qquad \text{to be dualized} \end{aligned}$$

where $\mathbf{w}$ is a random variable uniform on $[1, 2]$

For $a < 2$, we can construct a maximizing sequence of multipliers for the dual problem that does not converge in L^2.
(We are in the so-called *non relatively complete recourse* case, that is, the case where the constraints on $\mathbf{u}_1$ induce a stronger constraint on u_0)

An optimal multiplier is available in $(\mathrm{L}^\infty)^\star$...

Constraint qualification in $(\mathrm{L}^\infty, \mathrm{L}^1)$

From now on, we assume that

$$\mathcal{U} = \mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$$
$$\mathcal{V} = \mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m)$$
$$C = \{0\}$$

where the σ-algebra $\mathcal{F}$ is not finite modulo $\mathbb{P}$

We consider the pairing $(\mathrm{L}^\infty, \mathrm{L}^1)$ with the following topologies:

- $\sigma(\mathrm{L}^\infty, \mathrm{L}^1)$: weak* topology on L^∞ (coarsest topology such that all the L^1-linear forms are continuous),
- $\tau(\mathrm{L}^\infty, \mathrm{L}^1)$: Mackey-topology on L^∞ (finest topology such that the continuous linear forms are only the L^1-linear forms)

Weak* closedness of linear subspaces of L^∞

Proposition 2

Let $\Theta : \mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n) \to \mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m)$ *be a linear operator, and assume that there exists a linear operator* $\Theta^\dagger : \mathrm{L}^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m) \to \mathrm{L}^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ *such that:*

$$\left\langle \mathbf{v} \,, \Theta(\mathbf{u}) \right\rangle = \left\langle \Theta^\dagger(\mathbf{v}) \,, \mathbf{u} \right\rangle \,, \quad \forall \mathbf{u}, \ \forall \mathbf{v}$$

Then the linear operator Θ *is weak** *continuous*

Applications

- ▷ $\Theta(\mathbf{u}) = \mathbf{u} - \mathbb{E}\left(\mathbf{u} \mid \mathcal{B}\right)$: non-anticipativity constraints,
- ▷ $\Theta(\mathbf{u}) = A\mathbf{u}$ with $A \in \mathcal{M}_{m,n}(\mathbb{R})$: finite number of constraints

A duality theorem

$$(\mathcal{P}) \qquad \min_{\mathbf{u}\in\mathcal{U}} J(\mathbf{u}) \quad \text{s.t.} \quad \Theta(\mathbf{u}) = 0$$

with $J(\mathbf{u}) = \mathbb{E}\big(j(\mathbf{u},\mathbf{w})\big)$

Theorem 1

Assume that j is a convex normal integrand, that J is continuous in the Mackey topology at some point $\mathbf{u}_0$ such that $\Theta(\mathbf{u}_0) = 0$, and that Θ is weak* continuous on $\mathrm{L}^\infty(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^n)$
Then, $\mathbf{u}^\star \in \mathcal{U}$ is an optimal solution of Problem $(\mathcal{P})$ if and only if there exists $\lambda^\star \in \mathrm{L}^1(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m)$ such that

- $\mathbf{u}^\star \in \arg\min_{\mathbf{u}\in\mathcal{U}} \mathbb{E}\Big(j(\mathbf{u},\mathbf{w}) + \lambda^\star \cdot \Theta(\mathbf{u})\Big)$
- $\Theta(\mathbf{u}^\star) = 0$

Extension of a result given by R. Wets for non-anticipativity constraints

1. Decomposition and coordination
   - A bird's eye view of decomposition methods
   - (A brief insight into Progressive Hedging)
   - Spatial decomposition methods in the deterministic case
   - The stochastic case raises specific obstacles
2. Dual approximate dynamic programming (DADP)
   - Problem statement
   - DADP principle and implementation
   - Numerical results on a small size problem
3. Theoretical questions
   - Existence of a saddle point
   - Convergence of the Uzawa algorithm
   - Convergence w.r.t. information
4. Conclusion

Uzawa algorithm

$$(\mathcal{P}) \qquad \min_{\mathbf{u}\in\mathcal{U}} J(\mathbf{u}) \quad \text{s.t.} \quad \Theta(\mathbf{u}) = 0$$

with $J(\mathbf{u}) = \mathbb{E}\big(j(\mathbf{u},\mathbf{w})\big)$

The standard Uzawa algorithm

$$\mathbf{u}^{(k+1)} \in \arg\min_{\mathbf{u}\in\mathcal{U}^{\mathrm{ad}}} J(\mathbf{u}) + \big\langle \lambda^{(k)}, \Theta(\mathbf{u}) \big\rangle$$
$$\lambda^{(k+1)} = \lambda^{(k)} + \rho\,\Theta\big(\mathbf{u}^{(k+1)}\big)$$

makes sense with in the L^∞ setting, that is, the minimization problem is well-posed and the update formula is valid one

Note that all the multipliers $\lambda^{(k)}$ belong to $\mathrm{L}^\infty\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m\big)$, as soon as the initial multiplier $\lambda^{(0)} \in \mathrm{L}^\infty\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m\big)$

Convergence result

Theorem 2

Assume that

1. $J : \mathcal{U} \to \overline{\mathbb{R}}$ *is proper, weak** *l.s.c., differentiable and* a*-convex*
2. $\Theta : \mathcal{U} \to \mathcal{V}$ *is affine, weak** *continuous and* κ*-Lipschitz*
3. $\mathcal{U}^{\mathrm{ad}}$ *is weak** *closed and convex,*
4. *an admissible* $\mathbf{u}_0 \in \operatorname{dom} J \cap \Theta^{-1}(0) \cap \mathcal{U}^{\mathrm{ad}}$ *exists*
5. *an optimal* L^1*-multiplier to the constraint* $\Theta(\mathbf{u}) = 0$ *exists*
6. *the step* ρ *is such that* $0 < \rho < \frac{2a}{\kappa}$

Then, there exists a subsequence $\left\{\mathbf{u}^{(n_k)}\right\}_{k\in\mathbb{N}}$ *of the sequence generated by the Uzawa algorithm converging in* L^∞ *toward the optimal solution* $\mathbf{u}^\star$ *of the primal problem*

Remarks about these results

- The result is not as good as expected (global convergence)
- Improvements and extensions (inequality constraint) needed
- The Mackey-continuity assumption forbids the use of bounds
  - In order to deal with almost sure bound constraints, we can turn towards the work of R.T. Rockafellar and R. J-B Wets
  - In a series of 4 papers (stochastic convex programming), they have detailed the duality theory on two-stage and multistage problems, with the focus on non-anticipativity constraints
  - These papers require
    - a strict feasability assumption
    - a relatively complete recourse assumption

1. Decomposition and coordination
   - A bird's eye view of decomposition methods
   - (A brief insight into Progressive Hedging)
   - Spatial decomposition methods in the deterministic case
   - The stochastic case raises specific obstacles

2. Dual approximate dynamic programming (DADP)
   - Problem statement
   - DADP principle and implementation
   - Numerical results on a small size problem

3. Theoretical questions
   - Existence of a saddle point
   - Convergence of the Uzawa algorithm
   - Convergence w.r.t. information

4. Conclusion

Relaxed problems

Following the interpretation of DADP in terms of a relaxation of the original problem, and given a sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ of subfields of the σ-field $\mathcal{F}$, we replace the abstract problem

$$(\mathcal{P}) \qquad \min_{\mathbf{u}\in\mathcal{U}} J(\mathbf{u}) \quad \text{s.t.} \quad \Theta(\mathbf{u}) = 0$$

by the sequence of approximated problems:

$$(\mathcal{P}_n) \qquad \min_{\mathbf{u}\in\mathcal{U}} J(\mathbf{u}) \quad \text{s.t.} \quad \mathbb{E}\big(\Theta(\mathbf{u}) \bigm| \mathcal{F}_n\big) = 0$$

We assume the Kudo convergence of $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ toward $\mathcal{F}$:

$$\mathcal{F}_n \longrightarrow \mathcal{F} \iff \forall \mathbf{x} \in \mathrm{L}^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}),\ \mathbb{E}\big(\mathbf{x} \bigm| \mathcal{F}_n\big) \xrightarrow{\mathrm{L}^1} \mathbb{E}\big(\mathbf{x} \bigm| \mathcal{F}\big)$$

Convergence result

Theorem 3

Assume that

- ▷ $\mathcal{U}$ *is a topological space*
- ▷ $\mathcal{V} = \mathrm{L}^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m)$ *with* $p \in [1, +\infty)$
- ▷ J *and* Θ *are continuous operators*
- ▷ $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ *Kudo converges toward* $\mathcal{F}$

Then the sequence $\{\widetilde{J}_n\}_{n\in\mathbb{N}}$ *epi-converges toward* $\widetilde{J}$, *with*

$$\widetilde{J}_n = \begin{cases} J(\mathbf{u}) & \text{if } \mathbf{u} \text{ satisfies the constraints of } (\mathcal{P}_n) \\ +\infty & \text{otherwise} \end{cases}$$

1 Decomposition and coordination

2 Dual approximate dynamic programming (DADP)

3 Theoretical questions

4 Conclusion

Conclusion

- DADP method allows to tackle
  large-scale stochastic optimal control problems,
  such as those found in energy management
- A host of theoretical and practical questions remains open
- We would like to test DADP on (smart) grids, extending the works
  on "flower models" (Unit Commitment problem) and
  on "chained models" (hydraulic valley management)
  to "network models" (grids)