MIXED INTEGER NON-LINEAR PROGRAMS FEATURING "ON/OFF" CONSTRAINTS

APPLICATION IN TELECOMMUNICATIONS

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Plan

- Defining "on/off" constraints
- Defining "*perspective*" functions
- Connections to earlier works
- A new result
- Our motivating application: the delay constrained routing problem

Defining "on/off" constraints

- Considered Problems:

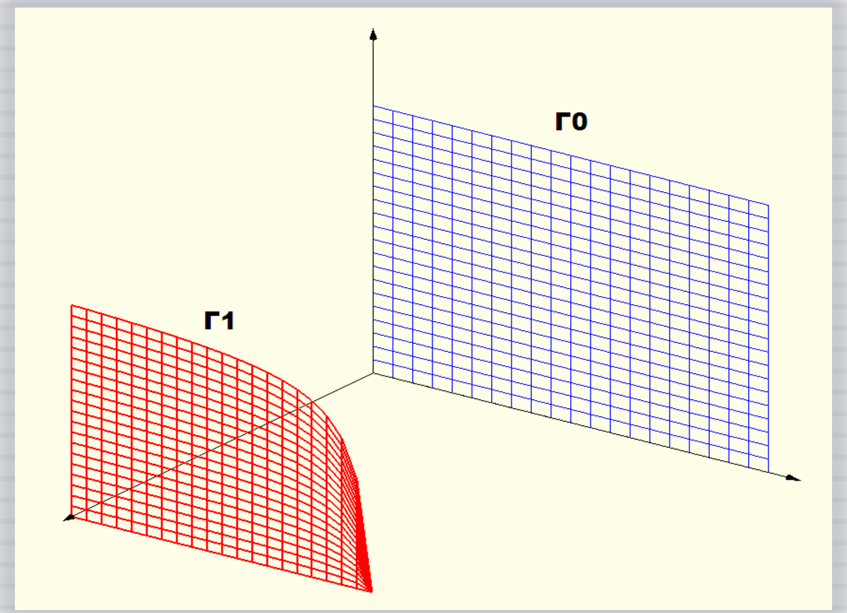
Given convex functions h, g and f_k:

$$
\begin{aligned}
& \min \ h(\mathbf{x}, \mathbf{y}, \mathbf{z}) \\
s.t. \quad & g(\mathbf{x}, \mathbf{y}, \mathbf{z}) \leq 0 \\
& f_k(\mathbf{x}) \leq 0 \text{ if } z_k \ = \ 1, \ \forall k \in \{1, \ 2, ..., n_k\}, \\
& \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}, \\
& \mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{N}^m, \mathbf{z} \in \{0,1\}^{n_k}.
\end{aligned}
$$

- The indicator variable z_k controls the activation of the k^{th} on/off constraint

Defining “on/off” constraints

- For each “on/off” constraint, the generated feasible region is a union of two disjoint sets



Defining "on/off" constraints

- Classical convex formulations rely on the Big-M approach:

$$
\begin{aligned}
& \min \;\; h(\mathbf{x},\mathbf{y},\mathbf{z}) \\
s.t. \;\; & g(\mathbf{x},\mathbf{y},\mathbf{z}) \leq 0 \\
& f_k(\mathbf{x}) \leq (1 - z_k)M, \;\; \forall k \in \{1,\;2,\ldots,n_k\}, \\
& \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}, \\
& \mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{N}^m, \mathbf{z} \in \{0,1\}^{n_k}.
\end{aligned}
$$

- Advantage: Compact models
- Inconvenient: Bad continuous relaxation

Defining "on/off" constraints

- The problem can be written as a Disjunctive Program:

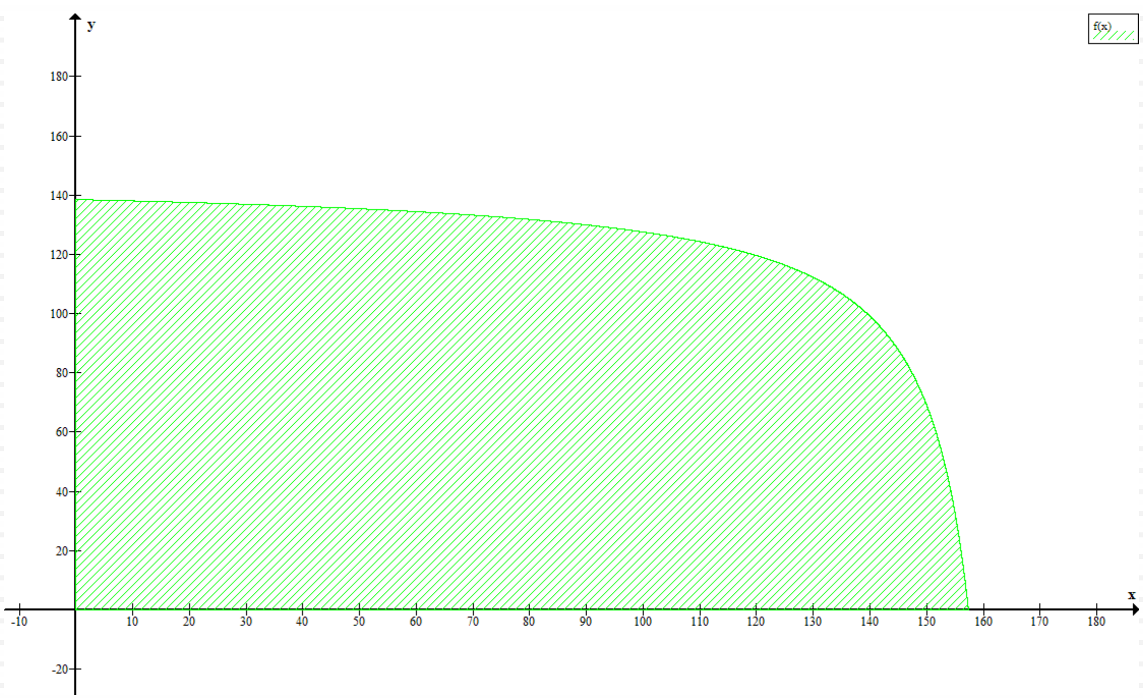
$$
\begin{aligned}
& \min \; h(\mathbf{x},\mathbf{y},\mathbf{z}) \\
s.t. \quad & g(\mathbf{x},\mathbf{y},\mathbf{z}) \le 0 \\
& (\mathbf{x}, z_k) \in \Gamma_0^k \cup \Gamma_1^k, \;\; \forall k \in \{1,\; 2, ..., n_k\}, \\
& \Gamma_0^k = \{\; (\mathbf{x},\;\; z_k) \in \mathbb{R}^n \times \{0,1\} : z_k = 0,\; \mathbf{l} \le \mathbf{x} \le \mathbf{u}\}, \\
& \Gamma_1^k = \{\; (\mathbf{x},\;\; z_k) \in \mathbb{R}^n \times \{0,1\} : z_k = 1,\; f_k(x) \le 0,\; \mathbf{l} \le \mathbf{x} \le \mathbf{u}\}, \\
& \mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{N}^m, \mathbf{z} \in \{0,1\}^{n_k}.
\end{aligned}
$$

- Can we formulate the convex hull of each disjunction?

Defining "*perspective*" functions

Given a convex function $f : \mathbb{R}^n \to \mathbb{R}$,
its perspective function denoted $\tilde{f} : \mathbb{R}^{n+1} \to \mathbb{R} \cup \{+\infty\}$ is defined by:

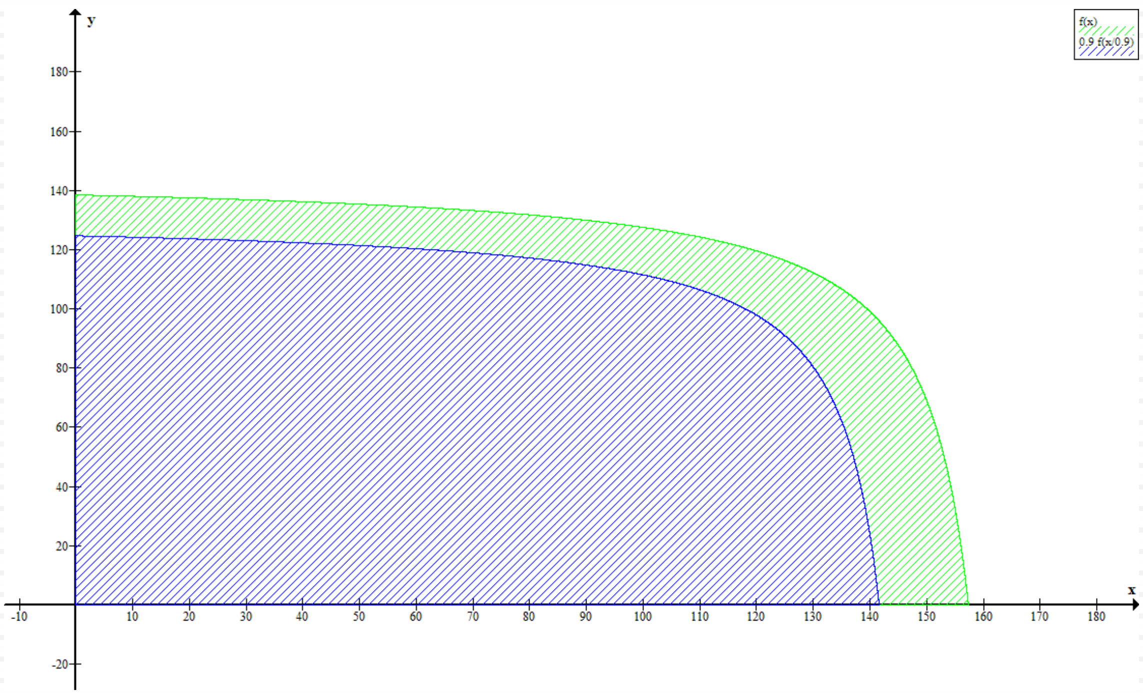
$$\tilde{f}(\mathbf{x}, z) \equiv \begin{cases} z f(\mathbf{x}/z) & \text{if } z > 0, \\ +\infty & \text{if } z \leq 0. \end{cases}$$



Defining "*perspective*" functions

$$\tilde{f} : \mathbb{R}^{n+1} \to \mathbb{R} \cup \{+\infty\}, \tilde{f}(\mathbf{x}, z) \equiv \begin{cases} z f(\mathbf{x}/z) & \text{if } z > 0, \\ +\infty & \text{if } z \leq 0. \end{cases}$$

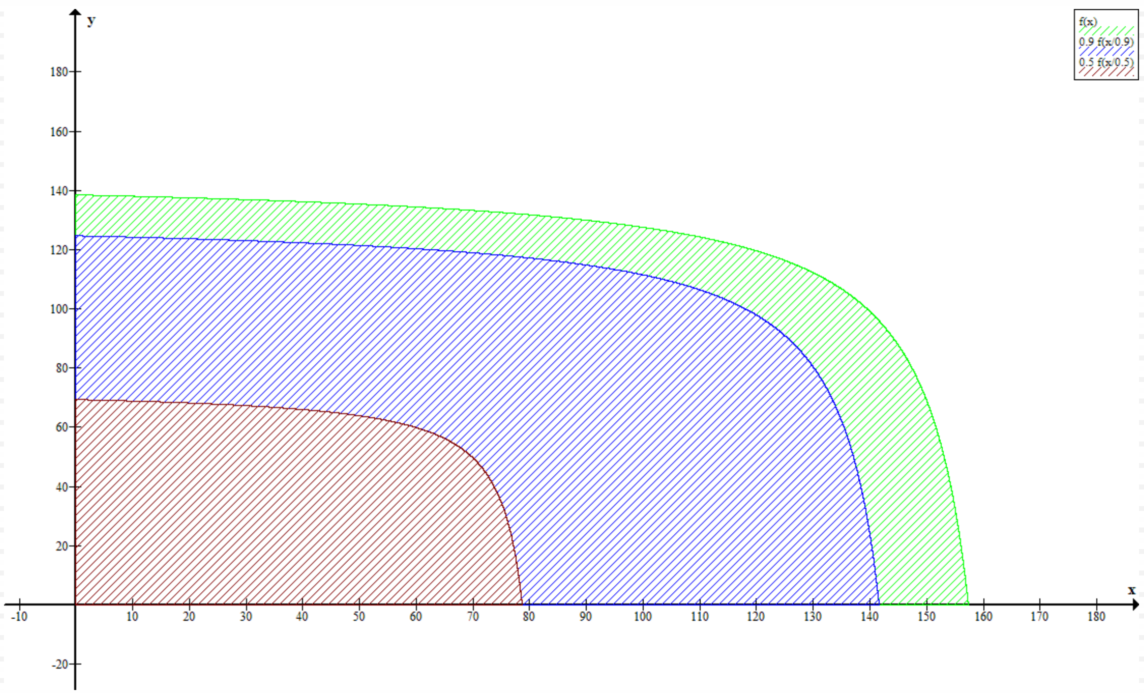
z = 0.9



Defining "*perspective*" functions

$$\tilde{f} : \mathbb{R}^{n+1} \to \mathbb{R} \cup \{+\infty\}, \tilde{f}(\mathbf{x}, z) \equiv \begin{cases} z f(\mathbf{x}/z) & \text{if } z > 0, \\ +\infty & \text{if } z \leq 0. \end{cases}$$

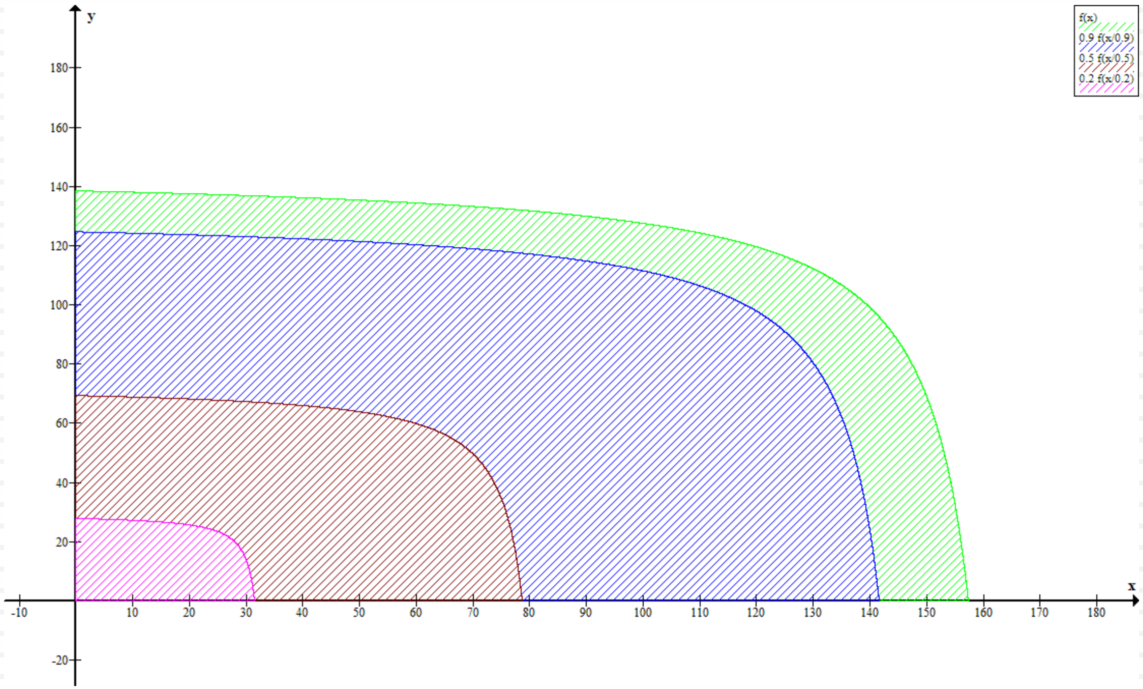
z = 0.5



Defining "*perspective*" functions

$$\tilde{f} : \mathbb{R}^{n+1} \to \mathbb{R} \cup \{+\infty\}, \tilde{f}(\mathbf{x}, z) \equiv \begin{cases} z f(\mathbf{x}/z) & \text{if } z > 0, \\ +\infty & \text{if } z \leq 0. \end{cases}$$

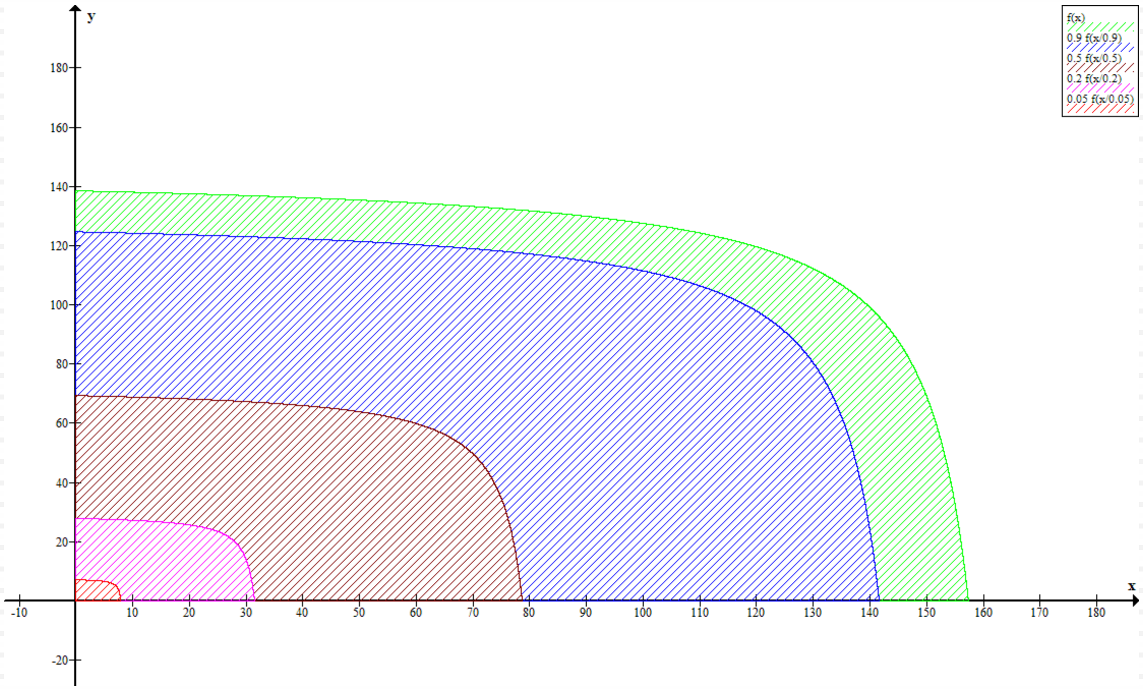
z = 0.2



Defining "*perspective*" functions

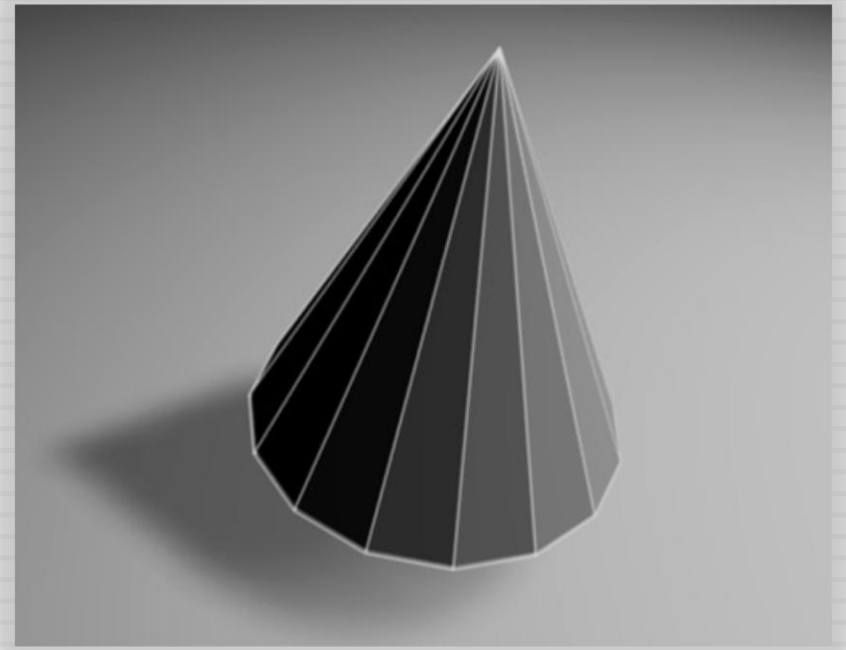
$$\tilde{f} : \mathbb{R}^{n+1} \to \mathbb{R} \cup \{+\infty\}, \tilde{f}(\mathbf{x}, z) \equiv \begin{cases} z f(\mathbf{x}/z) & \text{if } z > 0, \\ +\infty & \text{if } z \leq 0. \end{cases}$$

z = 0.05



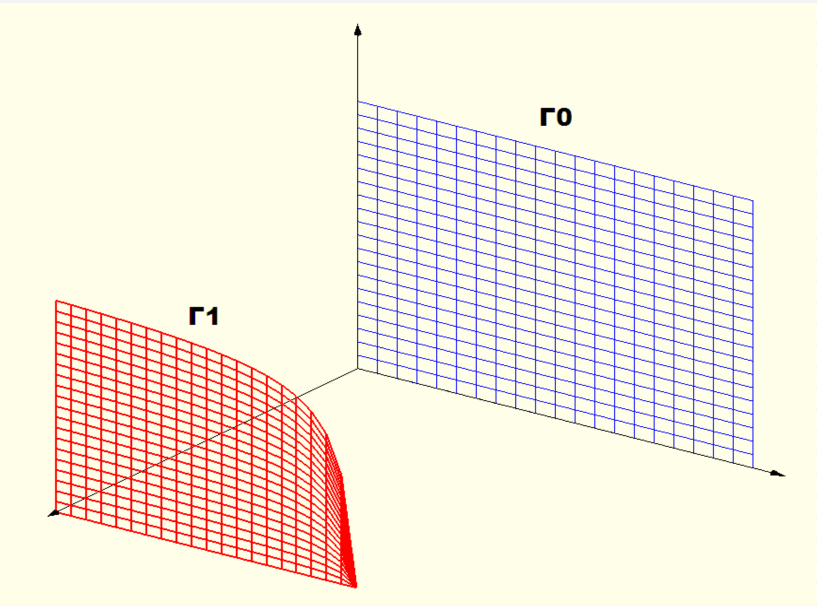
State of the art in convex analysis

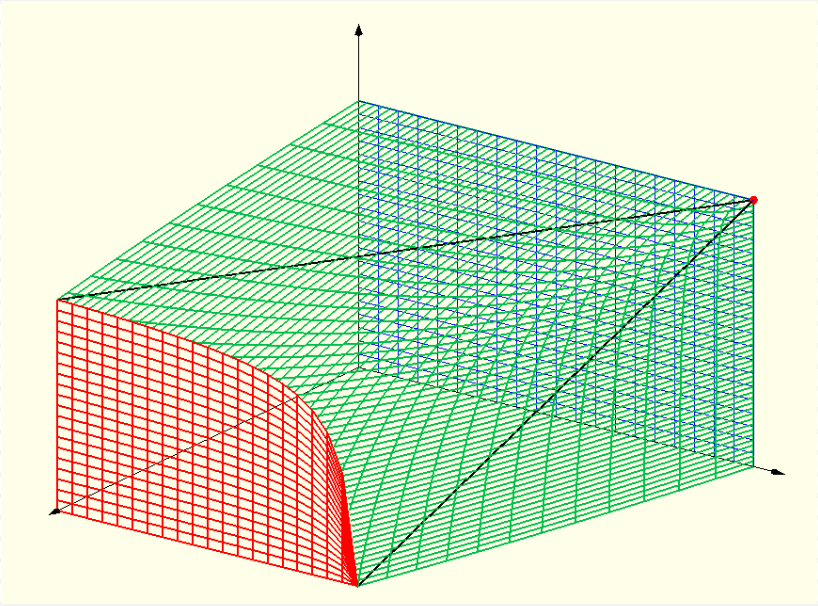
- Günlük and Linderoth (2008) defined $\mathrm{conv}(\Gamma_0 \cup \Gamma_1)$ in the space of original variables when Γ_0 is restrained to a single point.



A new challenge

- The union of a hyper-rectangle and a closed convex bounded set





A new challenge

- Ceria and Soares characterize the convex hull in an extended space
  - A relatively important number of added variables
  - Non efficient in practice (Heavy formulations)

- Can we formulate the convex hull in the space of original variables ?

A new challenge

- Definition:

Let $f : E \to \mathbb{R}$, $E \subseteq \mathbb{R}^n$, f is *independently increasing* (resp. *decreasing*) on the ith coordinate if:
$\forall x = (x_1, x_2, ..., x_i, ..., x_n) \in dom(f)$, $x' = (x_1, x_2, ..., x'_i, ..., x_n) \in dom(f)$ $s.t.$ $x'_i \geq x_i \Rightarrow f(x') \geq$ (resp. $\leq$) $f(x)$.

We say that f is *independently monotone* on the ith coordinate if it is independently increasing or independently decreasing on this given coordinate.

f is *order preserving* if it is independently monotone on each and every coordinate.

A new challenge

- Example:

Consider the following functions:

1. $f(x_1, x_2, x_3) = e^{(2x_1 - x_2)} + x_3$, $(x_1, x_2, x_3) \in \mathbb{R}^3$, f is independently increasing on coordinate 1 and 3, independently decreasing on coordinate 2, therefore it is an order preserving function.
2. $f(x, y) = x^4 + y^2$, $(x, y) \in \mathbb{R}^2$, the variation of f depends on the sign of the variables, f is not order preserving.
3. $f(x) = \sum_{i=1}^{n} \frac{1}{c_i - x_i}$, where $x \in \mathbb{R}^n$. Since f is a sum of univariate increasing functions, it is an order preserving function.

Additive functions which are sum of univariate monotone functions are commonly encountered order preserving functions.

A new result (simple version)

Let:

$f(E \rightarrow \mathbb{R},\ E \subseteq \mathbb{R}^n)$, be a closed convex function, i.i. on all coordinates,

$\Gamma_0 = \{\ (\mathbf{x},\ z) \in \mathbb{R}^n \times \{0,1\} : z = 0,\ \mathbf{l}^0 \leq \mathbf{x} \leq \mathbf{u}^0\}$,

$\Gamma_1 = \{\ (\mathbf{x},\ z) \in \mathbb{R}^n \times \{0,1\} : z = 1,\ f(x) \leq 0,\ \mathbf{l}^1 \leq \mathbf{x} \leq \mathbf{u}^1\}$,

$$\text{then } \boldsymbol{conv}\,(\boldsymbol{\Gamma}_0 \cup \boldsymbol{\Gamma}_1) = \boldsymbol{cl}(\boldsymbol{\Gamma}),$$

$$\text{where } \Gamma = \left\{ \begin{array}{l} (\mathbf{x},\ z) \in \mathbb{R}^{n+1} : \\ z q_S\,(x/z) \leq 0,\ \forall S \subset \{1, 2, ..., n\}, \\ z\mathbf{l}^1 + (1-z)\mathbf{l}^0 \leq \mathbf{x} \leq z\mathbf{u}^1 + (1-z)\mathbf{u}^0, \\ 0 < z \leq 1. \end{array} \right\},$$

$$\textit{with } q_S = (f \circ h_S),\ h_S(\mathbb{R}^n \rightarrow \mathbb{R}^n) \text{ defined by } (h_S(x))_i = \begin{cases} l_i^1\ \forall i \in S, \\ x_i - \frac{(1-z)u_i^0}{z}\ \forall i \notin S, \end{cases}$$

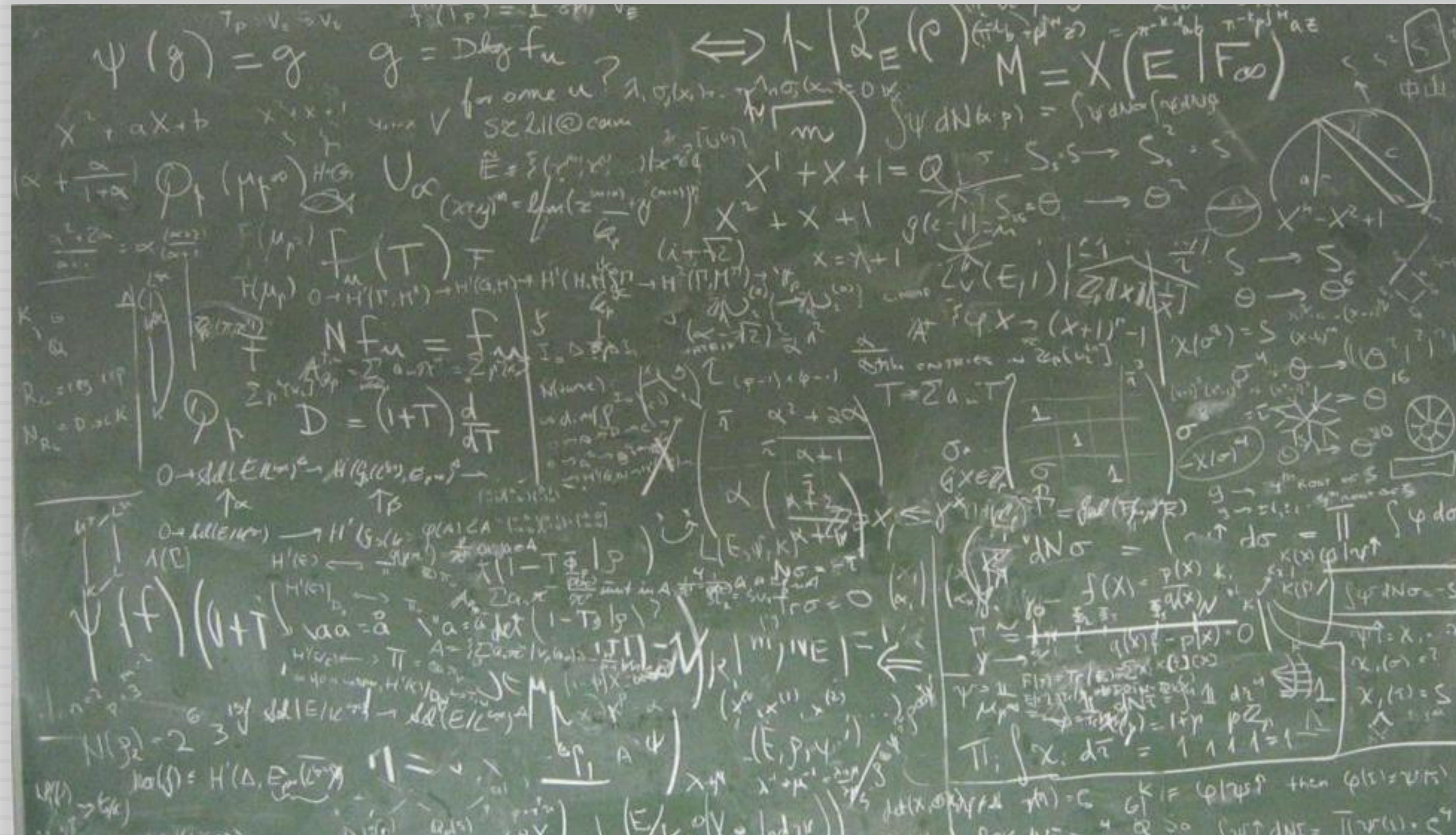
A new result (simple version)

- One constraint captivating the nonlinearity of the convex hull

$$\text{For } S = \emptyset,\ \Gamma_{\emptyset} = \left\{ \begin{array}{l} (\mathbf{x},\ z) \in \mathbb{R}^{n+1} : \\ zf\left(\frac{\mathbf{x}-(1-z)\mathbf{u}^{\mathbf{0}}}{z}\right) \leq 0, \\ z\mathbf{l}^{\mathbf{1}} + (1-z)\mathbf{l}^{\mathbf{0}} \leq \mathbf{x} \leq z\mathbf{u}^{\mathbf{1}} + (1-z)\mathbf{u}^{\mathbf{0}}, \\ 0 < z \leq 1. \end{array} \right\},$$

- $\Gamma_{\emptyset}$ coincide with the convex hull on an important region :
  - all points verifying the system $zl^1 + (1-z)u^0 \leq x$

Some elements of proof



MINLPs relaxations

- The Big-M formulation
  - Compact model
  - Bad relaxations
- The new disjunctive formulation
  - Compact model
  - Good relaxations

Our motivating application: the delay constrained routing problem

- Multi-Flow Routing under differentiated delay guarantees
- Different delays corresponding to different services
- The delay function is a non linear exponentially increasing function $\left(\frac{1}{c_e - f_e}\right)$
- A set of candidate paths given by traffic engineers
- Suitable for centralized routing protocols (implemented in backbone networks)

Our motivating application: the delay constrained routing problem

$$
\begin{aligned}
& z_k^i \in \{0,1\}, && \forall k \in K,\ \forall P_k^i \in P(k) \\
& \phi_k^i \in [0,1], && \forall k \in K,\ \forall P_k^i \in P(k) \\
& x_e \in \mathbb{R}, && \forall e \in E.
\end{aligned}
$$

Our motivating application: the delay constrained routing problem

$$\begin{aligned}
&\phi_k^i \leq z_k^i, && \forall k \in K,\ \forall P_k^i \in P(k) \\
&z_k^i \in \{0,1\}, && \forall k \in K,\ \forall P_k^i \in P(k) \\
&\phi_k^i \in [0,1], && \forall k \in K,\ \forall P_k^i \in P(k) \\
&x_e \in \mathbb{R}, && \forall e \in E.
\end{aligned}$$

Our motivating application: the delay constrained routing problem

$$\sum_{P_k^i \in P(k)} z_k^i \leq N, \qquad \forall k \in K$$

$$\phi_k^i \leq z_k^i, \qquad \forall k \in K, \ \forall P_k^i \in P(k)$$

$$z_k^i \in \{0, 1\}, \qquad \forall k \in K, \ \forall P_k^i \in P(k)$$

$$\phi_k^i \in [0, 1], \qquad \forall k \in K, \ \forall P_k^i \in P(k)$$

$$x_e \in \mathbb{R}, \qquad \forall e \in E.$$

Our motivating application: the delay constrained routing problem

$$
\begin{aligned}
& x_e \le c_e, \ \forall e \in E \\
& \sum_{e \in P_k^i} \frac{1}{c_e - x_e} \le \alpha_k, && \forall k \in K, \ \forall P_k^i \in P(k) \text{ if } z_k^i = 1 \\
& \sum_{P_k^i \in P(k)} z_k^i \le N, && \forall k \in K \\
& \phi_k^i \le z_k^i, && \forall k \in K, \ \forall P_k^i \in P(k) \\
& z_k^i \in \{0, 1\}, && \forall k \in K, \ \forall P_k^i \in P(k) \\
& \phi_k^i \in [0, 1], && \forall k \in K, \ \forall P_k^i \in P(k) \\
& x_e \in \mathbb{R}, && \forall e \in E.
\end{aligned}
$$

Our motivating application: the delay constrained routing problem

$$\sum_{k \in K} \sum_{P_k^i \ni l} \phi_k^i v_k \leq x_e, \qquad \forall e \in E$$

$$x_e \leq c_e, \ \forall e \in E$$

$$\sum_{e \in P_k^i} \frac{1}{c_e - x_e} \leq \alpha_k, \qquad \forall k \in K, \ \forall P_k^i \in P(k) \text{ if } z_k^i = 1$$

$$\sum_{P_k^i \in P(k)} z_k^i \leq N, \qquad \forall k \in K$$

$$\phi_k^i \leq z_k^i, \qquad \forall k \in K, \ \forall P_k^i \in P(k)$$

$$z_k^i \in \{0, 1\}, \qquad \forall k \in K, \ \forall P_k^i \in P(k)$$

$$\phi_k^i \in [0, 1], \qquad \forall k \in K, \ \forall P_k^i \in P(k)$$

$$x_e \in \mathbb{R}, \qquad \forall e \in E.$$

Our motivating application: the delay constrained routing problem

$$\sum_{i=1}^{n_k} \phi_k^i \geq 1, \qquad \forall k \in K$$

$$\sum_{k \in K} \sum_{P_k^i \ni l} \phi_k^i v_k \leq x_e, \qquad \forall e \in E$$

$$x_e \leq c_e, \ \forall e \in E$$

$$\sum_{e \in P_k^i} \frac{1}{c_e - x_e} \leq \alpha_k, \qquad \forall k \in K, \ \forall P_k^i \in P(k) \text{ if } z_k^i = 1$$

$$\sum_{P_k^i \in P(k)} z_k^i \leq N, \qquad \forall k \in K$$

$$\phi_k^i \leq z_k^i, \qquad \forall k \in K, \ \forall P_k^i \in P(k)$$

$$z_k^i \in \{0, 1\}, \qquad \forall k \in K, \ \forall P_k^i \in P(k)$$

$$\phi_k^i \in [0, 1], \qquad \forall k \in K, \ \forall P_k^i \in P(k)$$

$$x_e \in \mathbb{R}, \qquad \forall e \in E.$$

Our motivating application: the delay constrained routing problem

$$
\begin{aligned}
\min \quad & \sum_{e \in E} w_e x_e & \\
& \sum_{i=1}^{n_k} \phi_k^i \geq 1, & \forall k \in K \\
& \sum_{k \in K} \sum_{P_k^i \ni l} \phi_k^i v_k \leq x_e, & \forall e \in E \\
& x_e \leq c_e, \ \forall e \in E & \\
& \sum_{e \in P_k^i} \frac{1}{c_e - x_e} \leq \alpha_k, & \forall k \in K, \ \forall P_k^i \in P(k) \text{ if } z_k^i = 1 \\
& \sum_{P_k^i \in P(k)} z_k^i \leq N, & \forall k \in K \\
& \phi_k^i \leq z_k^i, & \forall k \in K, \ \forall P_k^i \in P(k) \\
& z_k^i \in \{0, 1\}, & \forall k \in K, \ \forall P_k^i \in P(k) \\
& \phi_k^i \in [0, 1], & \forall k \in K, \ \forall P_k^i \in P(k) \\
& x_e \in \mathbb{R}, & \forall e \in E.
\end{aligned}
$$

Our motivating application: the delay constrained routing problem

- The delay constraint is an on/off constraint !

$$\sum_{e \in P_k^i} \frac{1}{c_e - x_e} \leq \alpha_k, \ \forall k \in K, \ \forall P_k^i \in P(k) \ if \ z_k^i = 1$$

$$0 \leq x_e \leq u_e, \ \forall e \in P_k^i, \ if \ z_k^i = 0$$

- Candidate formulations:

$$\sum_{e \in P_k^i} \frac{1}{c_e - x_e} \leq M - z_k^i (M - \alpha_k), \ \forall k \in K, \ \forall P_k^i \in P(k)$$

$$\sum_{e \in P_k^i} \left(\frac{{z_k^i}^2}{z_k^i c_e - x_e + (1 - z_k^i) u_e} \right) - z_k^i \alpha_k \leq 0, \ \forall k \in K, \ \forall P_k^i \in P(k)$$

Computational experiments

Implemented models in Bonmin (open MINLP solver)

	|V|	|E|	|K|	P_bigM	P_proj
rdata1	60	280	100	(2.7 ; 0)	**(2.4 ; 0)**
rdata2	61	148	122	(25 ; 0)	**(13 ; 0)**
rdata3	100	600	200	([0.28%] ; 157748)	**(344; 5097)**
rdata4	34	160	946	([0.001%] ; 79807)	**(1525 ; 50583)**
rdata5	67	170	761	([0.43%] ; 138618)	**([0.03%] ; 202122)**
rdata6	100	800	500	([0.006%] ; 176413)	**(934 ; 19351)**

Mono-routing constraints 3 candidate paths per demand

Computational experiments

	|V|	|E|	|K|	P_bigM	P_proj
rdata1	60	280	100	(799.7 ; 12633)	**(220.8 ; 1922)**
rdata2	61	148	122	**(16.1 ; 0)**	(24.8 ; 0)
rdata3	100	600	200	([0.08%] ; 94194)	**(768.6 ; 5207)**
rdata4	34	160	946	([0.4%] ; 40820)	**([0.04%] ; 45492)**
rdata5	67	170	761	([1.2%] ; 16106)	**(5467.7 ; 17347)**
rdata6	100	800	500	([0.7%] ; 5880)	**(5392 ; 23867)**

Multiple-routing constraints 10 candidate paths per demand

Conclusion-Perspectives

- Results apply for a general class of MINLPs
- New efficient tight formulations
- Looking closely at the case of linear functions : new non-trivial MIP cuts

Questions