Interior Point Methods in Mathematical Programming

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Journées en l'honneur de Pierre Huard
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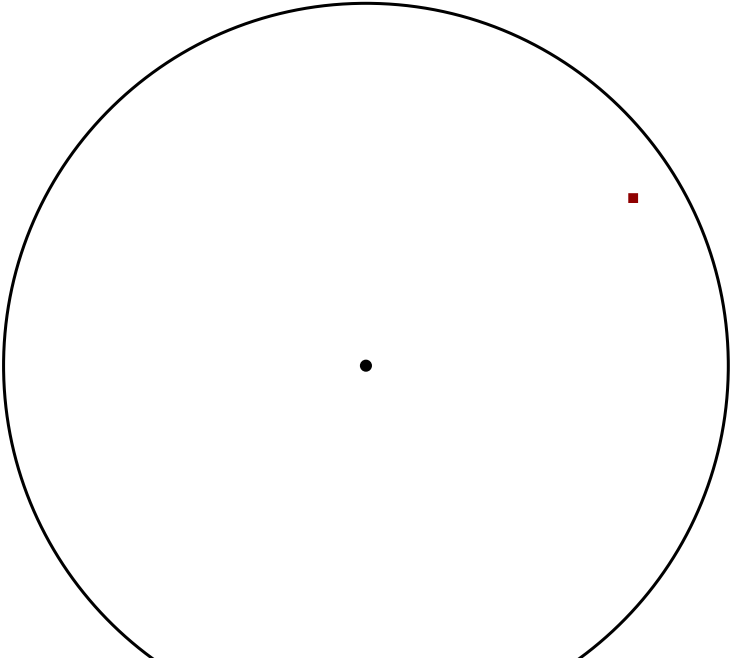

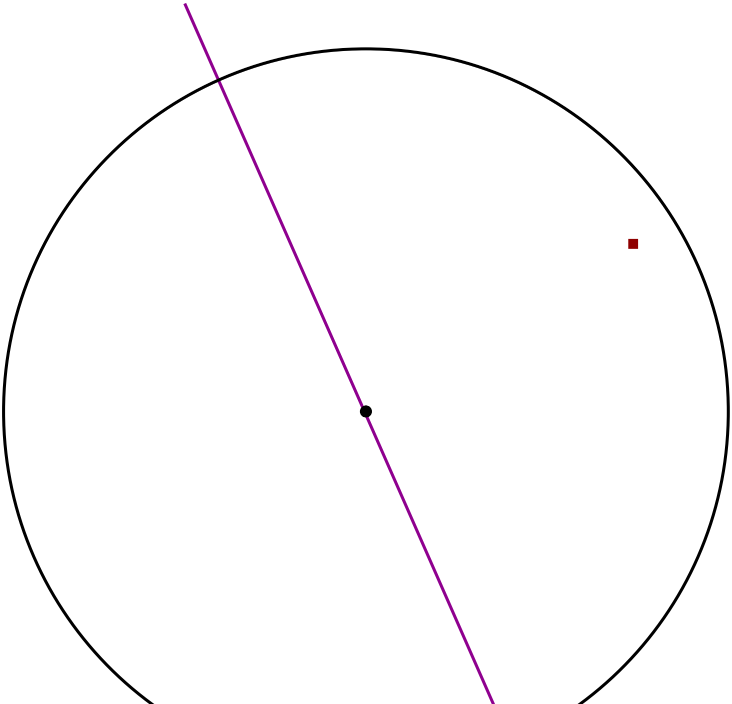

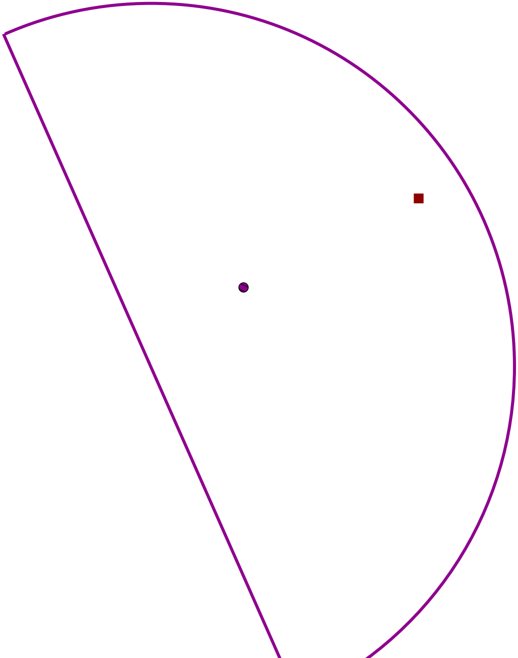

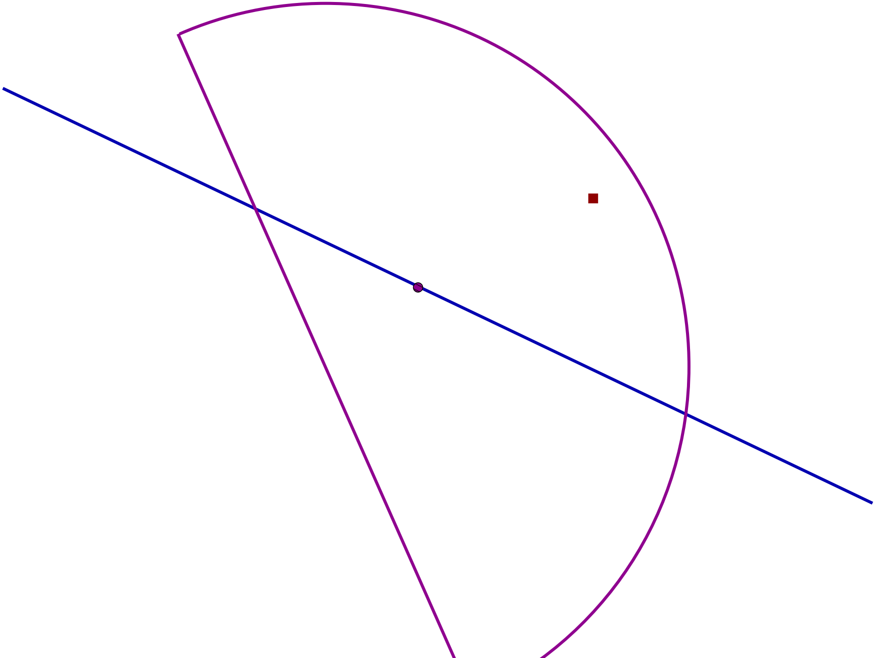

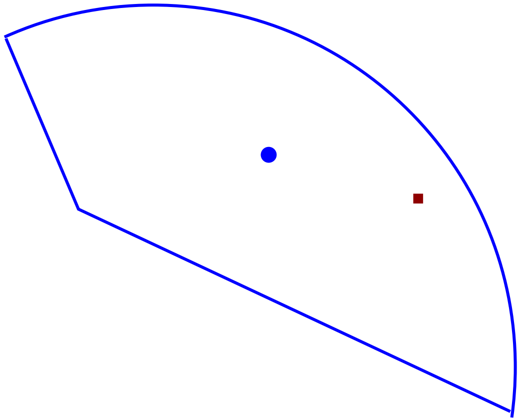

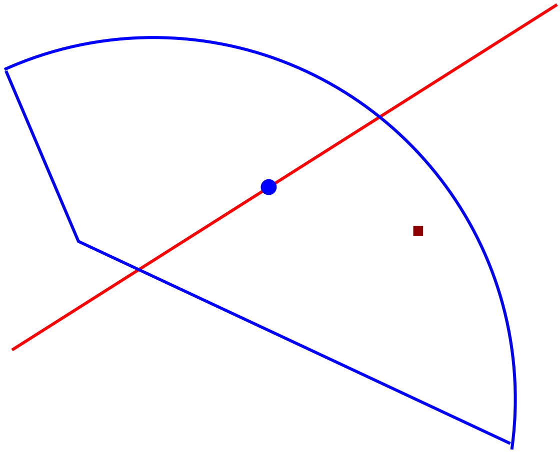

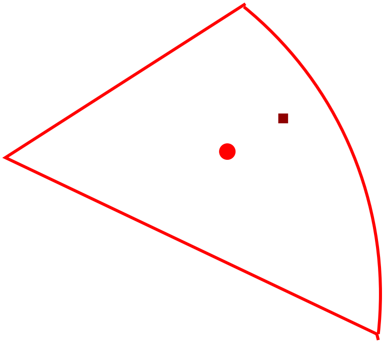

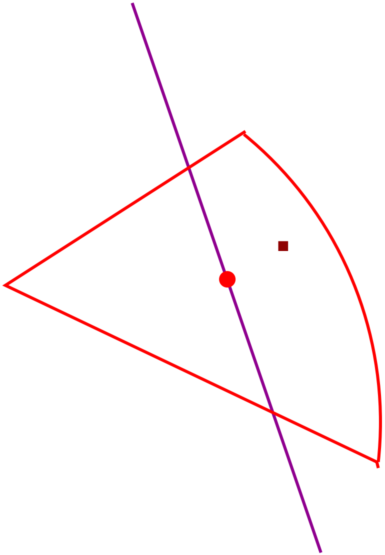

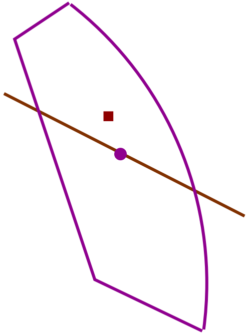

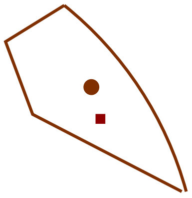

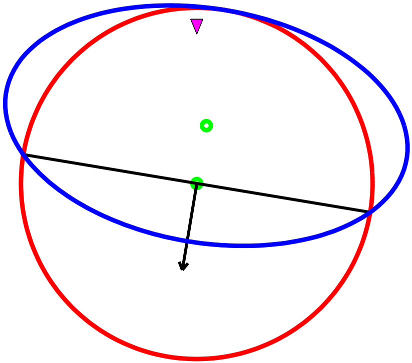

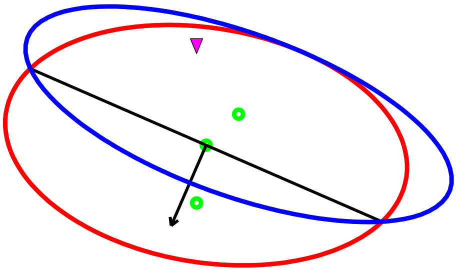

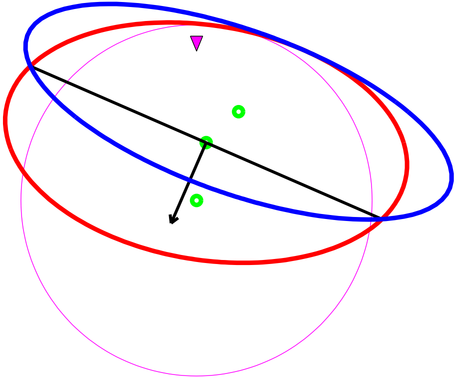

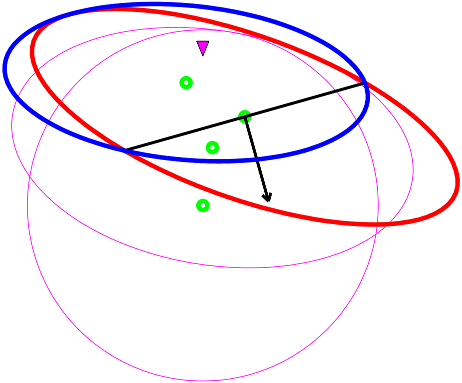

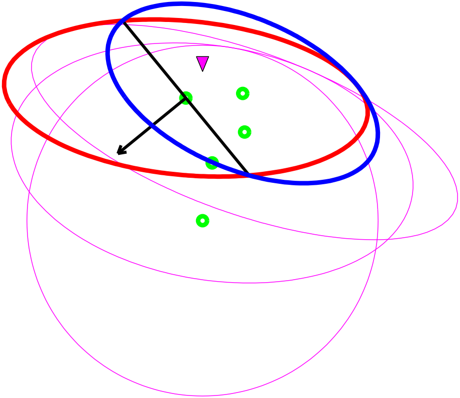

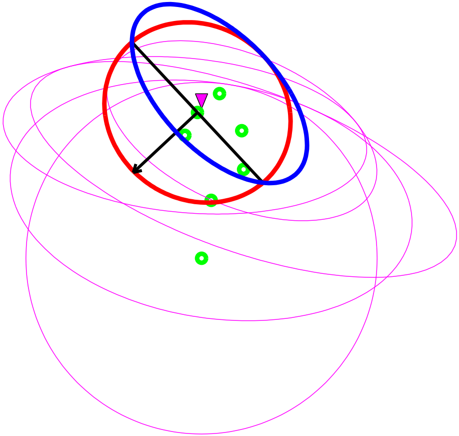

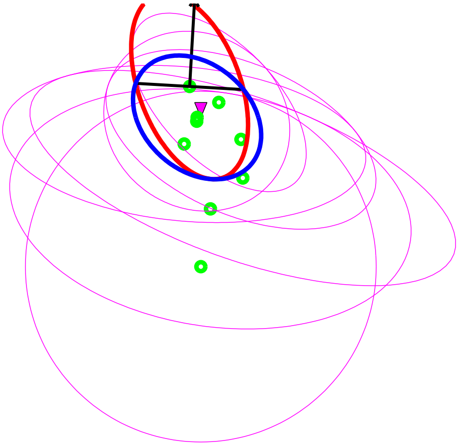

Definition of interior point

General non linear programming problem

Equality and Inequality

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & g(x) \leq 0 \\ & h(x) = 0 \end{aligned}$$

Equality and non-negativity

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f(x) = 0 \\ & x_I \geq 0 \end{aligned}$$

- Interior point: $x > 0$.
- Solution: $f(x) = 0$.
- Feasible solution: $f(x) = 0$, $x_I \geq 0$.
- Interior (feasible) solution: $f(x) = 0$, $x_I > 0$.

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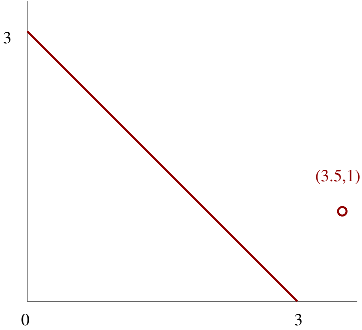
Example of interior infeasible point

Inequality

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & x \leq 3 \\ & x \geq 0 \end{array}$$

Equality and non-negativity

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & x_1 + x_2 = 3 \\ & x_1, x_2 \geq 0 \end{array}$$



The Affine-Scaling direction

Projection matrix

Given $c \in \mathbb{R}^n$ and a matrix A,
c can be decomposed as

$$c = P_A c + A^T y,$$

where $P_A c \in \mathcal{N}(A)$ is the projection of c into $\mathcal{N}(A)$.

The Affine-Scaling direction

Linearly constrained problem:

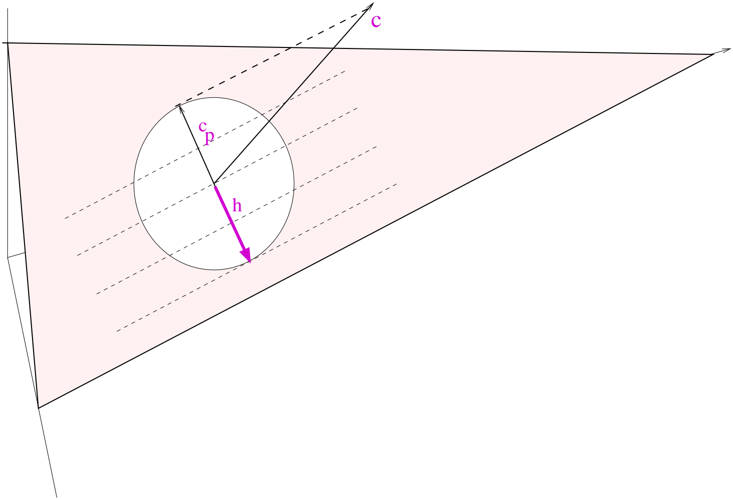
$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

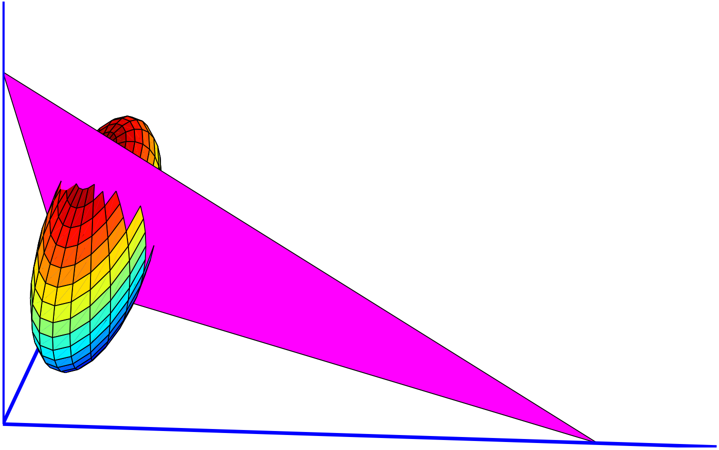
Define $c = \nabla f(x^0)$. The projected gradient (Cauchy) direction is

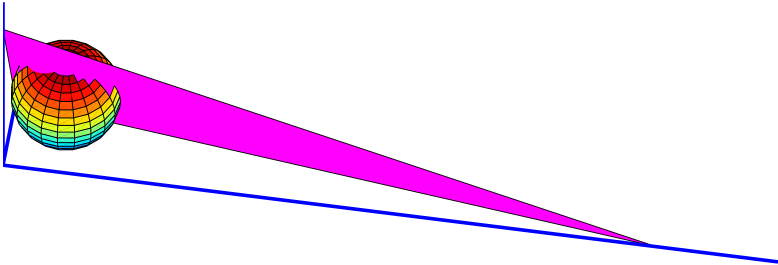
$$c_P = P_A c,$$

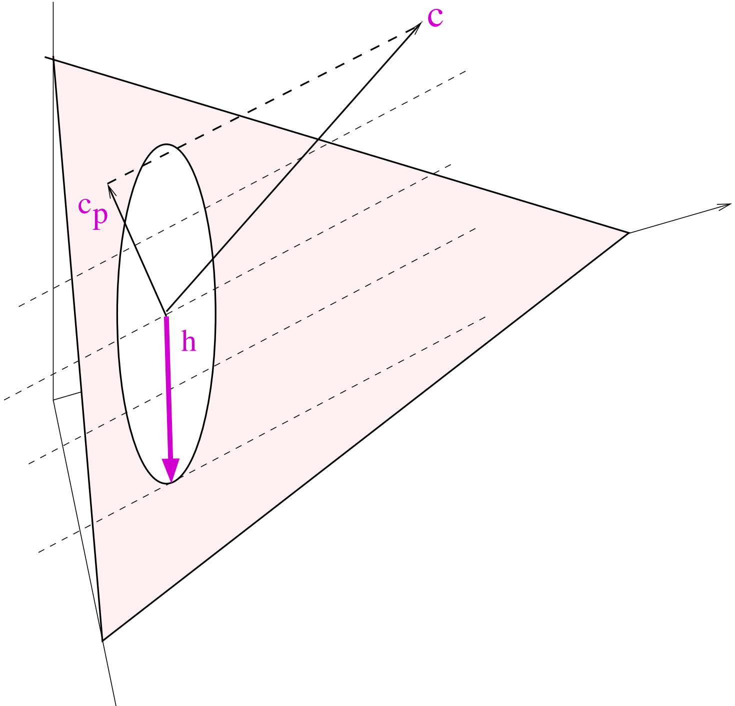
and the steepest descent direction is $d = -c_P$. It solves the trust region problem

$$\text{minimize}\{c^T h \mid Ah = 0, \ \|d\| \leq \Delta\}.$$


c
c
p
h






c
c p
h

The Affine-Scaling direction

Given a feasible point x_0, $X = diag(x_0)$ and $c = \nabla f(x_0)$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \qquad \begin{array}{rcl} x & = & X\bar{x} \\ d & = & X\bar{d} \end{array} \qquad \begin{array}{ll} \text{minimize} & (Xc)^T \bar{x} \\ \text{subject to} & AX\bar{x} = b \\ & \bar{x} \geq 0 \end{array}$$

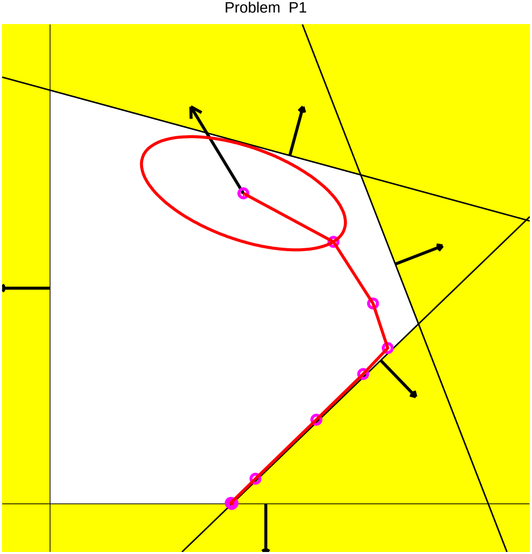
Scaled steepest descent direction:

$$\begin{array}{rcl} \bar{d} & = & -P_{AX}Xc \\ d & = & X\bar{d} = -XP_{AX}Xc \end{array}$$

Dikin's direction:

$$\begin{array}{rcl} \bar{d} & = & -P_{AX}Xc \\ d & = & -X\bar{d}/\|\bar{d}\|. \end{array}$$

Dikin's algorithm



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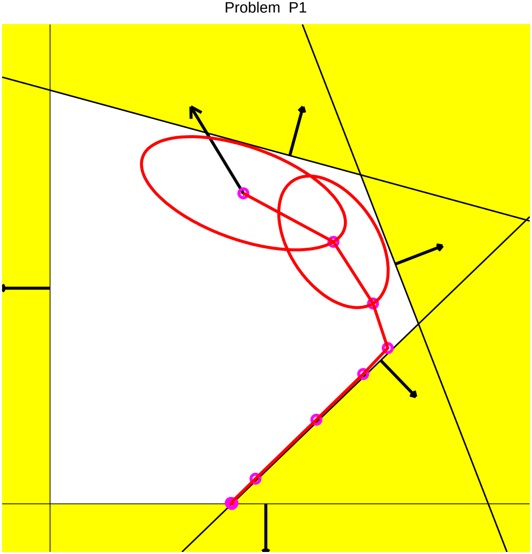

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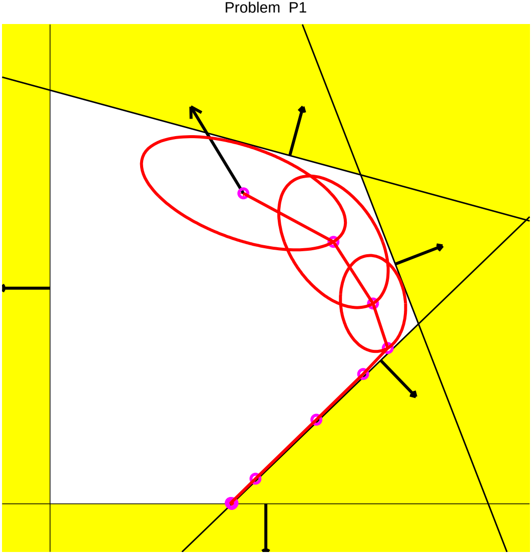

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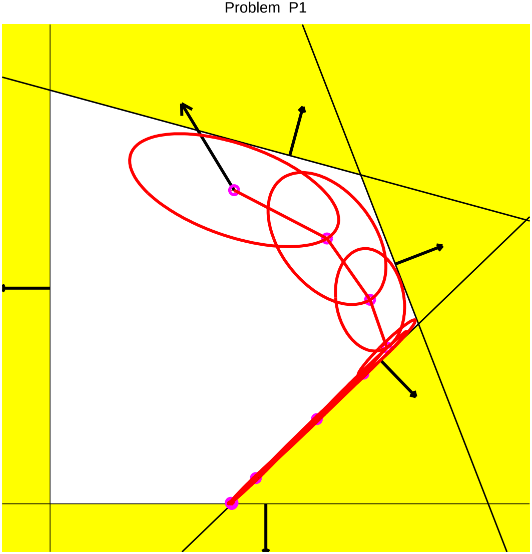
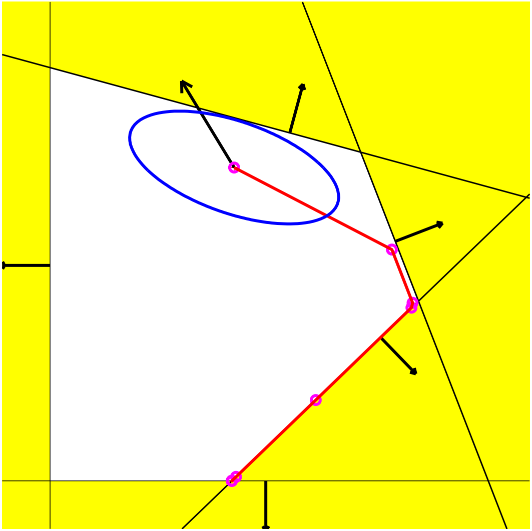


Affine scaling algorithm

Problem P1



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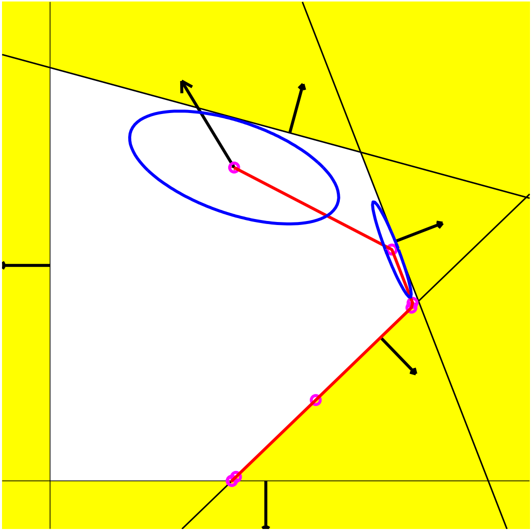

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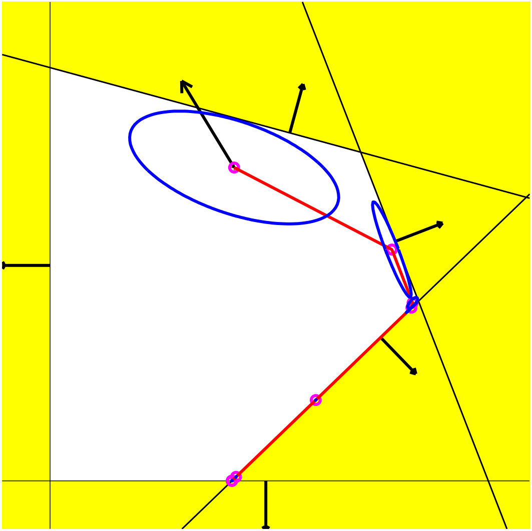

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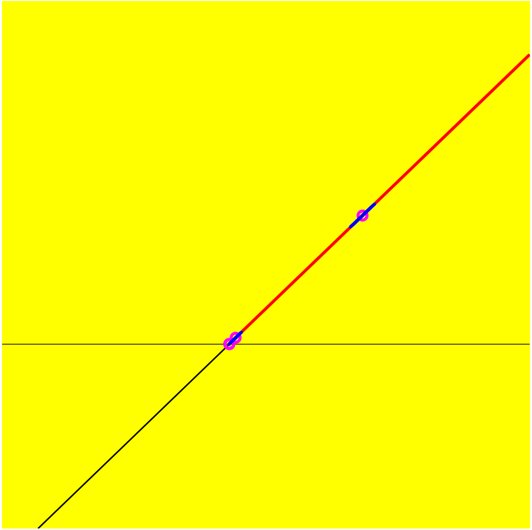

The logarithmic barrier function

$$x \in \mathbb{R}^n_{++} \mapsto p(x) = -\sum_{i=1}^{n} \log x_i.$$

Scaling: for a diagonal matrix $D > 0$

$$\begin{aligned} p(Dx) &= p(x) + \text{ constant}, \\ p(Dx) - p(Dy) &= p(x) - p(y). \end{aligned}$$

Derivatives:

$$\begin{aligned} \nabla p(x) &= -x^{-1} & \qquad \nabla p(e) &= -e \\ \nabla^2 p(x) &= X^{-2} & \qquad \nabla^2 p(e) &= I. \end{aligned}$$

At $x = e$, the Hessian matrix is the identity, and hence the Newton direction coincides with the Cauchy direction.

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At $x = e$, the Hessian matrix is the identity, and hence the Newton direction coincides with the Cauchy direction.

At any $x > 0$, the affine scaling direction coincides with the Newton direction.

The penalized function in linear programming

For $x > 0$, $\mu > 0$ and $\alpha = 1/\mu$,

$$f_\alpha(x) = \alpha c^T x + p(x) \quad \text{or} \quad f_\mu(x) = c^T x + \mu p(x)$$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \qquad \begin{array}{ll} \text{minimize} & c^T x + p(x) \\ \text{subject to} & Ax = b \\ & \bar{x} \geq 0 \end{array}$$

- For $\alpha \geq 0$ f_α is strictly convex and grows indefinitely near the boundary of the feasible set.
- Whenever the minimizers exist, they are defined uniquely by

$$x_\alpha = \text{argmin}_{x \in \Omega} f_\alpha(x).$$

- In particular, if Ω is bounded, x_0 is the analytic center of Ω
- If the optimal face of the problem is bounded, then the curve

$$\alpha > 0 \mapsto x_\alpha$$

is well defined and is called the primal central path.

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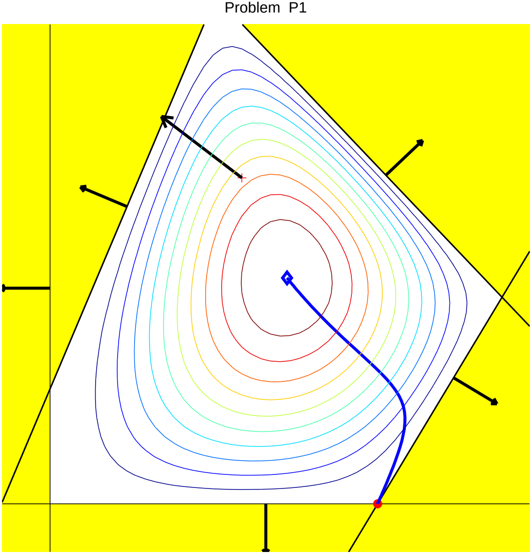
The central path



Equivalent definitions of the central path

There are four equivalent ways of defining central points:

- Minimizers of the penalized function:

$$\text{argmin}_{x\in\Omega} f_\alpha(x).$$

- Analytic centers of constant cost slices

$$\text{argmin}_{x\in\Omega}\{p(x) \mid c^T x = K\}$$

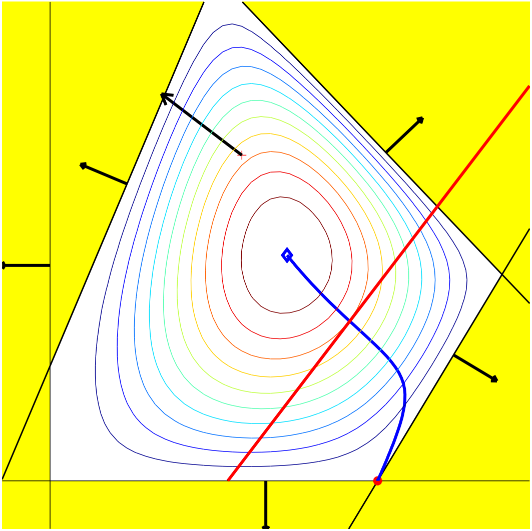
- Renegar centers: Analytic centers of Ω with an extra constraint $c^T x \leq$.

$$\text{argmin}_{x\in\Omega}\{p(x) - \log(K - c^T x) \mid c^T x < K\}$$

- Primal-dual central points (seen ahead).

Constant cost slices

Enter the new cut position (one point) and then the initial point



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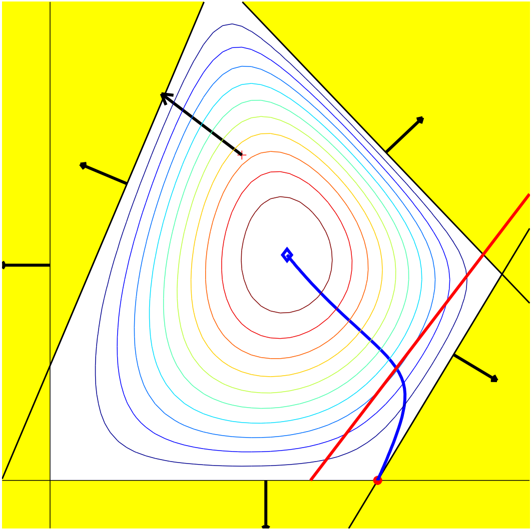

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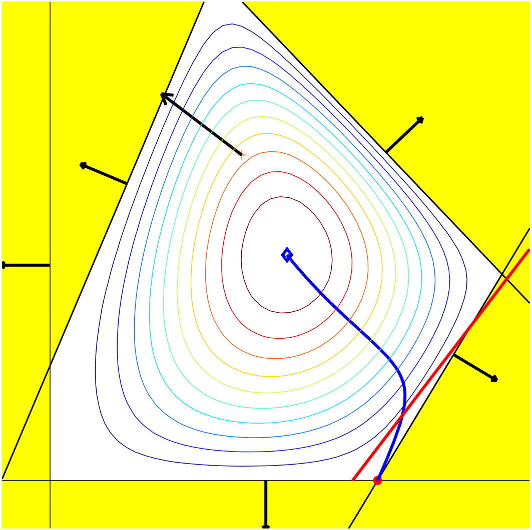
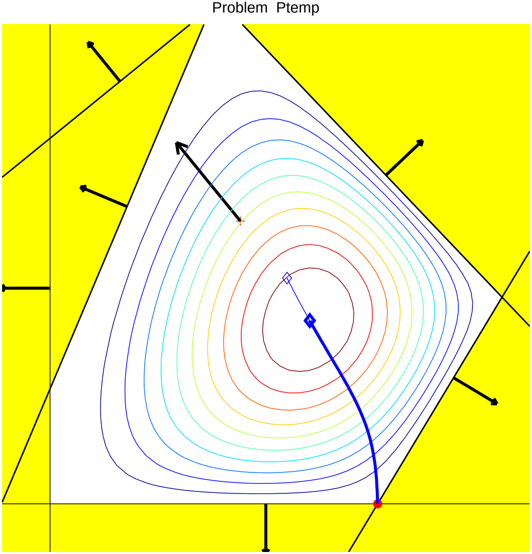


Renegar cuts



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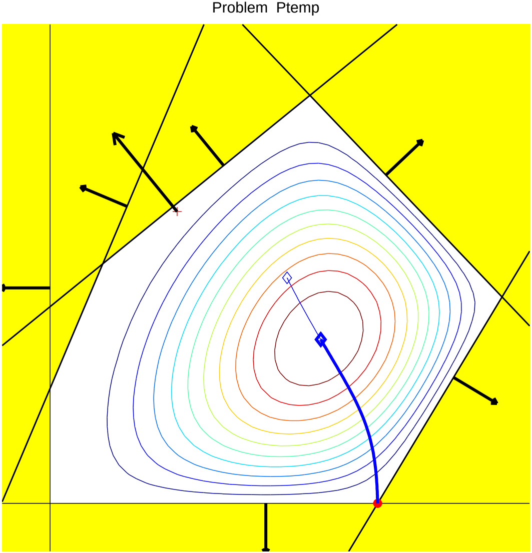

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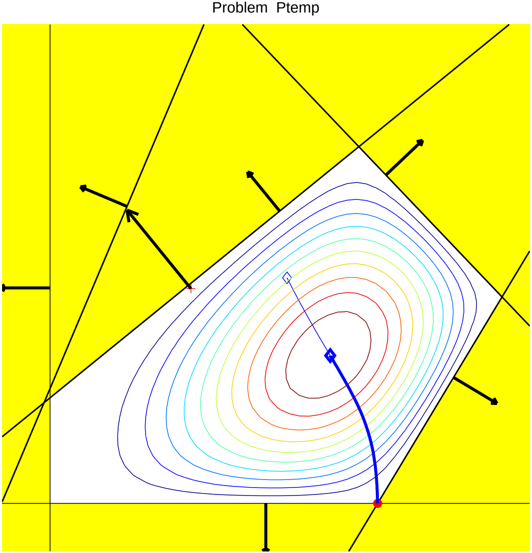

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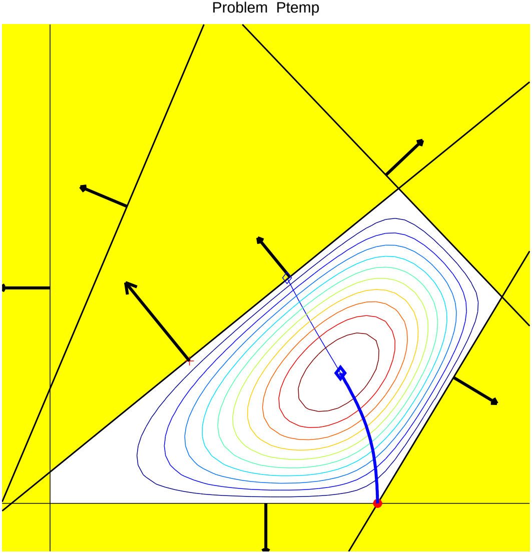

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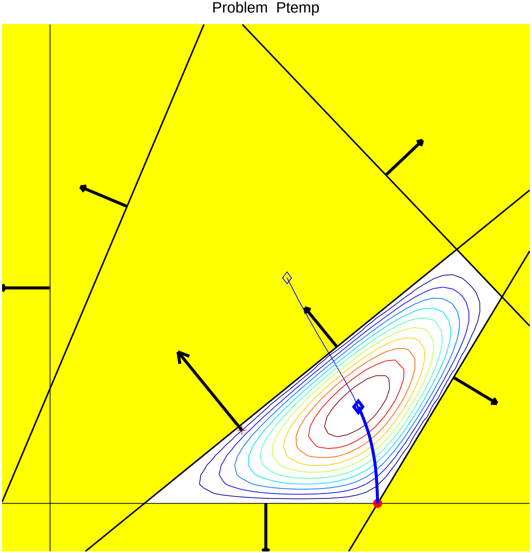

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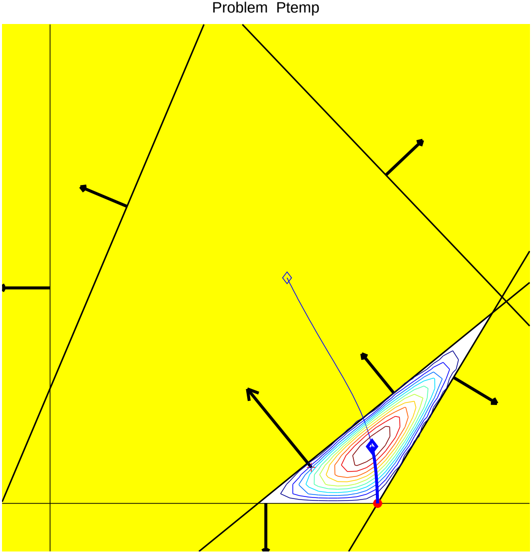

Centering

The most important problem in interior point methods is the following:

Centering problem

Given a feasible interior point x^0 and a value $\alpha \geq 0$, solve approximately the problem

$$\text{minimize}_{x \in \Omega^0} \, \alpha c^T x + p(x).$$

The Newton direction from x^0 coincides with the affine-scaling direction, and hence is the best possible. It is given by

$$\begin{aligned} d &= X\bar{d}, \\ \bar{d} &= -P_{AX}X(\alpha c - x^{-1}) = -\alpha P_{AX}Xc + P_{AX}e. \end{aligned}$$

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Efficiency of the Newton step for centering

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We define the Proximity to the central point as

$$\delta(x,\alpha) = \|\bar{d}\| = \| -\alpha P_{AX}Xc + P_{AX}e\|.$$

The following important theorem says how efficient it is:

Theorem

Consider a feasible point x and a parameter α. Let $x^+ = x + d$ be the point resulting from a Newton centering step. If $\delta(x,\alpha) = \delta < 1$, then $\delta(x^+,\alpha) < \delta^2$.

If $\delta(x,\alpha) \leq 0.5$, then this value is a very good approximation to the euclidean distance between e and $X^{-1}x_\alpha$, i. e., between x and x_α in the scaled space.

Efficiency of the Newton step for centering

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If $\delta(x,\alpha) \leq 0.5$, then this value is a very good approximation to the euclidean distance between e and $X^{-1}x_\alpha$, i. e., between x and x_α in the scaled space.

Primal results as we saw are important to give a geometrical meaning to the procedures, and to develop the intuition. Also, these results can be generalized to a large class of problems, by generalizing the idea of barrier functions.

From now on we shall deal with primal-dual results, which are more efficient for linear and non-linear programming.

The Linear Programming Problem

LP

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

LD

$$\begin{aligned} \text{maximize} \quad & b^T y \\ \text{subject to} \quad & A^T y \leq c \end{aligned}$$

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KKT: multipliers y, s

$$\begin{aligned} A^T y + s &= c \\ Ax &= b \\ xs &= 0 \\ x, s &\geq 0 \end{aligned}$$

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KKT: multipliers y, s

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Primal-dual optimality

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Duality gap

For x, y, s feasible,

$$c^T x - b^T y = x^T s \geq 0$$

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(LP) has solution x and (LD) has solution y, s if and only if the optimality conditions have solution x, y, s.

Primal-dual centering

Let us write the KKT conditions for the centering problem (now using μ instead of $\alpha = 1/\mu$).

$$\begin{array}{ll} \text{minimize} & c^T x - \mu \sum \log x_i \\ \text{subject to} & Ax = b \\ & x > 0 \end{array}$$

A feasible point x is a minimizer if and only if the gradient of the objective function is orthogonal to the null space of A, which means

$$c - \mu x^{-1} = -A^T y,$$

for some $y \in \mathbb{R}^m$. Defining $s = \mu x^{-1}$, we get the conditions for a primal-dual center:

Primal-dual center

$$\begin{aligned} xs &= \mu e \\ A^T y + s &= c \\ Ax &= b \\ x, s &> 0 \end{aligned}$$

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Generalization

Let us write the KKT conditions for the convex quadratic programming problem

$$\begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2} x^T H x \\ \text{subject to} & Ax = b \\ & x > 0 \end{array}$$

The first KKT condition is written as

$$c + Hx - A^T y - s = 0$$

To obtain a symmetrical formulation for the problem, we may multiply this equation by a matrix B whose rows for a basis for the null space of A. Then $BA^T y = 0$, and we obtain the following conditions conditions:

$$\begin{array}{rcl} xs & = & 0 \\ -BHx + Bs & = & Bc \\ Ax & = & b \\ x, s & \geq & 0 \end{array}$$

Horizontal linear complementarity problem

In any case, the problem can be written as

$$\begin{aligned} xs &= 0 \\ Qx + Rs &= b \\ x, s &\geq 0 \end{aligned}$$

This is a linear complementarity problem, which includes linear and quadratic programming as particular problems. The techniques studied here apply to these problems, as long as the following monotonicity condition holds:

For any feasible pair of directions (u, v) such that $Qu + Rv = 0$, we have $u^T v \geq 0$.

The optimal face: the optimal solutions must satisfy $x_i s_i = 0$ for $i = 1, \dots, n$. This is a combinatorial constraint, responsible for all the difficulty in the solution.

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Primal-dual centering: the Newton step

Given x, s feasible and $\mu > 0$, find

$$\begin{aligned} x^+ &= x + u \\ s^+ &= x + v \end{aligned}$$

such that

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Primal-dual centering: Proximity measure

Given x, s feasible and $\mu > 0$, we want

$$xs = \mu e \text{ or equivalently } \frac{xs}{\mu} - e = 0$$

The actual error in this equation gives the proximity measure:

Proximity measure

$$x, s, \mu \mapsto \delta(x, s, \mu) = \|\frac{xs}{\mu} - e\|.$$

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$$\delta(x^+, s^+, \mu) < \frac{1}{\sqrt{8}} \frac{\delta^2}{1-\delta}.$$

In particular, if $\delta \leq 0.7$, then $\delta(x^+, s^+, \mu) < \delta^2$.

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Primal-dual path following: Traditional approach

- Assume that we have x, s, μ such that (x, s) is feasible and $\delta(x, s, \mu) \leq \alpha < 1$
- Choose $\mu^+ = \gamma\mu$, with $\gamma < 1$.
- Use Newton's algorithm (with line searches to avoid infeasible points) to find (x^+, s^+) such that $\delta(x^+, s^+, \mu^+) \leq \alpha$

Neighborhood of the central path

Given $\beta \in (0, 1)$, we define the neighborhood $\eta(\alpha)$ as the set of all feasible pairs (x, s) such that for some $\mu > 0$

$$\delta(x, s, \mu) \leq \beta$$

The methods must ensure that all points are in such a neighborhood, using line searches

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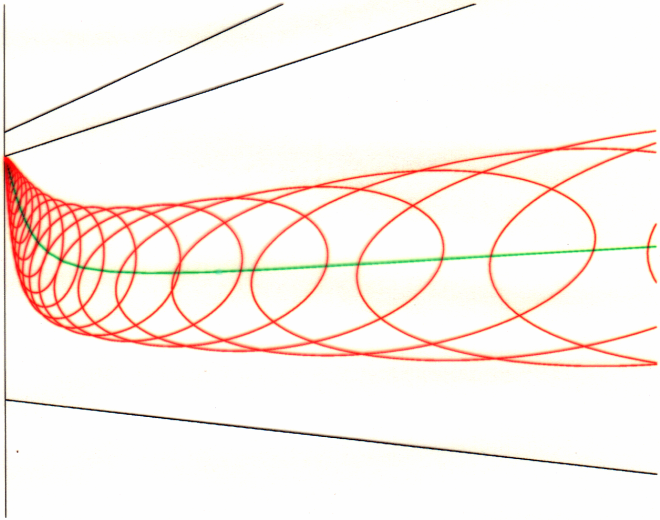
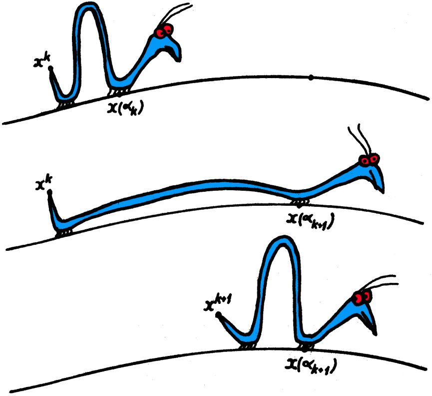
A neighborhood of the central path



Short steps

Using γ near 1, we obtain short steps. With $\gamma = 0.4/\sqrt{n}$, only one Newton step is needed at each iteration, and the algorithm is polynomial: it finds a solution with precision 2^{-L} in $O(\sqrt{n}L)$ iterations.



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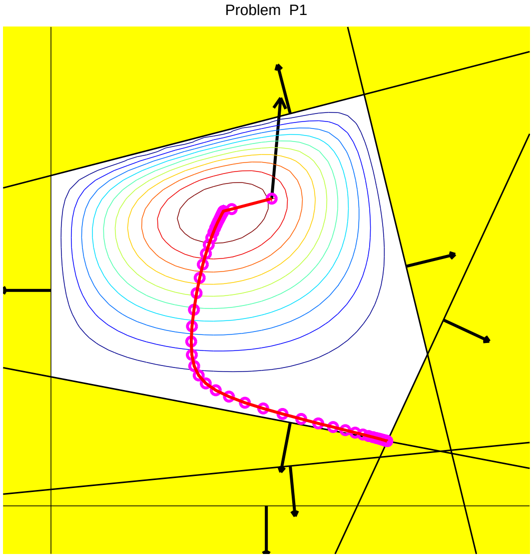
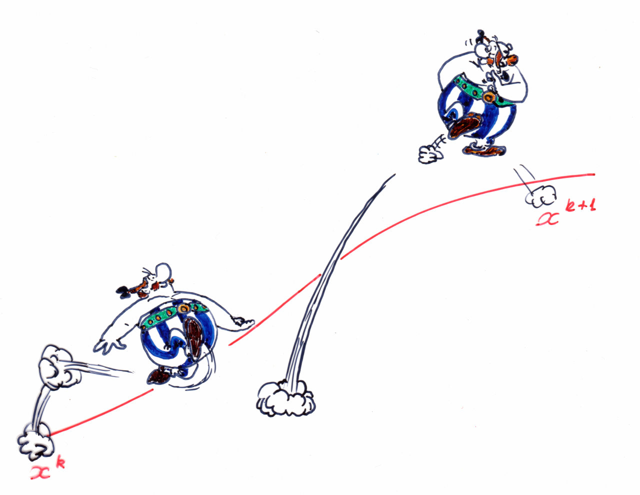
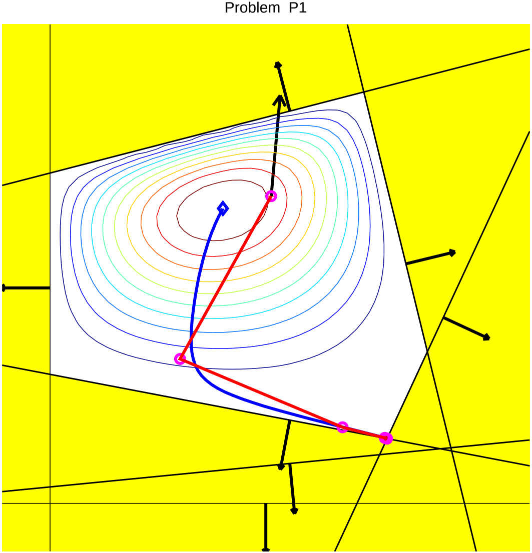


Large steps

Using γ small, say $\gamma = 0.1$, we obtain large steps. This uses to work well in practice, but some sort of line search is needed, to avoid leaving the neighborhood. Predictor-corrector methods are better, as we shall see.



Large steps

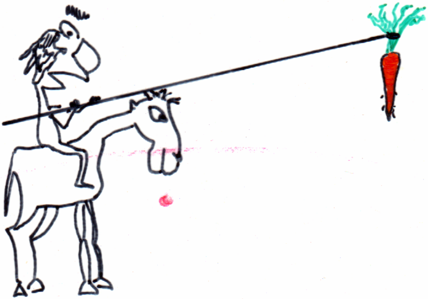


Adaptive methods

Assume that (x, s) feasible is given in $\eta(\beta)$, but no value of μ is given. Then we know:

- if (x, s) is a central point, then $xs = \mu e$ implies $x^T s = n\mu$. Hence the best choice for μ is $\mu = s^T s/n$.
- If (x, s) is not a central point, the value $\mu(x, s) = x^T s/n$ gives a parameter value which in a certain sense is the best possible.
- An adaptive algorithm does not use a value of μ coming from a former iteration: it computes $\mu(x, s)$ and then chooses a value $\gamma\mu(x, s)$ as new target.
- The target may be far. Compute a direction (u, v) and follow it until

$$\delta(x + \lambda u, s + \lambda v, \mu(x + \lambda u, s + \lambda v)) = \beta$$



Predictor-corrector methods

Alternate two kinds of iterations:

- Predictor: An iteration starts with (x, s) near the central path, and computes a Newton step (u, v) with goal $\gamma\mu(x, s)$, γ small.
- Follow it until

$$\delta(x+\lambda u, s+\lambda v, \mu(x+\lambda u, s+\lambda v)) = \beta$$

- Corrector: Set $x^+ = x+\lambda u$, $s^+ = s+\lambda v$, compute $\mu = \mu(x^+, s^+)$ and take a Newton step with target μ

- If the predictor uses $\gamma = 0$, it is called the affine scaling step. It has no centering, and tries to solve the original problem in one step.
- Using a neighborhood with $\beta = 0.5$, the resulting algorithm (the Mizuno-Todd-Ye algorithm) converges quadratically to an optimal solution, keeping the complexity at its best value of $O(\sqrt{n}L)$ iterations.

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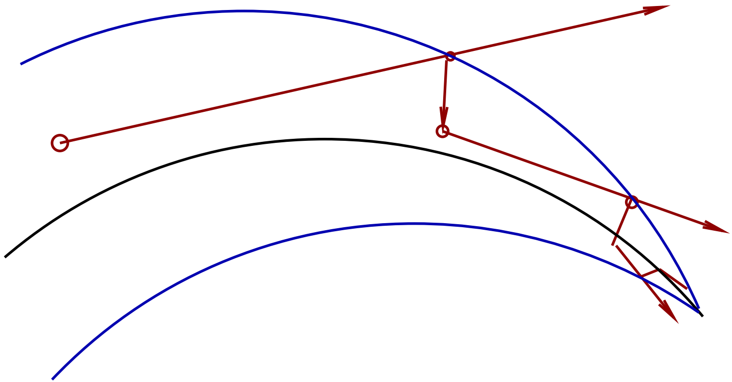

Mehrotra Predictor-corrector method: second order

When computing the Newton step, we eliminated the nonlinear term uv in the equation

$$\begin{aligned} xs + su + xv + uv &= \mu e \\ Qu + Rv &= 0 \end{aligned}$$

The second order method corrects the values of u, v by estimating the value of the term uv by a predictor step.

- Predictor: An iteration starts with (x, s) near the central path, and computes a Newton step (u, v) with goal μ^+, small. The first equation is

$$xv + su = -xs + \mu^+ e$$

- Compute a correction $(\Delta u, \Delta v)$ by

$$x\Delta v + s\Delta u = -uv.$$

- Line search: Set $x^+ = x + \lambda u + \lambda^2 \Delta u$, $s^+ = s + \lambda v + \lambda^2 \Delta v$, by a line search so that $\delta(x^+, s^+, \mu(x^+, s^+)) = \beta$.

Mehrotra Predictor-corrector