A branch-and-cut algorithm for the Ring-Star Problem

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Outline

- Definition and application
- Previous work
- Our formulation
- Polyhedral analysis
- Premilinary experiments
- Conclusion

Definition

Let $G = (V, E, A)$ be a mixed complete graph where

- $V = \{v_1, v_2, \ldots, v_n\}$,
- E is the edge set and A is the arc set.

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There are :

- nonnegative *ring cost* c_{ij} associated with each edge $v_i v_j \in E$,
- nonnegative *assignment cost* d_{ij} associated with each arc $(v_i, v_j) \in A$.

Ring Star Problem (RSP)

Finding a simple cycle (ring) containing the root v_1 and an assignment of all other nodes (which do not belong to the ring) to some node on the cycle such that the sum of ring cost and assignment cost is minimized. The problem is NP-hard.

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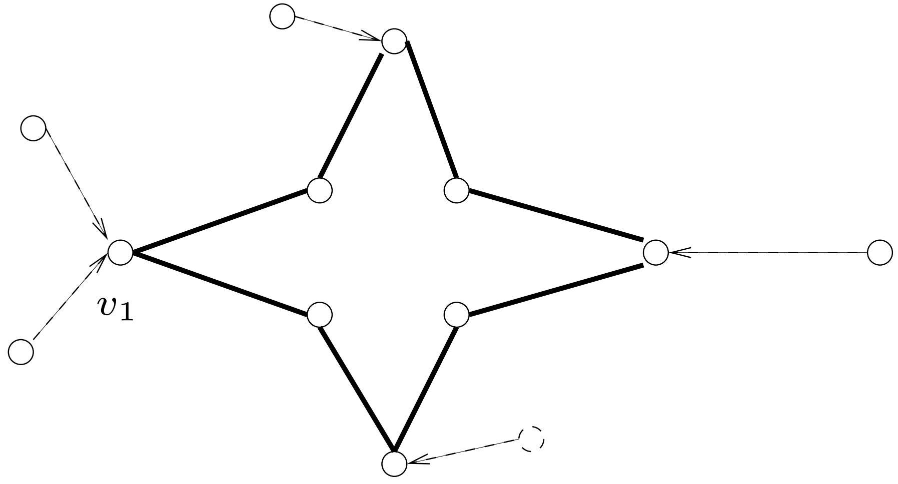

A solution

Variables and objective function

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Objective function:

$$\text{minimize} \sum_{v_i v_j \in E} c_{ij} x_{ij} + \sum_{(v_i, v_j) \in A} d_{ij} y_{ij}$$

Applications

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- Location, transport: Facilities are installed on the nodes of the ring and the remaining nodes are assigned to those facilities. The supplying cost is the cost of the Hamiltonian cycles on node-facility and the serving cost is the assignment cost of remaining vertices. The sum of the two costs must be minimized.

Previous work

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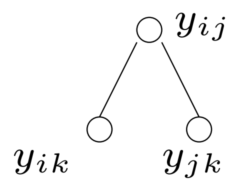
The cycle polytope here is the convex hull of the incidence vectors of all the cycles of G. Remark that the cycles considered in the problem is those containing v_1.

Some observations

The stable set polytope associated to the incompatiblity of the variables y_{ij}, y_{ik} and y_{ij}, y_{jk}.



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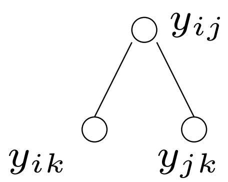

- The cycle polytope is very complex. In our knowledge, complete characterization of its dominant, the cycle polyhedron which correspond to polynomial cases (when costs are nonnegative) is not known (Bauer 1997). Coefficients of facet-defining inequalities are exponential on the number of nodes (Nguyen et al. 2001).

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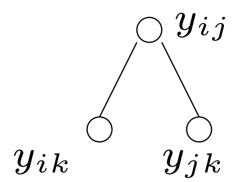

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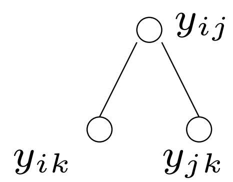

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- The same observations on the stable set polytope.

Our approach on the variables x

The cycle in a solution of RSP must contain $v_1 = s$. If we add an artificial node $v_1' = t$ which is a clone of v_1, the cycle can be transformed to a st-path.

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This can have advantages because the dominance of the st-paths

$$conv(x + \mathbb{R}^{|E|} | \text{x is incidence vector of a } st\text{-path})$$

is completely characterized by

$$x(\delta(S)) \geq 1 \; S \subset V, \; s \in S \text{ and } t \in V \setminus S$$

These inequalities are called st-*cut inequalities*.

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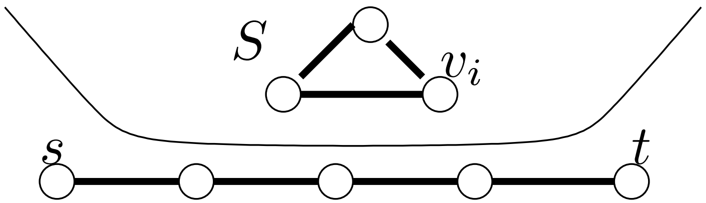

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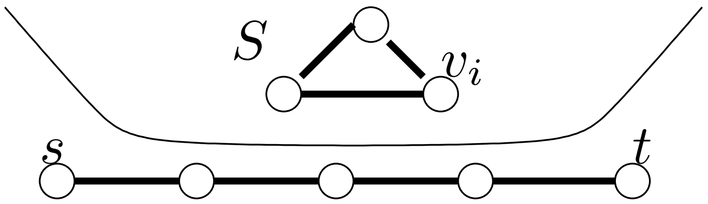


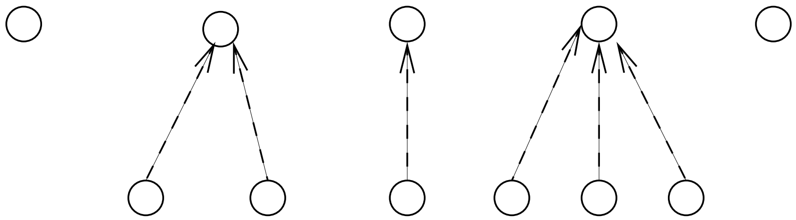
Thus we would obtain solutions with one st-path + disjoint cycles. We add the following constraints to cut off them :

$$x(\delta(S)) \geq 2y_{ii} \; \forall S \subset V \setminus \{s,t\} \text{ and } \forall v_i \in S$$

Let us call *connexity inequalities* these inequalities.

Our approach on the variables y

In a solution of RSP, on the variables y we can have following configuration:



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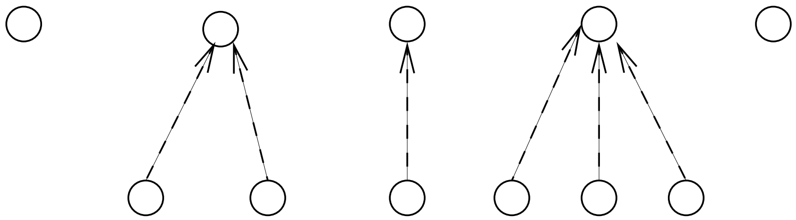

An isolated vertex v_i having $y_{ii} = 1$. If we consider y_{ii} as an arc from v_i to some artificial root node v_0, a solution on variables y will be a directed spanning tree with paths from root v_0 to every other node of length at most 2 (with inversion on the arc's orientation).

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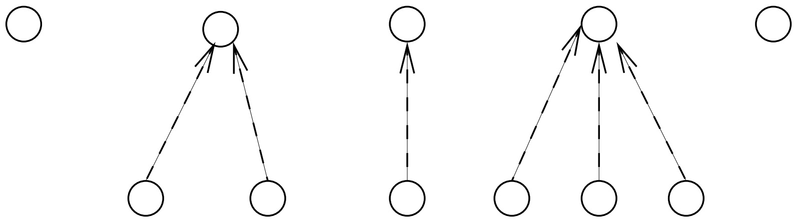

An isolated vertex v_i having $y_{ii} = 1$. If we consider y_{ii} as an arc from v_i to some artificial root node v_0, a solution on variables y will be a directed spanning tree with paths from root v_0 to every other node of length at most 2 (with inversion on the arc's orientation). These spanning trees are called *hop-constrained spanning trees*.

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We consider the hop-constrained spanning trees with $H = 2$.

The Hop-constrained Minimum Spanning Tree Problem (HMST) with $H = 2$ is NP-hard as it is equivalent to the Simple Uncapaciated Facility Location.

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G. Dahl (1998) gave the following integer formulation for the problem:

$$\sum_{i \in V \setminus \{j\}} y_{ij} = 1, \quad j \in V \setminus \{0\}$$

$$y_{0j} \geq y_{jk}, \quad \text{for every arc (j,k)}$$

$$y_{ij} \in \{0, 1\}$$

It is easy to convert this formulation to an integer formulation for the variable y in RSP.

Remarks and formulation

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In an optimal solution of RSP, if all the variables x are integer then the

Integer formulation and dimension of R

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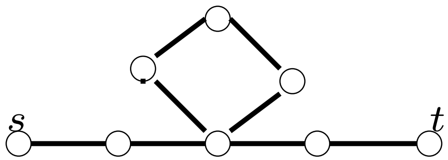
The dimension of RSP is $|E| + |A| - (4n - 4) = |E| + |A| - 4n + 4$.

Facets for RSP : Blossom inequalities

A T-join with $T = \{s, t\}$ is composed by a st-path and eventually some additional cycles.



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Thus a T-join dominates a st-path.
The convex hull of the T-joins is completely characterized by $0 \leq x_{ij} \leq 1$ and the *blossom* inequalities

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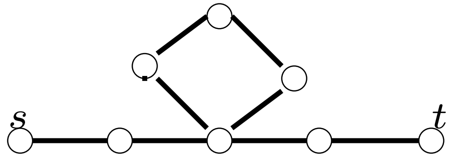

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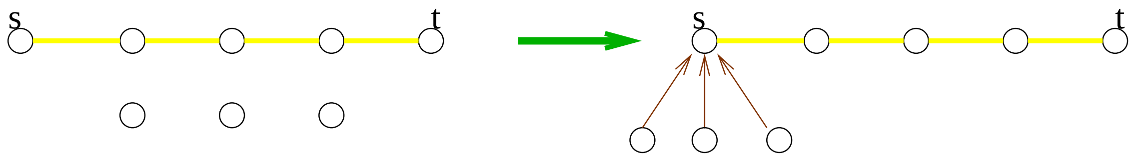
From the cycle polytope, Labbé et al. obtain a subset of blossom inequalities, the 2-matching inequalities which require F to be a matching.

Facets for st-PP $\Rightarrow$ facets for RSP

Let $a^T x \geq b$ define a facet of st-PP. The $|E| - 2$ st-paths affinely independent satisfying $a^T x = b$ transformed to solutions of RSP:



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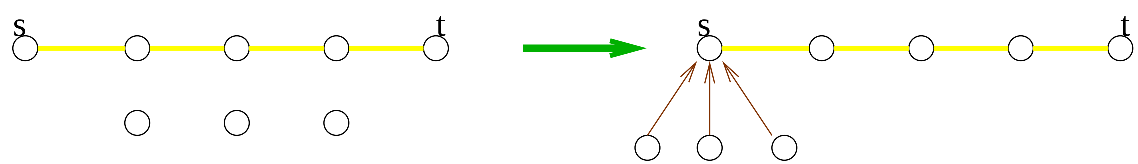

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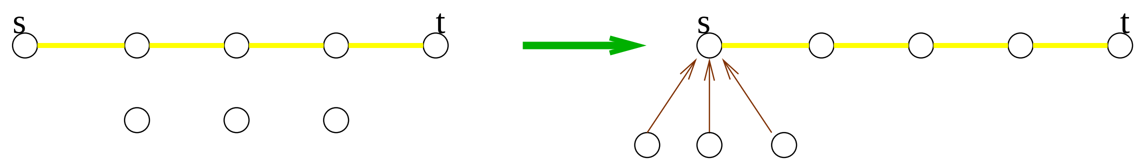
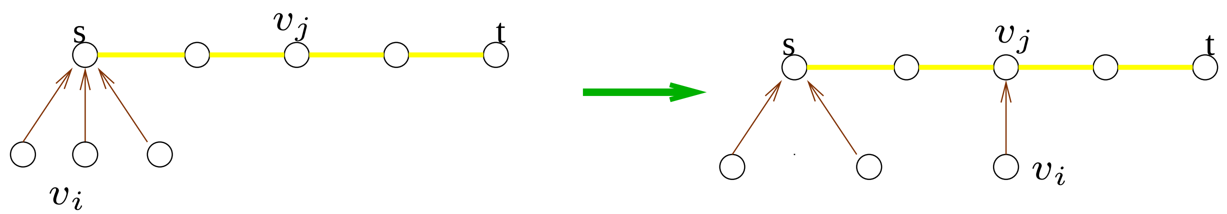


Each arc (v_i, v_j) correspond to the following point:



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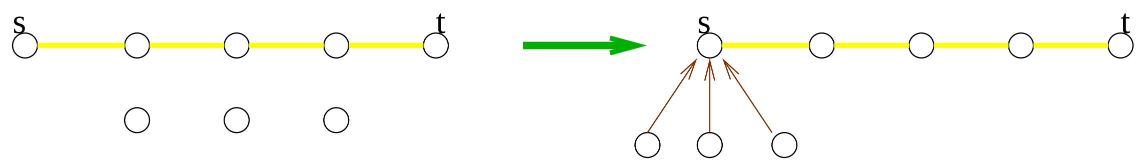

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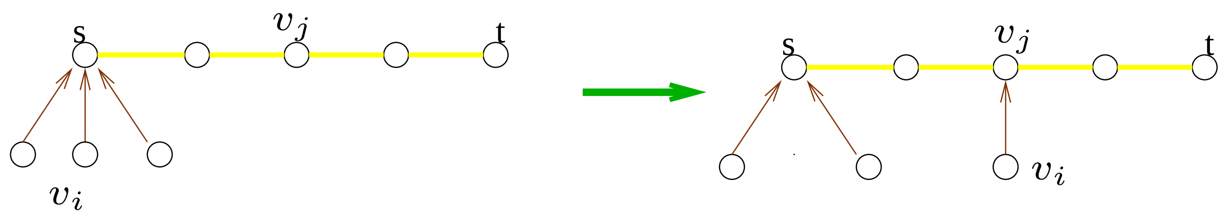

From that, we obtain $|\bar{y}| = |A| - (n + 2n - 4 + n - 2) = |A| - 4n + 6$ points affinely independent. In total, we have $|E| + |A| - 4n + 4$ points affinely indenpendent $\Rightarrow a^T x \geq b$ defines a facet for RSP.

Our branch-and-cut algorithm

About the code of Labbe et al. (2004)

- Branch-and-Cut framework: ABACUS,
- Linear Solver: CPLEX 6.0
- used cuts : connexity, 2-matching and some other "small" inequalities.
- Very good initial heuristic: Often very near optimal solution (101%), primal heuristic.

About our code

- Branch-and-Cut framework: COIN/Bcp,
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Premilinary experiments

- We have tested our code on TSP instances with at most 100 nodes.
- In general, the code is from 5 until 10 times faster than the code of Labbé et al.
- There is one instance on that our code is slower!

Conclusion and future works

- We have established the correspondance between facets of RSP and the ones of it's projections respectively on x-space and y-space.
- Our formulation allows to derive interesting facets for RSP included some well-studied inequalities: st-cut, blossom.
- Better separate the blossom inequalities.
- Better initial and primal heuristics.
- New inequalities for the st-path polytopes and the HMST polytope.