SDP successes in the telecom world

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orange™

Contents...

- SDP for fairness/wireline
- SDP for radio
- SDP for Webworld

The Use of Semi-Definite Programming in the Telecommunication Context

We model a network with

- V a set of vertices,
- E a set of edges on V,
- $d_{u,v}$ the demand of traffic associated to a source/destination pair (u, v) in the demand set D.

Today's assumptions:

- The traffic can be splitted (no monorouting).
- All the demand can be routed through the network, even with a margin on each link.
- We try to optimize some global parameter over the network.

Therefore, a solution of the problem can be given by a flow $\phi^e_{u,v}, e \in E, (u,v) \in D$, and surplus margins $\delta_{u,v}, (u,v) \in D$ and $\kappa_e, e \in E$

Constraints

We constrain the network to

- route for each demand $(u, v) \in D$, a rate of $(1 + \delta_{u,v})d_{u,v}$,
- restrain the capacity used in $(1 - \kappa_e)C_e$.

Our set of constraints is:

$$\forall (u,v) \in D \quad \forall e \in E \qquad \phi^e_{u,v} \geq 0 \qquad \delta_{u,v} \geq 0 \qquad \kappa_e \geq 0$$

$$\forall (u,v) \in D \qquad \sum_{\substack{w \in V \\ (u,w) \in E}} \phi^{(u,w)}_{u,v} \geq \sum_{\substack{w \in V \\ (w,u) \in E}} \phi^{(w,u)}_{u,v} + (1+\delta_{u,v}) d_{u,v}$$

$$\forall (u,v) \in D \quad \forall x \in V - \{u,v\} \qquad \sum_{\substack{w \in V \\ (x,w) \in E}} \phi^{(x,w)}_{u,v} \geq \sum_{\substack{w \in V \\ (w,x) \in E}} \phi^{(w,x)}_{u,v}$$

$$\forall (u,v) \in D \qquad \sum_{\substack{w \in V \\ (v,w) \in E}} \phi^{(v,w)}_{u,v} + (1+\delta_{u,v}) d_{u,v} \geq \sum_{\substack{w \in V \\ (w,v) \in E}} \phi^{(w,v)}_{u,v}$$

$$\forall e \in E \qquad \sum_{(u,v) \in D} \phi^e_{u,v} \leq (1-\kappa_e) C_e$$

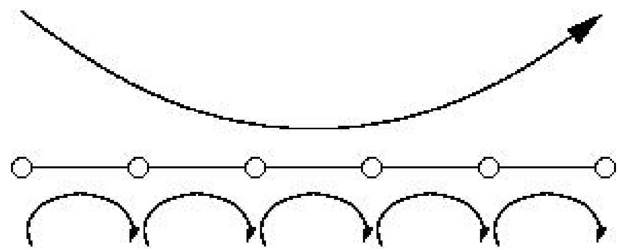
One (our?) goal

A natural goal for an operator is to

Maximize the revenue

Small (but important) consideration

A link can be shared by several connections:



C'est trop injuste !

Links to social politics

There are two natural ways to optimize the way the network will carry the load (Mo and Walrand 1998):

The network-aware optimization

$$\text{Maximize } \frac{1}{1-\alpha} \sum_{e \in E} \kappa_e^{1-\alpha}, \quad \alpha \geq 0, \alpha \neq 1.$$

The user-aware optimization

$$\text{Maximize } \frac{1}{1-\alpha} \sum_{(u,v) \in D} \delta_{u,v}{}^{1-\alpha}, \quad \alpha \geq 0, \alpha \neq 1.$$

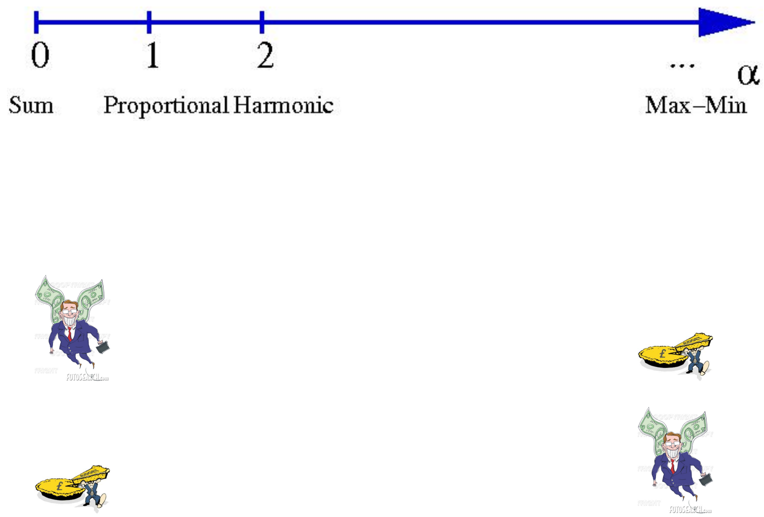
The special case $\alpha = 1$, in both cases, corresponds to what is called the *proportional fairness*, and can be formulated respectively as:

$$\text{Maximize} \prod_{e \in E} \kappa_e,$$

and

$$\text{Maximize} \prod_{(u,v) \in D} \delta_{u,v}.$$

Sense of this optimization (for users): redistribute the social revenue.



”All communities divide themselves into the few and the many. The first are the rich and well-born; the other the mass of the people... turbulent and changing, they seldom judge or determine right. Give therefore to the first class a distinct, permanent share in the Government.”

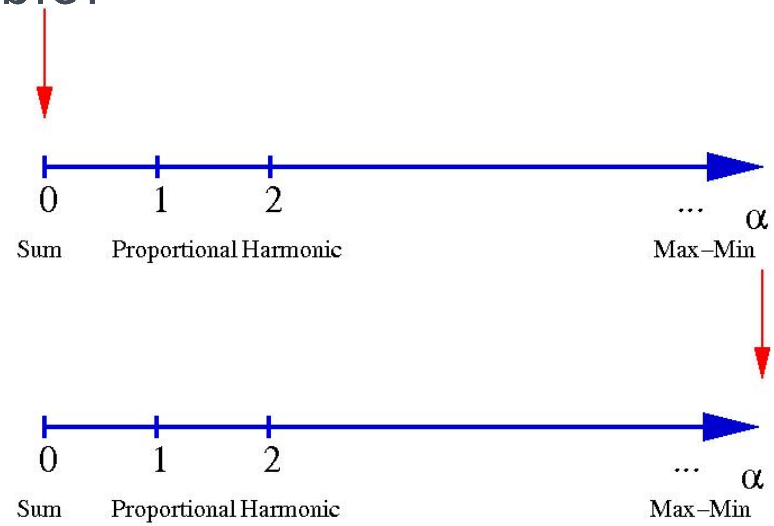
- Alexander Hamilton

”De chacun selon ses capacités, à chacun selon ses besoins.”

- Etienne Cabet

What is LP feasable?

What is LP feasable?



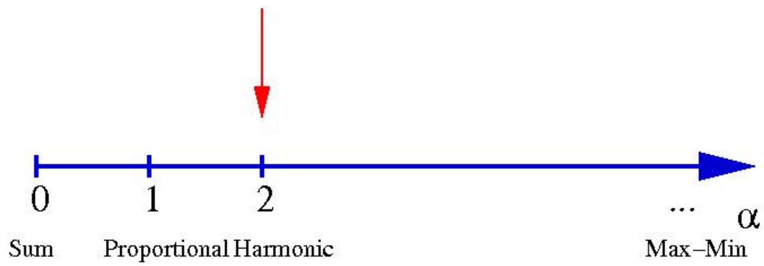
Usage of SDP for fairness criteria

Proposition 1. *Let x, y and z be three positive real numbers. Then*

$$\begin{pmatrix} x & z \\ z & y \end{pmatrix} \succeq 0 \textit{ if and only if } xy \geq z^2.$$

In particular, if one sets $z = 1$, we obtain constraints of the form

$$y \geq \sum_{i=1}^{i=n} \frac{1}{x_i}.$$



Thanks to an idea of Nemirovski, we have:

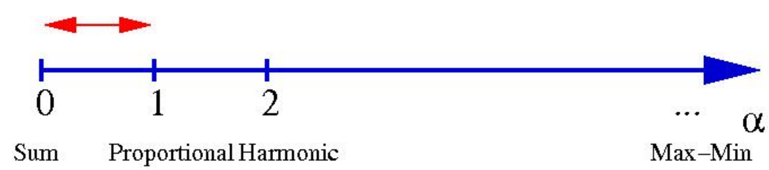
Proposition 2. *Let x and y be two real positive number. It is possible, using SDP constraints, to bound x and y by the relation*

$$y \leq x^{k/2^p},$$

with $p \in \mathbb{N}$ and $k \in \{0, \ldots, 2^p - 1\}$.

In other words, if $\alpha < 1$ is approximated by some $1 - k/2^p$, then

$$y \leq \sum_{i=1}^{i=n} {x_i}^{1-\alpha}$$



Proof Now let $a_1, \ldots, a_p$ be a series of $0/1$ integers, such that

$$k = \sum_{i=1}^{i=p} a_i 2^{i-1}.$$

We note $y_0 = 1$, and submit $y_1, \ldots, y_p$ to the following contraints:

$$\begin{cases} \begin{pmatrix} y_{i-1} & y_i \\ y_i & x \end{pmatrix} \succeq 0 & \text{if } a_i = 1 \\ \begin{pmatrix} y_{i-1} & y_i \\ y_i & 1 \end{pmatrix} \succeq 0 & \text{if } a_i = 0 \end{cases}$$

Then, obviously, $y_i^2 \leq y_{i-1} x^{a_i}$, and if $y_1, \ldots, y_{p-1}$ are submitted to no other constraints, we have:

$$y_p \leq x^{\sum_{i=1}^{i=p} a^i / 2^{p+1-i}} = x^{k/2^p}. \square$$

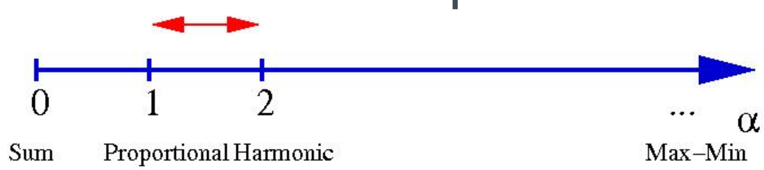
What about values of α greater than one? A simple solution is also given in the following.

Proposition 3. *Let x and y be two real positive numbers. It is possible, using SDP constraints, to bound x and y by the relation*

$$y \geq \frac{1}{x^\beta},$$

where $\beta = k/2^p$, with $p \in \mathbb{N}$ and $k \in \{0, \ldots, 2^p - 1\}$.

Obviously, that solves our problem for $\alpha \in (1;2)$.



Proof Let z be an intermediate variable. Using proposition 2, one can set $z \leq x^\beta$. Also one can write

$$\begin{pmatrix} y & 1 \\ 1 & z \end{pmatrix} \succeq 0$$

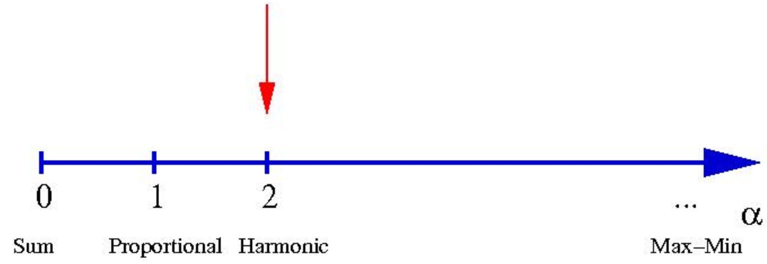
which leads to $yz \geq 1$. Then x and y are bounded by the unique relation: $yx^\beta \geq 1$, hence the result. $\square$

Proposition 4. *Let x and y be two real positive numbers. It is possible, using SDP constraints, to bound x and y by the relation*

$$y \geq \frac{1}{x^{1/\beta}},$$

where $\beta = k/2^p$, with $p \in \mathbb{N}$ and $k \in \{0, \ldots, 2^p - 1\}$.

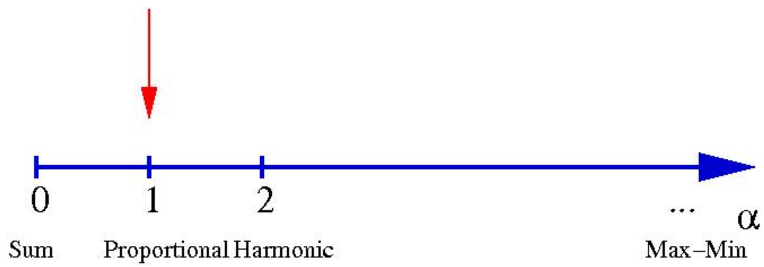
That proposition covers the cases $\alpha \in (2; +\infty)$.



Proportional Fairness

Proposition 5. *Let* y*, and* $x_1, \ldots, x_n$ *be real positive numbers. Then using SDP constraints, it is possible to bound these numbers by the relation*

$$y^{2^{\lceil \log_2(n) \rceil}} \leq \prod_{i=1}^{i=n} x_i.$$

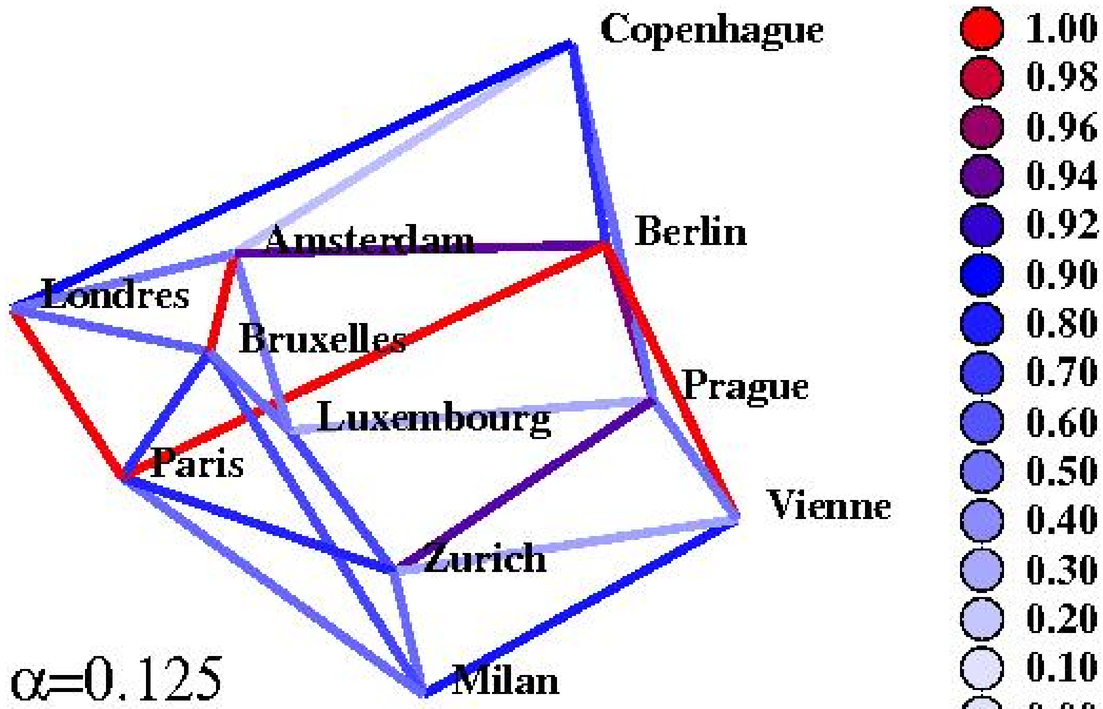


Proof Let p be the smallest integer such that $2^p \geq n$. We construct a family of real positive variables $y_{i2^k+1,(i+1)2^k}$ with $1 \leq k \leq p$, $i \in \{0, \ldots, 2^{p-k}-1\}$ $l \geq 0$, satisfying the following constraints:

$$\begin{pmatrix} y_{2i2^{k-1}+1,(2i+1)2^{k-1}} & y_{i2^k+1,(i+1)2^k} \\ y_{i2^k+1,(i+1)2^k} & y_{(2i+1)2^{k-1}+1,(2i+2)2^{k-1}} \end{pmatrix} \succeq 0,$$

where we note $y_{j,j} = x_j$ for $j \in \{1, \ldots, n\}$, and $y_{j,j} = 1$ for $j \in \{n+1, \ldots, 2^p\}$. $\square$

Experiments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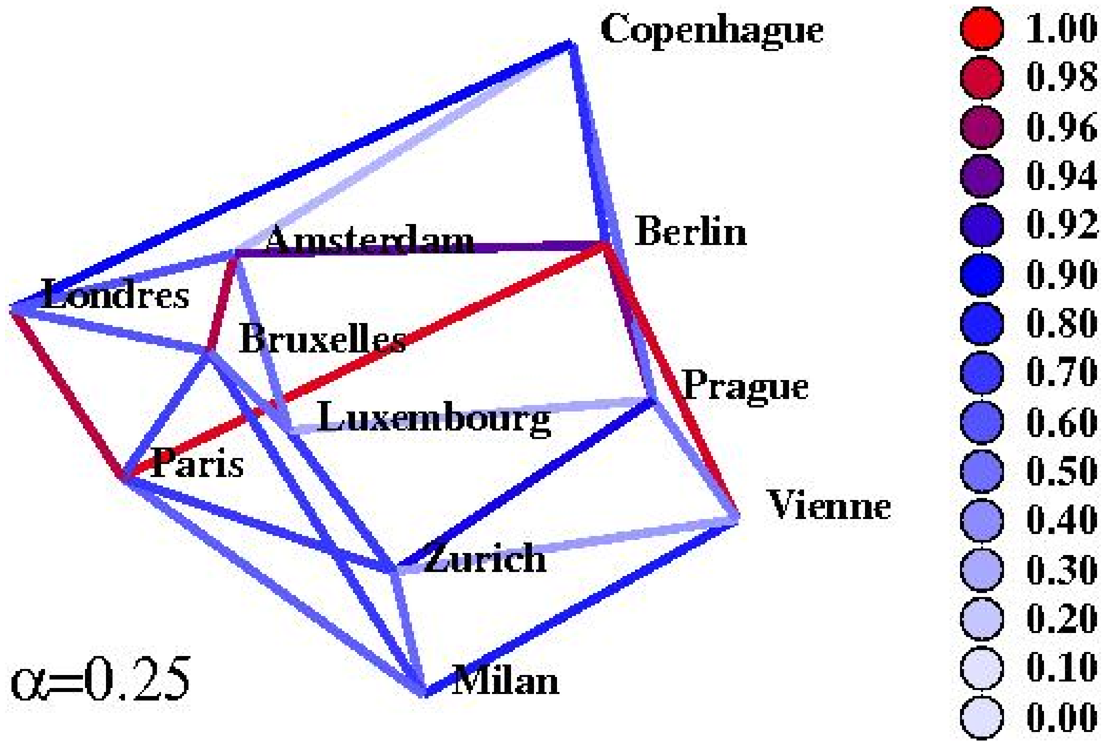
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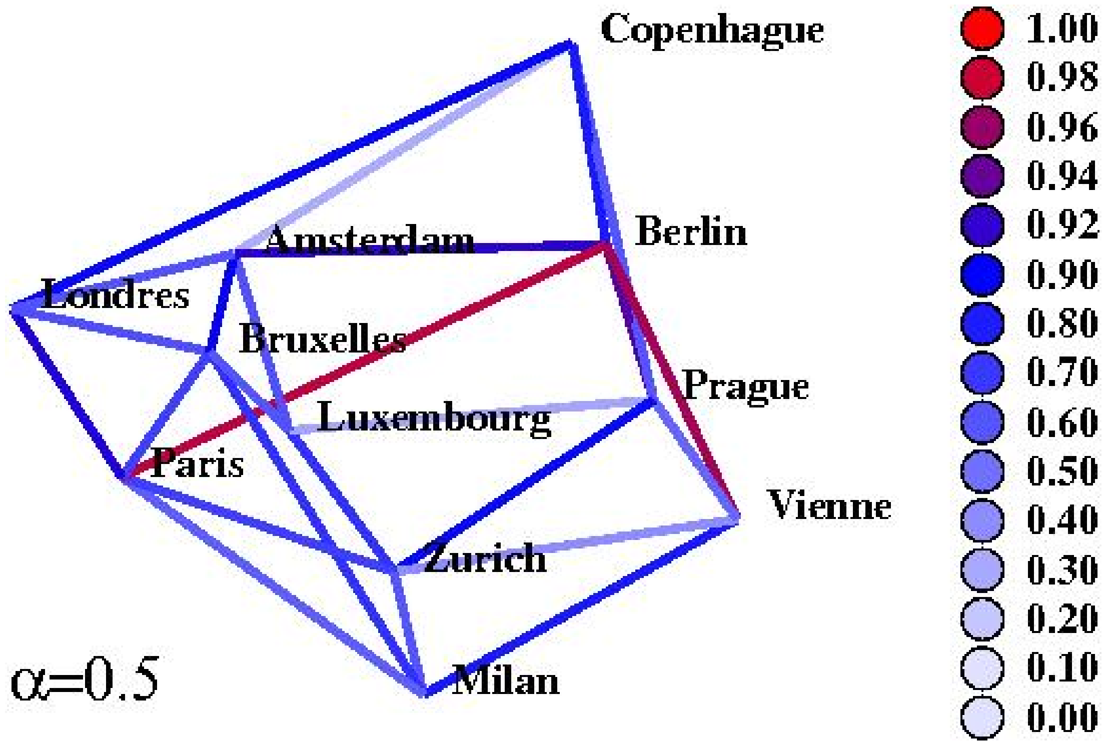
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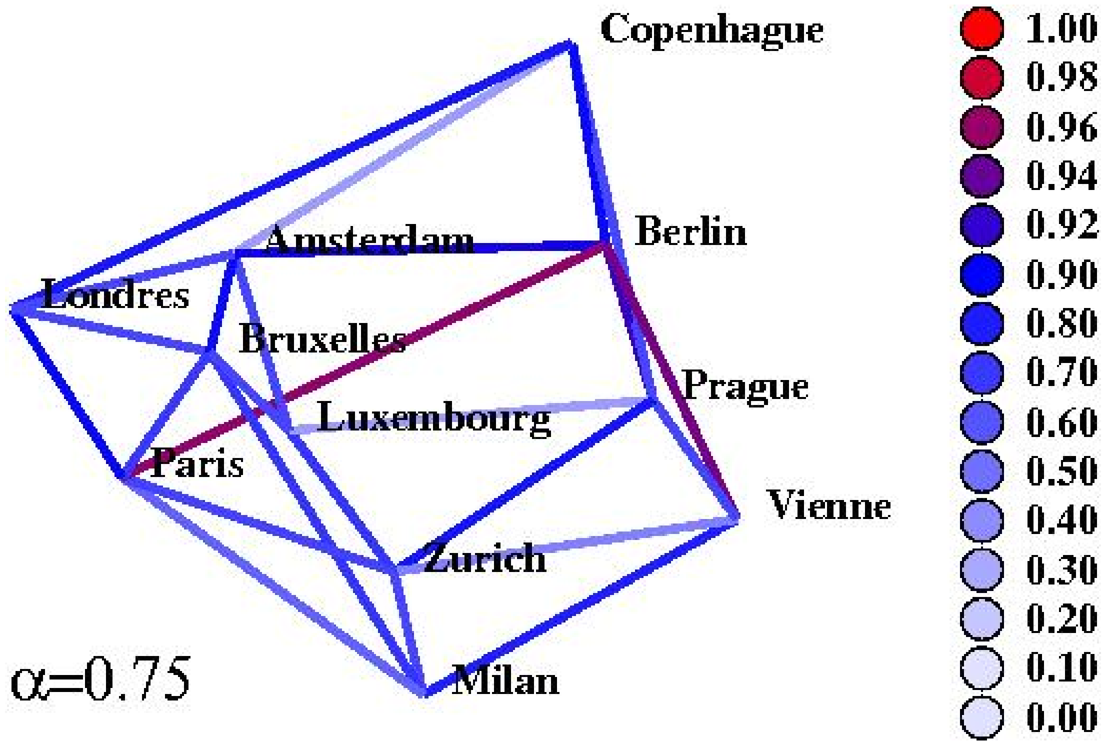
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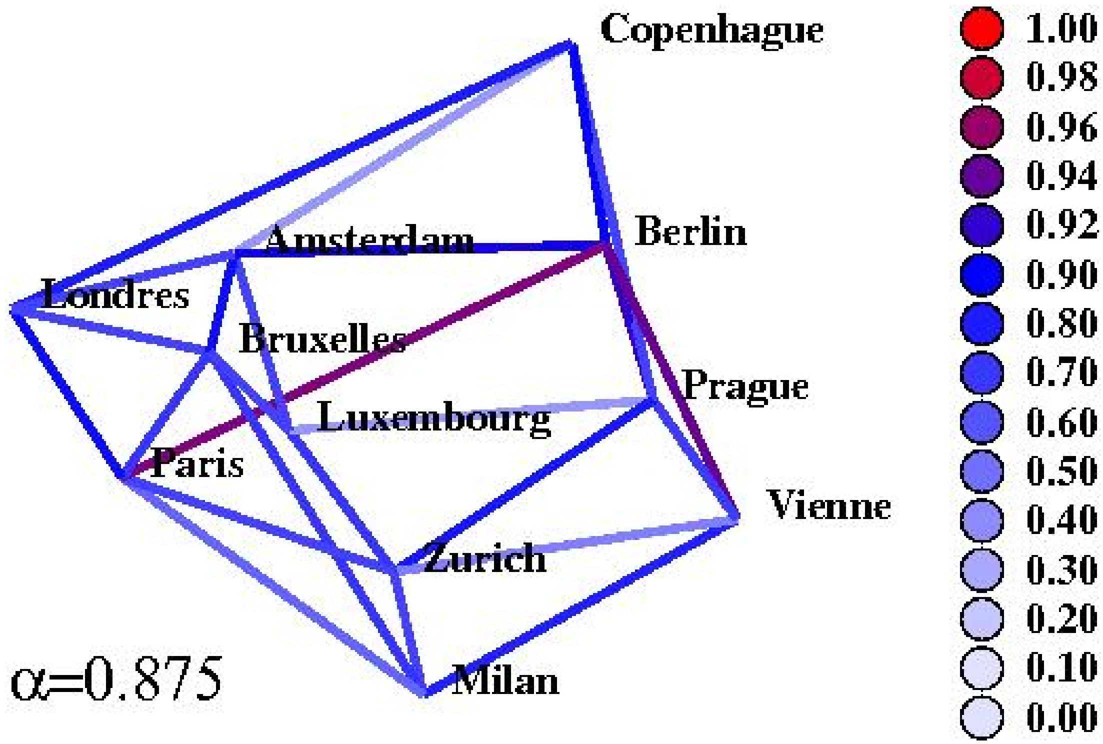
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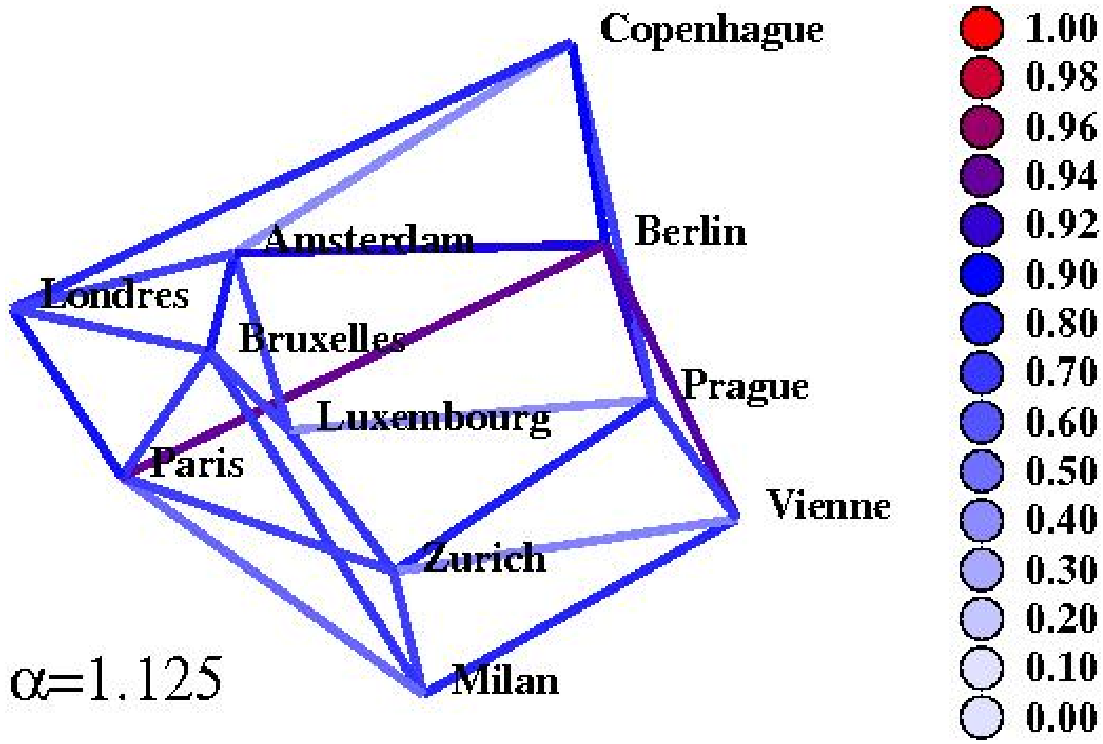
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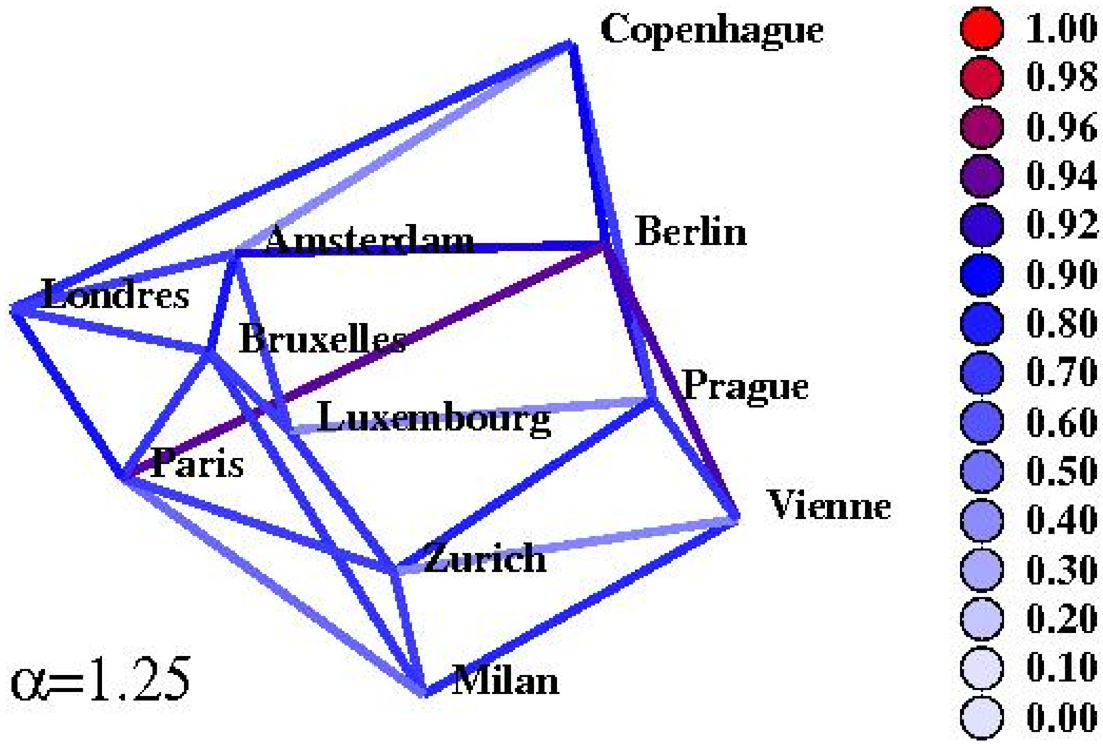
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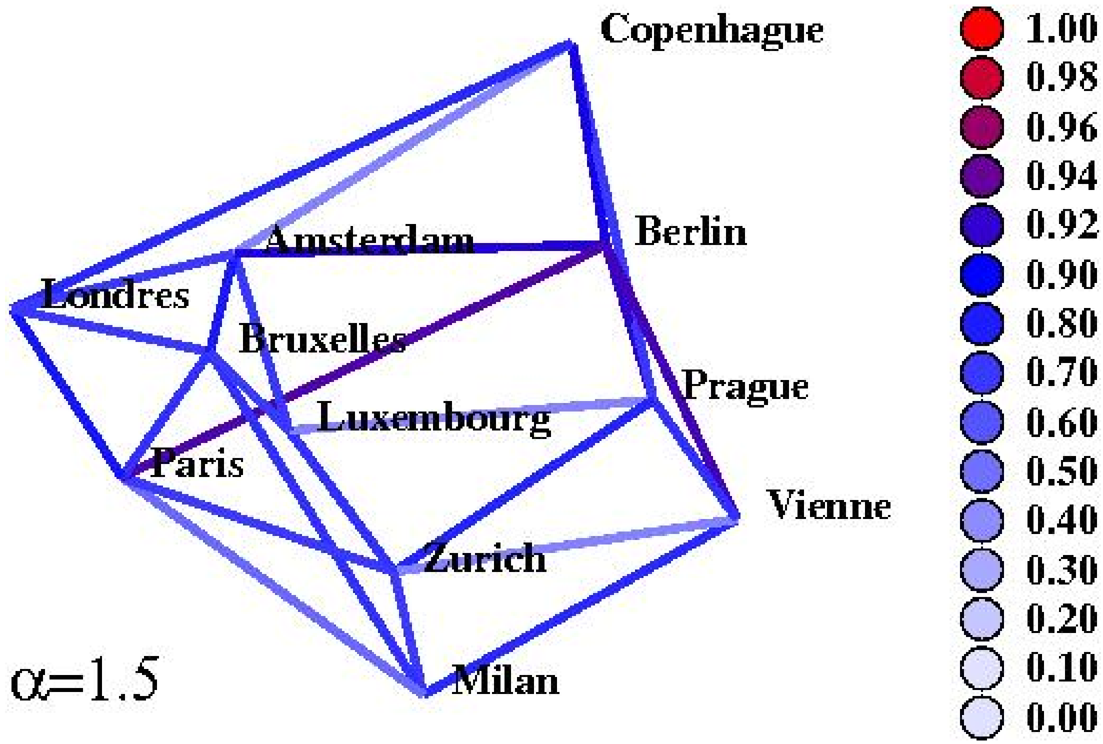
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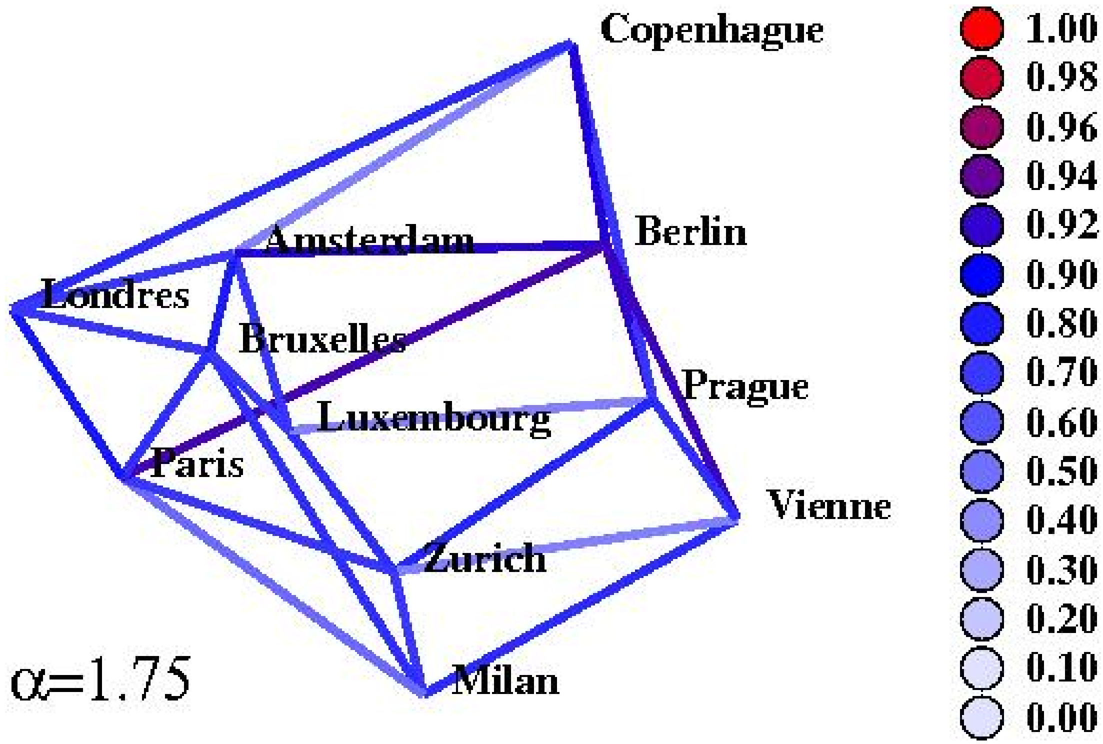

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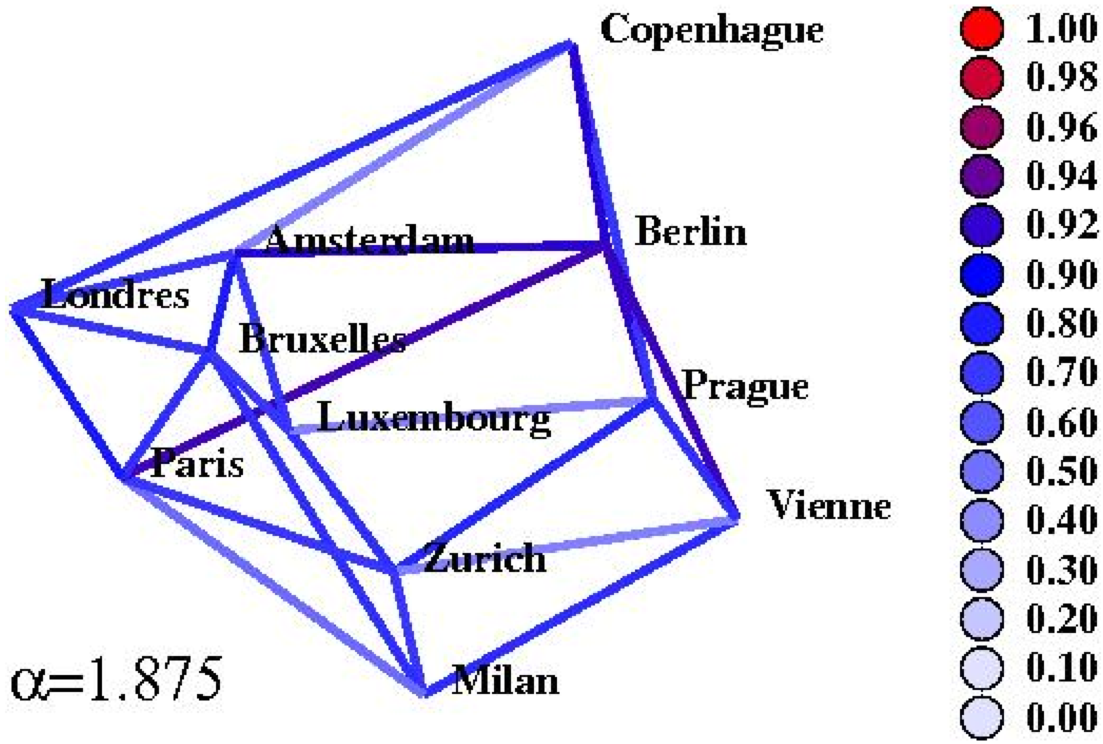
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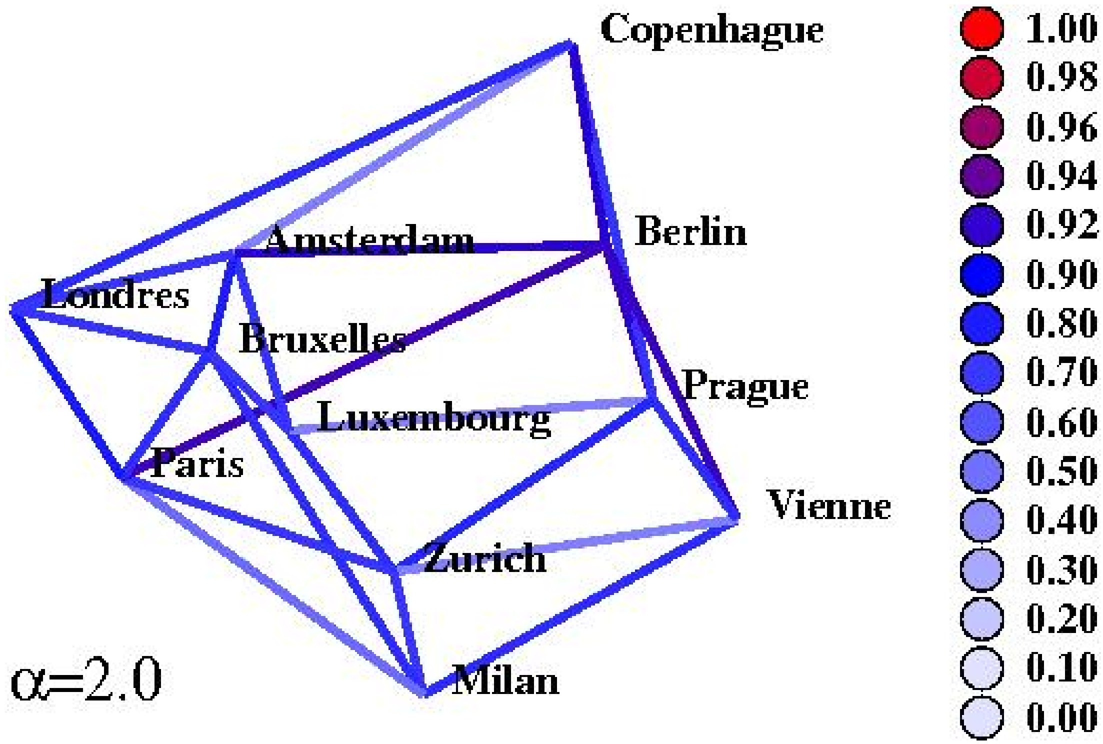
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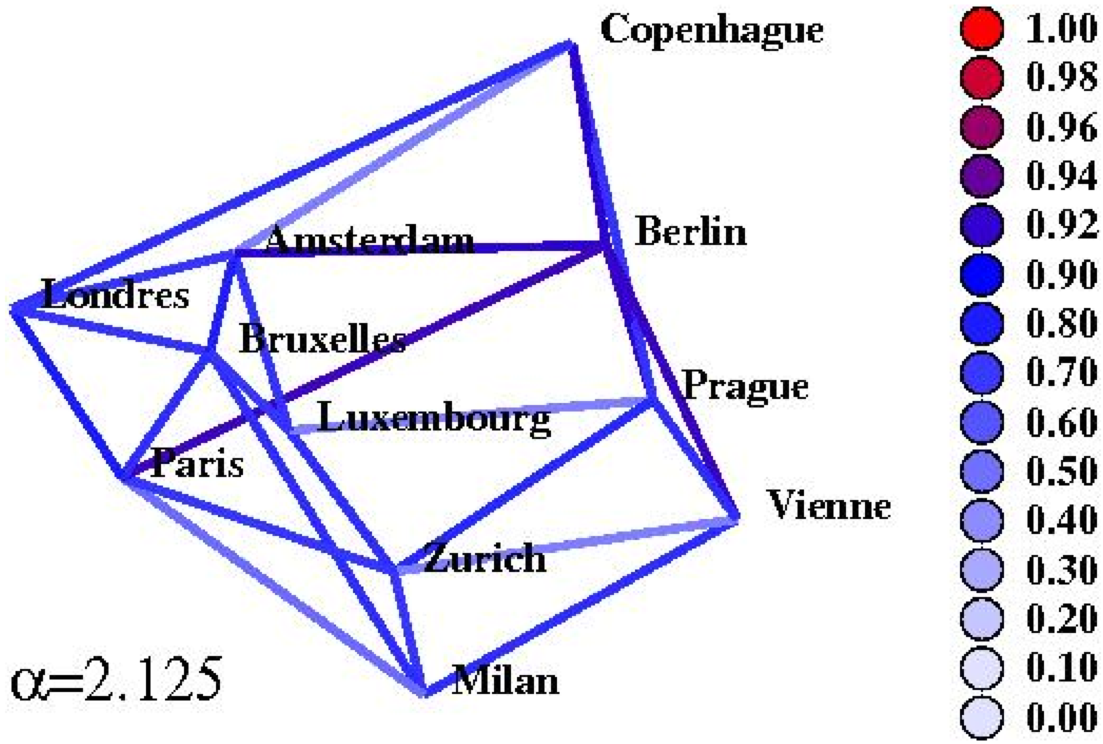
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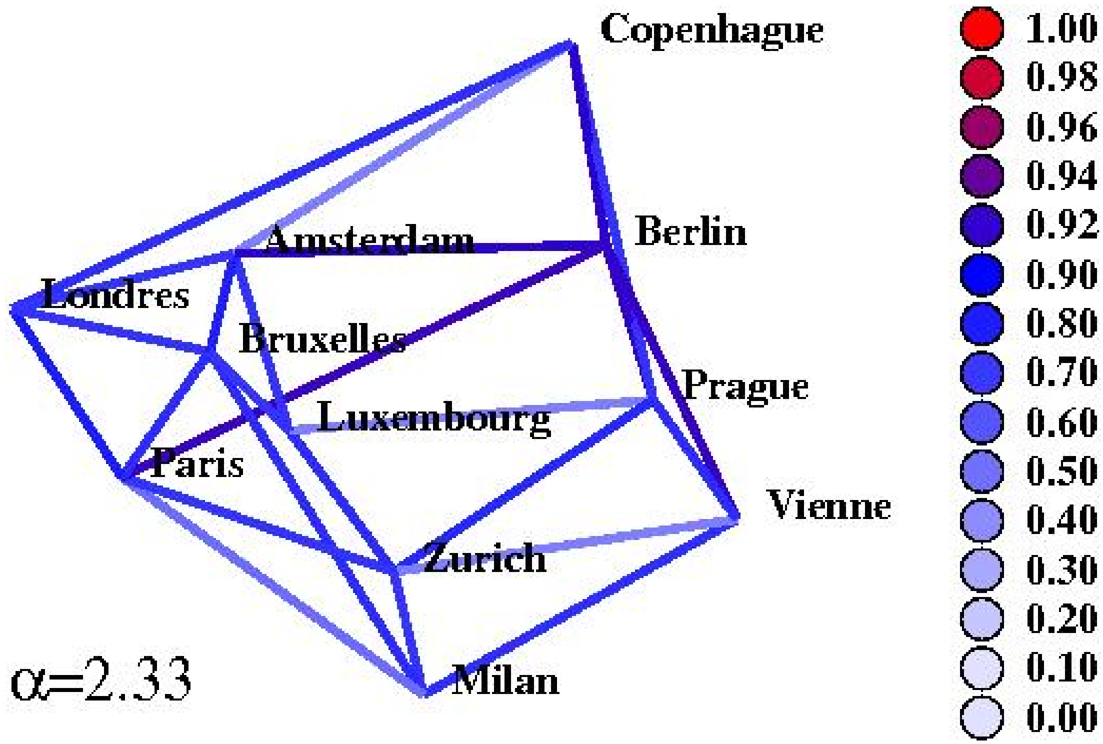
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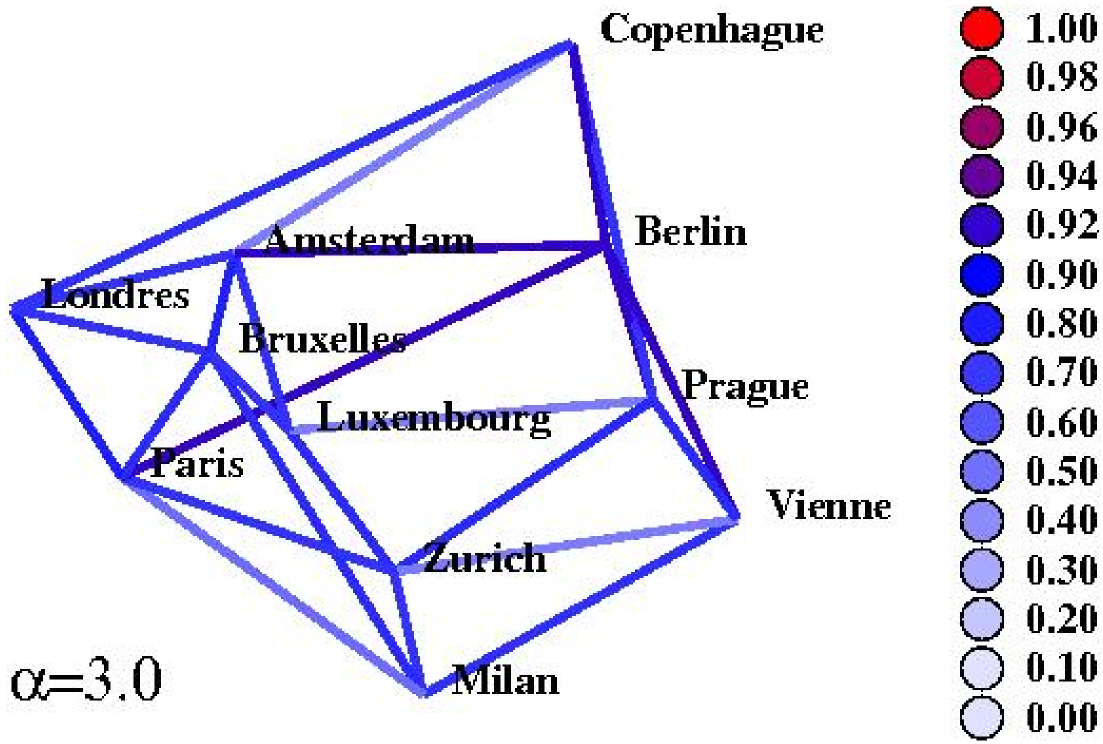
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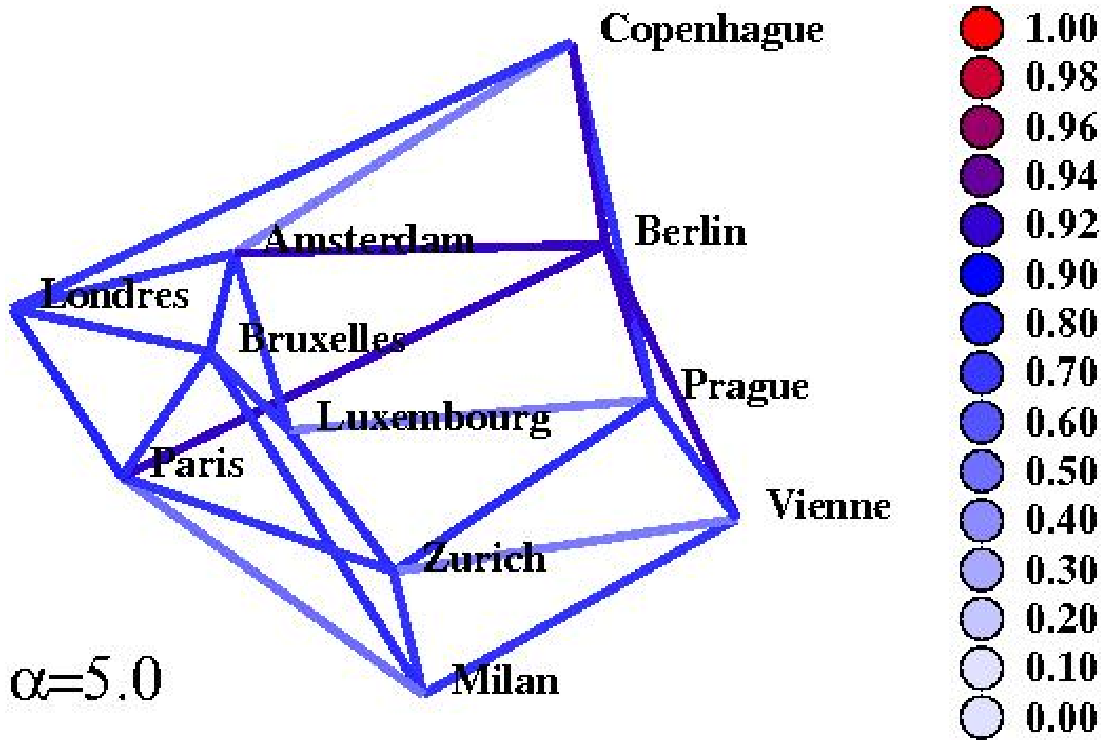
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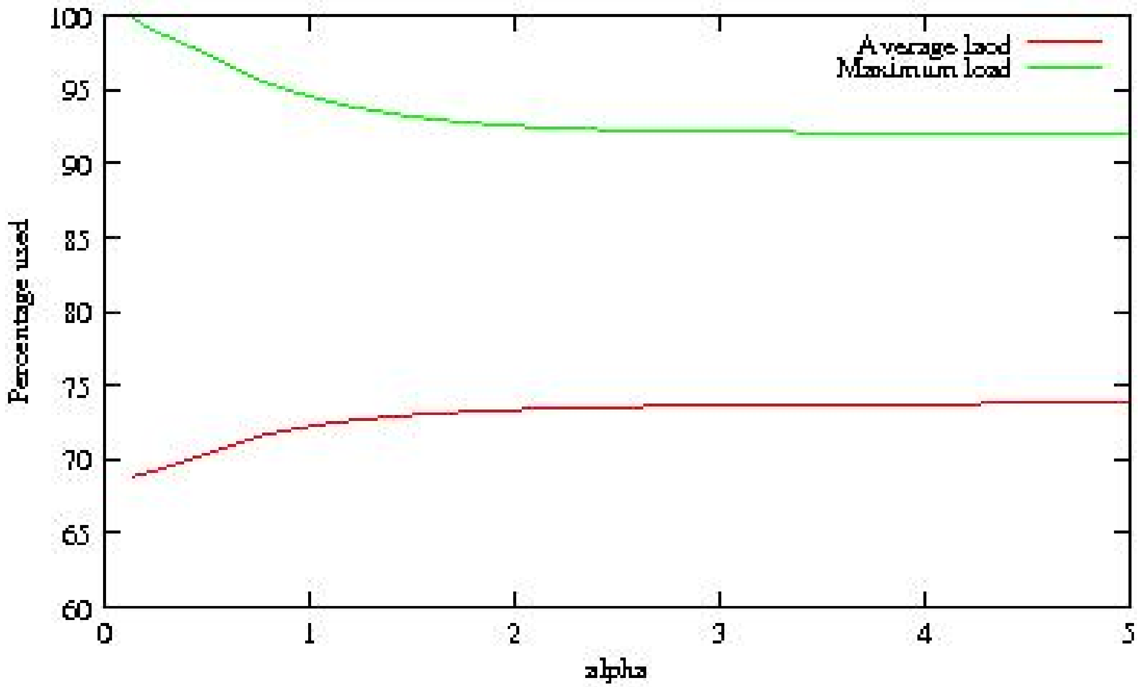
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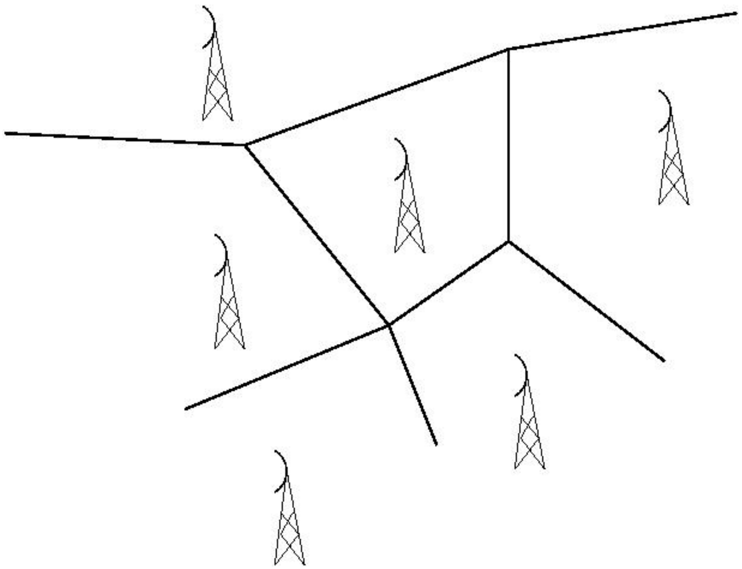
Two different kinds of networks



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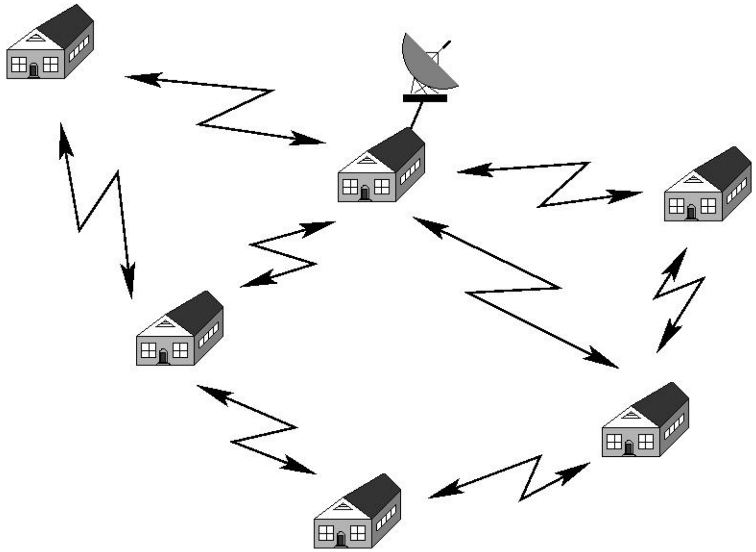


meshed

Main characteristics

Cellular networks:

- Operator-driven
- Planification required to share ressources

Meshed networks:

- "Who's paying what?" problem
- Coverage issues (nice probabilistic models)

Main QoS issues for radio

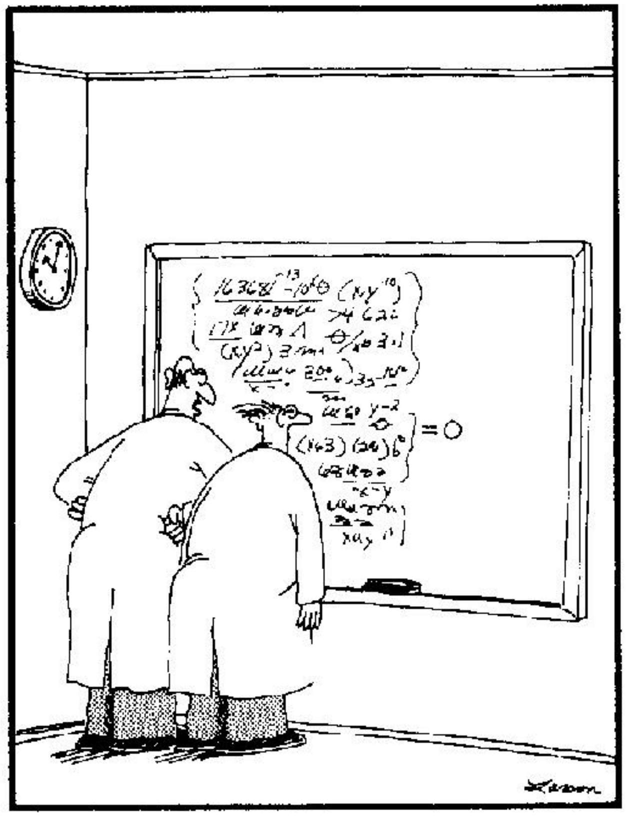
→ Fairness

→ Fast reconfiguration (meteo, business, week-end time, . . .)

→ Scarce ressources

→ Reliability/accessibility

How to put this in equations?



"No doubt about it, Ellington—we've mathematically expressed the purpose of the universe. Gad, how I love the thrill of scientific discovery!"

A fairness formulation

There exists a natural formula that generalizes several 'fairness' objective functions (Mo and Walrand 1998):

$$\text{Maximize } \frac{1}{1-\alpha}\sum_{i\in I} r(i)^{1-\alpha}, \quad \alpha \geq 0, \alpha \neq 1.$$

The special case $\alpha = 1$, corresponds to *proportional fairness*, and can be described as:

$$\text{Maximiser } \prod_{i\in I} r(i).$$

Modeling of an uplink problem (1/2)

Datas:

MR_i	Minimum required rate for i
PR_i	Maximum required rate for i
δ_i	Thresshold of C/I for i (depends on the BER)
$\overline{p}_i$	Maximum power for i
ν_{a_i}	Noise factor for the base a_i
$g_{a_i,j}$	Gain on j for base a_i

Variables:

$r(i)$	Rate assigned to i
p_i	Power assigned to i

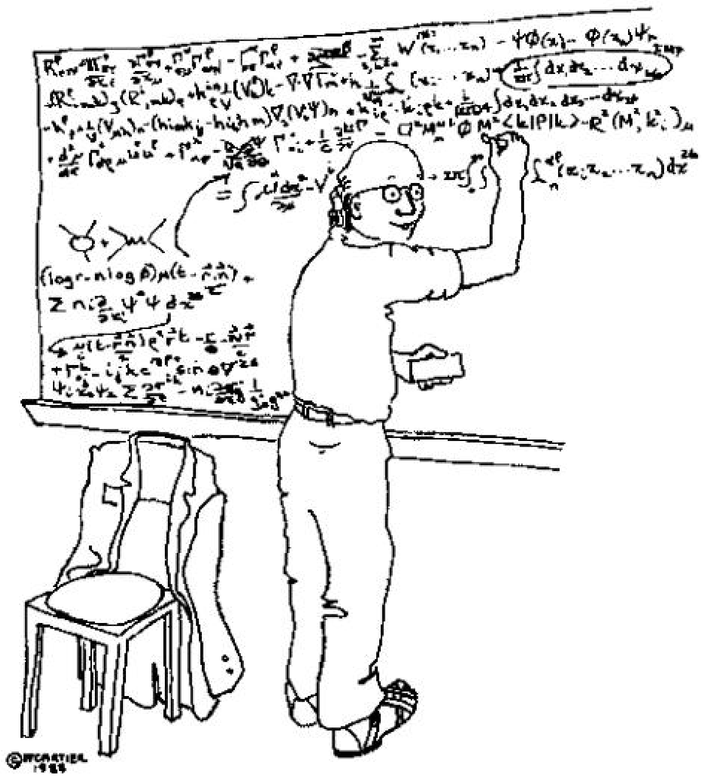
Modeling of an uplink problem (2/2)

$$\left\{\begin{array}{c} MR_i \le r(i) \le PR_i, \\ \\ 0 \le p_i \le \overline{p}_i, \\ \\ \delta_i r(i) \le \dfrac{g_{a_i,i} p_i}{\nu_{a_i} + \sum_{j \ne i} g_{a_i,j} p_j}, \end{array}\right. \qquad 1 \le i \le N.$$

A fair assignment can then be obtained by solving the following problem:

$$\text{Maximize} \sum_{i=1}^{N} \frac{r(i)^{1-\alpha}}{1-\alpha}. \tag{1}$$

It is even not convex!



"At this point we notice that this equation is beautifully simplified if we assume that space–time has 92 dimensions."

Next step: SDP

Perron root of a matrix

$$\rho(X) = \min\{\lambda \geq 0 : \lambda \cdot I - X \succeq 0\}.$$

Theorem 1. *Let X be a symetric square root matrix with nonneative entries.*

Then the system

$$(I - X)p = b$$

with $b > 0$ has a solution $p \geq 0$ iff $\rho(X) > 0$

Note that it is a rewriting of SIR conditions!

Bibliographical notes

1992	Bambos and Pottie	Powers with no white noise in polynomial time
1993	Foschini and Miljanic/Hanly	Powers in polynomial time
1994	Yun and Messerschmitt	Analytical solution
1999	Leelahakriengkrai and Agrawali	Power/rates is non-convex
2001	Andersin and Henrikson	Patent for B&B solution
2002	Won Lee, Mazumbar, and Shroff	Downling using branch-and-price (almost linear)
2004	Altman, Touati and G.	Solution usind SDP
2006	Galtier	Uplink using combined branch-and-price/analytical in linear time

A first simplification...

Given an adequate change of variables: $\forall i \in [1..N],\ \rho(i) = \dfrac{r_i}{1+\delta_i r(i)}$,and manipulating somehow the equations, the problem turns to:

$$\begin{cases} \dfrac{MR_i}{1+\delta_i MR_i} \leq \rho(i) \leq \dfrac{PR_i}{1+\delta_i PR_i}, \\ 0 \leq \nu_{a_i}\delta_i\rho(i) \leq g_{a_i,i}\bar{p}_i \left(1 - \sum_{j=1}^{N} \delta_j \rho(j)\right) \end{cases}$$

It is linear!

But the objective function becomes: $\displaystyle\sum_{i=1}^{N} \frac{1}{1-\alpha} \left(\frac{\rho(i)}{1-\delta_i\rho(i)} \right)^{1-\alpha}$.

Problem reformulation

$$\text{Maximize } \sum_{i=1}^{p} f_i(\lambda_i)$$

$$\text{under the constraints } \begin{array}{ll} \lambda_i - m_i \geq 0 & \forall i \in \{1, \ldots, p\} \\ M_i - \lambda_i \geq 0 & \forall i \in \{1, \ldots, p\} \\ C - \sum_{i=1}^{p} \lambda_i \geq 0 & \forall i \in \{1, \ldots, p\}. \end{array}$$

where p is an integer, f_i a concave function, continuously derivable for $i \in \{1, \ldots, p\}$, and C, m_i, M_i be positive real numbers, $(i \in \{1, \ldots, p\})$.

Characteristics of the optimal solution λ^*

Analyzing the Kuhn-Tucker conditions, noting the dual coefficients μ_i, ν_i $i \in \{1, \ldots, p\}$ and ρ, for each i, $i \in \{1, \ldots, p\}$, three cases can occur:

- $\lambda_i^* = m_i$. Then $\nu_i = 0$ and we have $-f_i'(\lambda_i^*) - \mu_i + \rho = 0$, which gives $f_i'(\lambda_i^*) \leq \rho$.
- $\lambda_i^* = M_i$. In this case $\mu_i = 0$ and then $-f_i'(\lambda_i^*) + \nu_i + \rho = 0$, which gives $f_i'(\lambda_i^*) \geq \rho$.
- $\lambda_i^* \in]m_i, M_i[$. We then have $\nu_i = 0$ and $\mu_i = 0$, so $f_i'(\lambda_i^*) = \rho$.

The key of the problem

It consists in *backtracking* the value of ρ. We can associate, to each ρ, a capacity used, as:

$$\varphi(\rho) = \sum_{i:\rho > f_i'(m_i)} m_i + \sum_{i:\rho < f_i'(M_i)} M_i + \sum_{i:f_i'(M_i) \leq \rho \leq f_i'(m_i)} (f_i')^{-1}(\rho),$$

and then all the problem consists in solving:

$$\varphi(\rho^*) = C.$$

An algorithm

- By dichotomy find an interval $]y_1; y_2[$ for ρ where $\varphi(\rho^*) = C$ occurs and for all i, $m_i \notin]y_1; y_2[$ and $M_i \notin]y_1; y_2[$.

- Once this interval is found, compute

$$\rho^* = \left[\sum_{\substack{i: f_i'(M_i) \le y_1 \\ y_2 \le f_i'(m_i)}} (f_i')^{-1} \right]^{-1} \left(C - \sum_{i: y_2 > f_i'(m_i)} m_i - \sum_{i: y_1 < f_i'(M_i)} M_i \right).$$

- Backtrack corresponding values for λ^*.

It is almost linear!

Conclusion & further work

→ Many UMTS radio problems can get avantadge of linear/convex optimization tools.

→ We can test and readjust very quickly the capacity for an individual cell.

→ Apply this to all the cells respects the constraints, at the price of (rarely) degrading the capacity.

Perspectives:

- More than one cell.
- More on real implementation.

Contents...

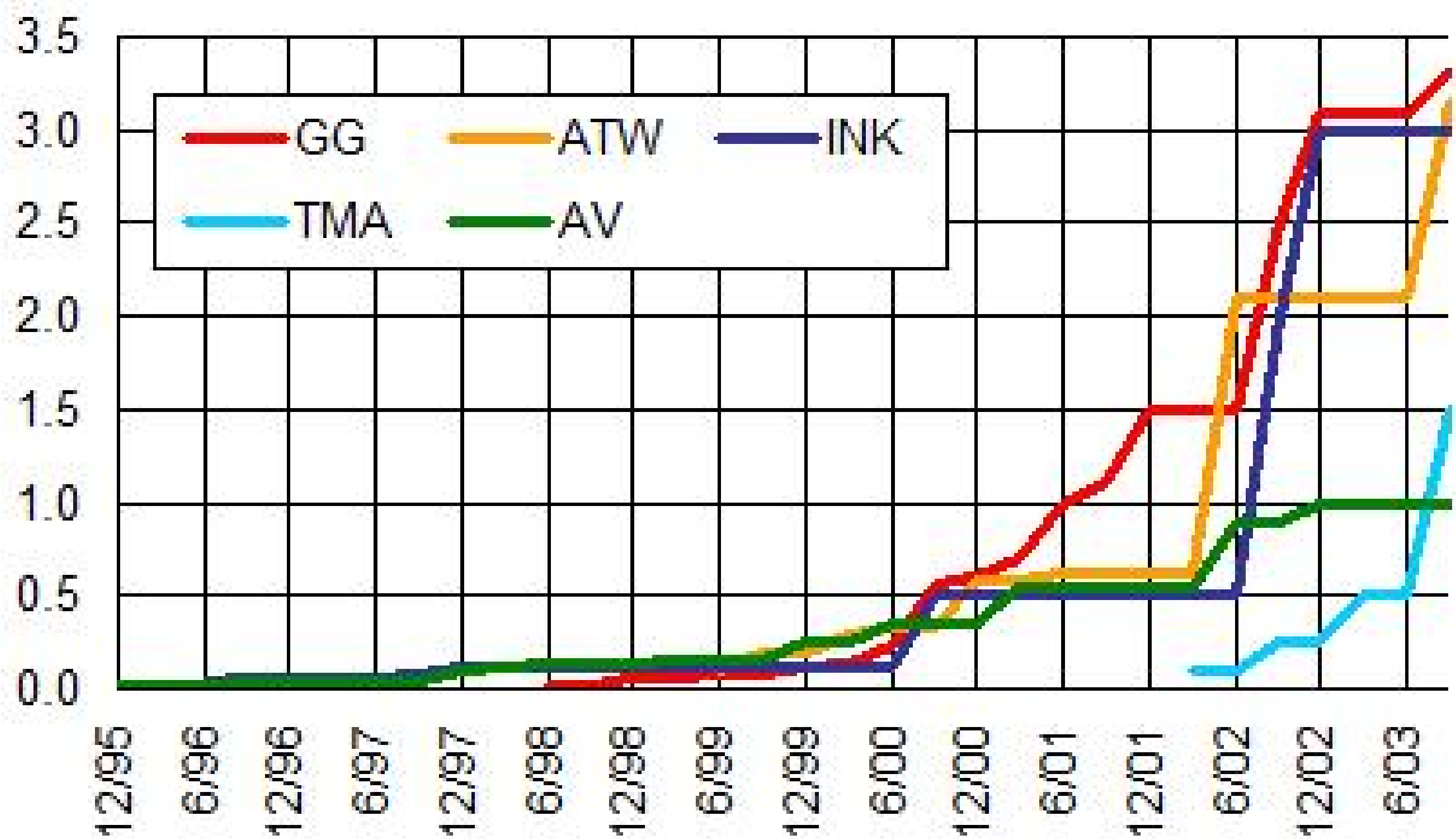
- SDP for fairness/wireline
- SDP for radio
- SDP for Webworld

the search engines: a gateway to the web?

www.searchenginegenie.com

10^6 searches	per day	per month
Google	91	2,733
Yahoo	60	1,792
MSN	28	845
AOL	16	486
Ask	13	378
others	6	166
total	213	6,400

A massive set of data



in *billions* of links

Google principle

Method based on the **rank** of a page

$$\rho(p) = K \left(\alpha \sum_{q \in \Gamma^-(p)} \frac{\rho(q)}{|\delta^+(q)|} + (1-\alpha)E(p) \right),$$

where

K is the normalization parameter,

α is a balancing parameter,

$E(p)$ is the probability that a user (uniformly) falls on p.

our approach

→ employ our **graph theory** background

→ give **alternative approaches** to search the web

the PACK 237: cartographie Pages Web

- brevet 04843 (A. Laugier) : randonnée
- brevet 05618 (A. Laugier, S. Raymond) : mineur
- brevet 05711 (J. Galtier) : webworld

→ integrated in the PAC by PIV

Randonnée and Mineur methods

$\longrightarrow$ Randonnée

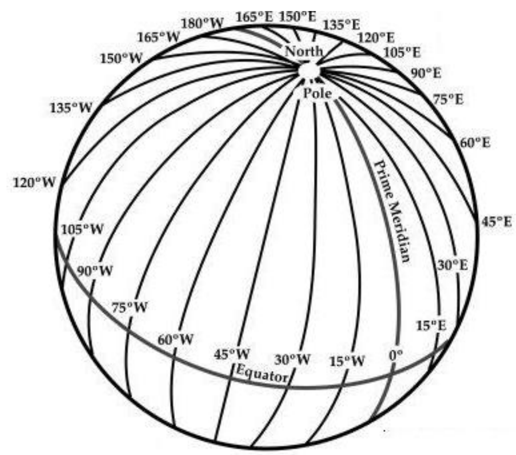
- Look-up of graph expanders
- Extension by random walks

$\longrightarrow$ Mineur

- Computation of powers of adjacency matrix
- Elimination of poorly connected nodes

idea of webworld

WWW $\mapsto$



URL $\longrightarrow$ a point in the unit sphere

the web pages are viewed as **particules**.

2 particules attract each other iff one cites the other or they are cocited. otherwise, they repel each other.

a good sphere: the Goemans-Williamson spectrahedron

Given a set of vectors $Y_i \in \mathbb{R}^d$, $i \in \{1, \ldots, n\}$,

$$r_d = \max \sum_{\{i,j\} \in E} \delta_{i,j} \frac{1 - Y_i \cdot Y_j}{2}$$

$$\text{subject to} ||Y_i|| = 1, \qquad i \in \{1, \ldots, n\}$$

- $r_1 < r_2 < \cdots < r_n$, $r_n = r_k$ for $k \geq n$,
- For $d = n$, SDP problem, for $d = 1$, $r_1 = MAXCUT$, $r_d < 0.87856\ r_1$.

connections to web problem

we are given an undirected graph $G = (V, E)$.

each edge $\{u, v\}$ is associated to a weight $w_{u,v} \in [0, 1]$, that emphasizes the **degree of connection between the pages**.

we set δ as follows: $\delta_{u,v} = \left| \begin{array}{ll} 1 & \text{if } \{u, v\} \notin E \\ 1 - w_{u,v} & \text{if } \{u, v\} \in E. \end{array} \right.$

two equivalent Goemans-Williamson problems (KKT)

$$\max \sum_{\{i,j\}\in E} \delta_{i,j} \frac{1 - Y_i \cdot Y_j}{2}$$

(1) subject to $Y_i \in \mathbb{R}^d, ||Y_i|| = 1, i \in \{1, \ldots, n\}$

(2) subject to $Y_i \in \mathbb{R}^d, ||Y_i|| \leq 1, i \in \{1, \ldots, n\}$

(3) subject to $Y_i \in \mathbb{R}^d, Y_i = -\sum_{\{j,i\}\in E} \delta_{i,j} Y_j / || \sum_{\{j,i\}\in E} \delta_{i,j} Y_j ||,\ i \in \{1, \ldots, n\}$

the trick!

we associate to each vertex u a point $X(u)$ in the 3D unit sphere.

problem: maximize $\displaystyle\sum_{\{u,v\}\in E} \delta_{u,v}||X(u)-X(v)||^2.$

the algorithm of Webworld

the algorithm used is **incremental**.

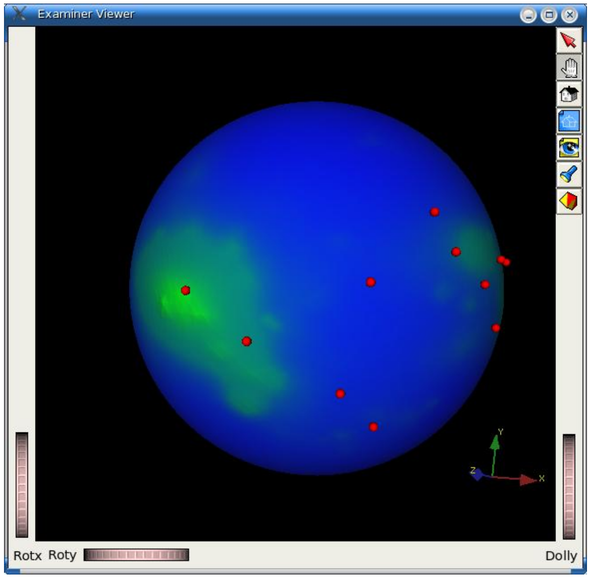
we update each vertex u as follows:

- compute the value $Y(u) := \sum_{v \in V - \{u\}} \delta_{u,v} X(v)$,
- if $Y(u) \neq 0$, we perform the operation $X(u) := -Y(u)/||Y(u)||$,
- if $Y(u) = 0$, choose $X(u)$ at random.

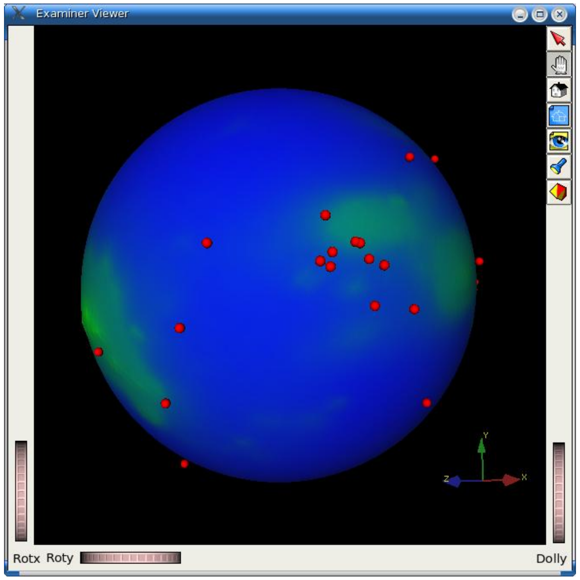
$\longrightarrow$ we increase the estimated **Shannon** capacity for the graph. in practice, **3 min** are required to update 97.10^6 **nodes**.

some results

the green parts of the planet represent the concentrated areas of sites, while the blue parts are the spase areas. the red bullets are the answers in voila.fr for the request.



request "orange"



request "france telecom"

Conclusion

SDP is fun!!!

- open doors to convex programming
- easy to use, many solvers
- a shortcut to new fields