Traffic routing in congested networks

Equilibrium, Efficiency, and Dynamics

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Outline

Background

Model and preliminaries

The PoA in practice

Algorithms and dynamics

Traffic...

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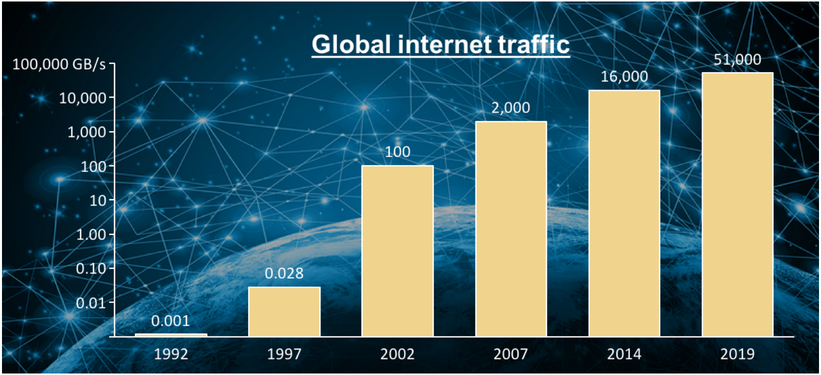
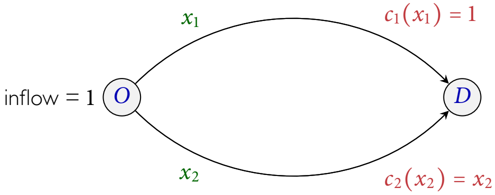


Pigou's example (1920's)



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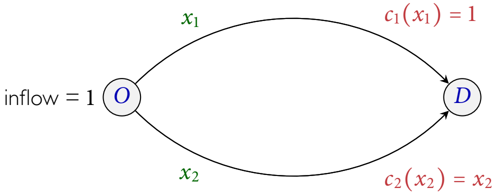

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$$c_1(x_1) = c_2(x_2) \tag{Eq}$$

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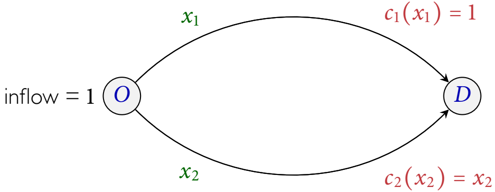

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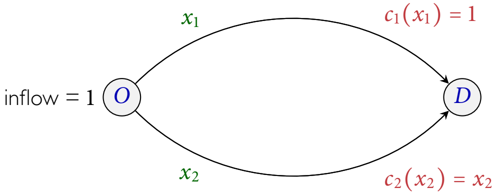

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$$\text{Price of Anarchy} \equiv \frac{C(\text{Eq})}{C(\text{Opt})} = \frac{4}{3}$$

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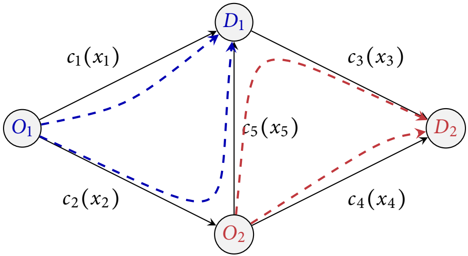

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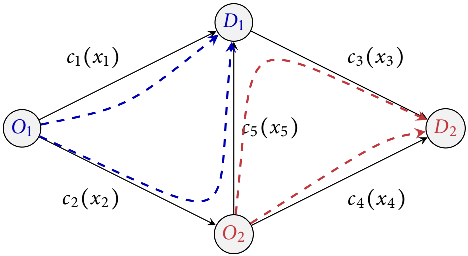

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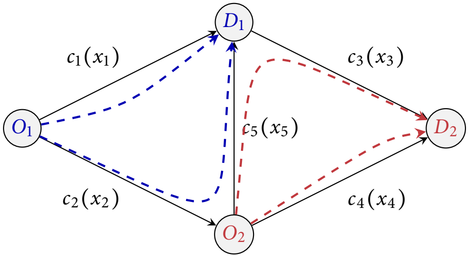

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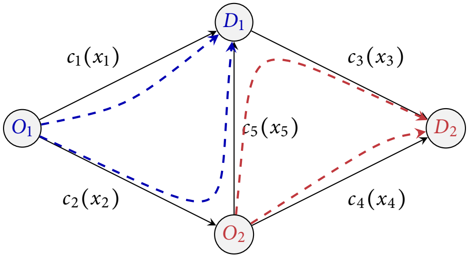

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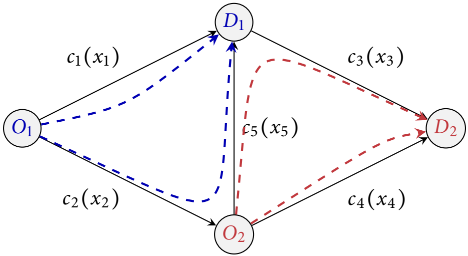

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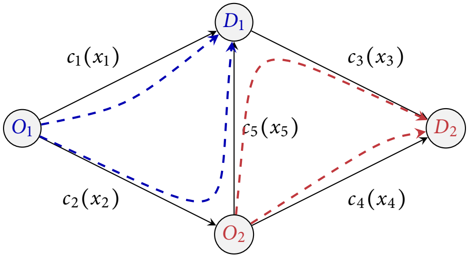

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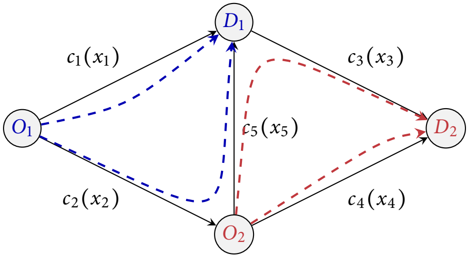

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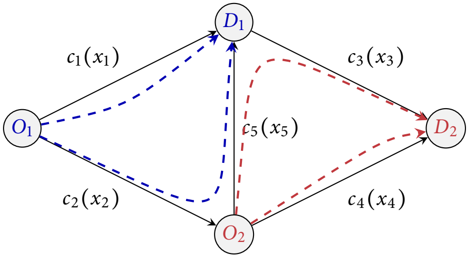

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- Nonatomic routing game: $\Gamma = (\mathcal{G}, \mathcal{I}, \{m^i\}_{i \in \mathcal{I}}, \{\mathcal{P}^i\}_{i \in \mathcal{I}}, \{c_e\}_{e \in \mathcal{E}})$

Fair/Envy-free traffic assignment

Wardrop's routing principle (Wardrop, 1952):

> *"At equilibrium, the delays along all utilized paths are equal and no higher than those that would be experienced by a traffic element going through an unused route"*

Fairness: *all traffic elements experience the same latency*

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$$c_{p^i}(f^*) \leq c_{q^i}(f^*) \quad \text{for all } i \in \mathcal{I} \text{ and all } p^i, q^i \in \mathcal{P}^i \text{ such that } f^*_{p^i} > 0$$

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Characterization of Wardrop equilibria (Beckmann et al, 1956):

$$\begin{aligned} &\text{minimize} && \sum_{e \in \mathcal{E}} C_e(x_e) \\ &\text{subject to} && x_e = \sum_{p \ni e} f_p, \; f \in \mathcal{F} \end{aligned} \tag{WE}$$

where $C_e(x_e) = \int_0^{x_e} c_e(w)\, dw$ is the *primitive* of c_e.

Optimum traffic assignment

Efficient flows minimize aggregate latency in the network

$$\begin{aligned} &\text{minimize} && C(f) = \sum_{p \in \mathcal{P}} f_p c_p(f) \\ &\text{subject to} && f \in \mathcal{F} \end{aligned} \tag{LM}$$

The price of anarchy

- Efficient routing: $\mathrm{Opt}(\Gamma) = \min_{f \in \mathcal{F}} C(f)$
- Equilibrium routing: $\mathrm{Eq}(\Gamma) = C(f^*)$, with f^* a Wardrop equilibrium

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Gap in efficiency measured by the *price of anarchy* (Koutsoupias & Papadimitriou, 1999)

$$\mathrm{PoA}(\Gamma) = \frac{\mathrm{Eq}(\Gamma)}{\mathrm{Opt}(\Gamma)}$$

$\mathrm{PoA}(\Gamma) \geq 1$ with equality **iff fair routing is also efficient**

How bad is selfish routing?

Theorem (Roughgarden & Tardos, 2002)

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Remarks

- Independent of network topology
- Valid for any number of O/D pairs
- Envy-free routing can become **arbitrarily bad:** $d/\log d \to \infty$ as $d \to \infty$
- Sharpness: for any traffic inflow $M = \sum^i m^i$, these bounds can be realized

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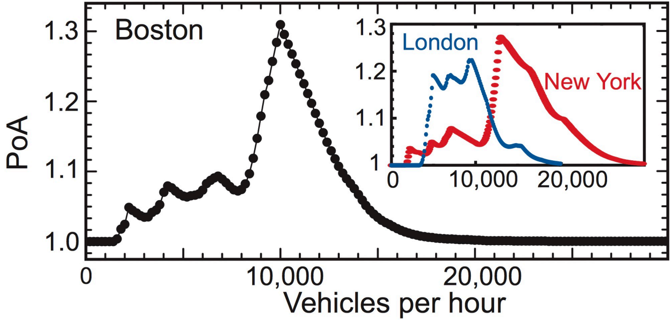
How bad is selfish routing in practice?

- PoA bounds $\Longrightarrow$ delicately tuned worst-case instances
- In typical networks, $\mathrm{PoA} \approx 1$ when the traffic is light or heavy (Youn et al., 2008; O'Hare et al., 2016; Monnot et al., 2017)



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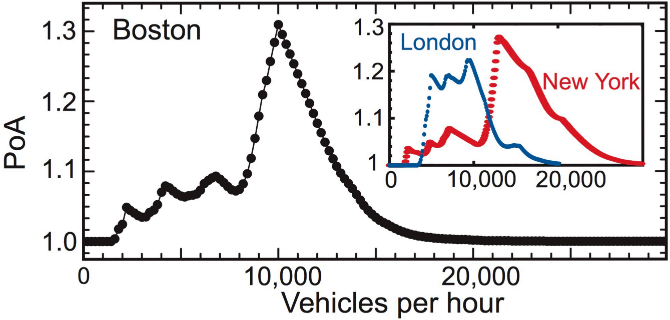
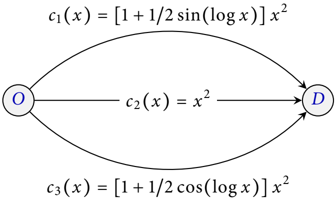
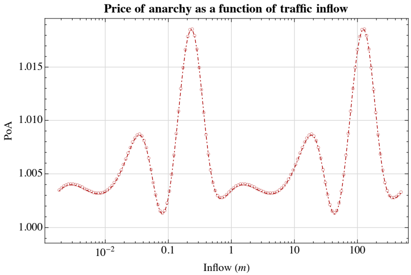


Is this always the case?

No!





- *Single O/D pair*
- *Three parallel links (no Braess-type shenanigans)*
- C^∞*-smooth, convex cost functions with polynomial growth*

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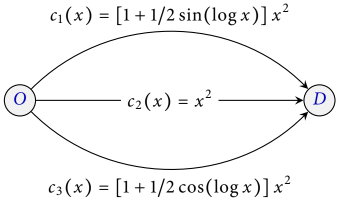

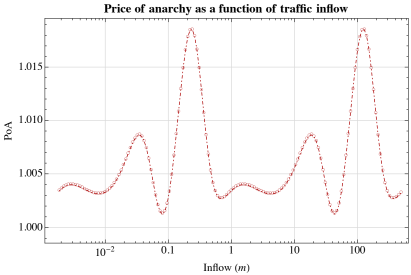

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Proposition (Colini-Baldeschi, Cominetti, M & Scarsini, 2017)

In the above network, $\mathrm{PoA}(\Gamma_M) \geq a > 1$ *for all values of the traffic inflow* M

What's wrong with this simple example?

Pathological oscillations

Main problem: Cost functions scale very irregularly (albeit polynomially!):

$$\lim_{t\to\{0,\infty\}} \frac{c_e(tx)}{c_e(t)} \quad \text{does not exist}$$

$\Longrightarrow$ Oscillations that are very **dense** (in light traffic) or very **wide** (in heavy traffic)

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Sanity check: *such oscillations are not observed in practice*

Regular variation

Definition (Karamata, 1930's)

A function $g\colon [0,\infty) \to (0,\infty)$ is called *regularly varying at $\omega \in \{0,\infty\}$* if

$$\lim_{t\to\omega} \frac{g(tx)}{g(t)} \quad \text{is finite and nonzero for all } x \geq 0. \tag{RV}$$

- $\omega = 0$: *light traffic limit*
- $\omega = \infty$: *heavy traffic limit*

Examples

1. Affine functions: $g(x) = ax + b$
2. Polynomials: $g(x) = \sum_{k=1}^{d} a_k x^k$
3. Asymptotic polynomials: $g(x) \sim x^q$ for some $q \geq 0$
4. Real-analytic at ω; logarithms; etc.

NB: Stronger than asking $g(x) = \Theta(x^q)$ (counterexample satisfies this)

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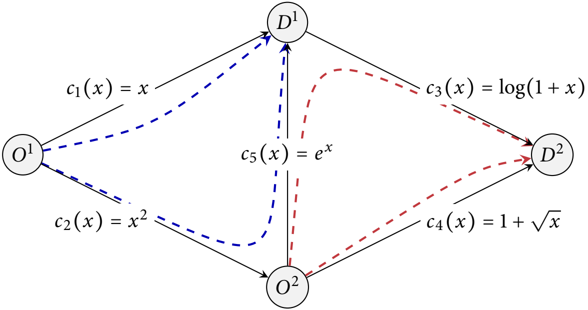
- *Network index:* $\alpha = \min_{p\in\mathcal{P}} \alpha_p$ (bottleneck caused by *slowest* pair)
- *Tight network:* $\alpha \in (0, \infty)$

NB: Edges/paths/pairs that are slow in heavy traffic can be fast in light traffic and vice versa

Benchmarks, light and heavy

Example: different benchmarks in a Wheatstone network



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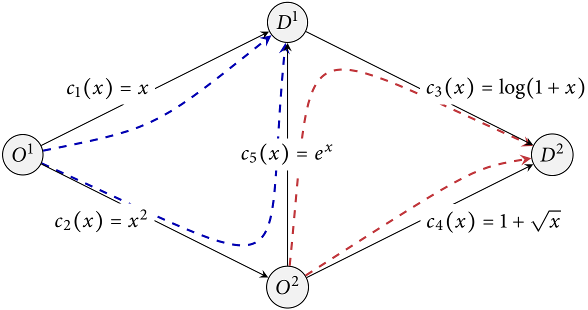

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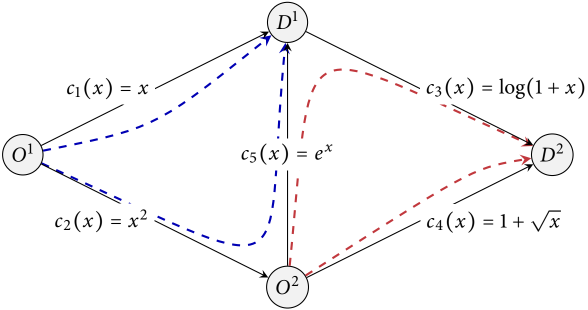

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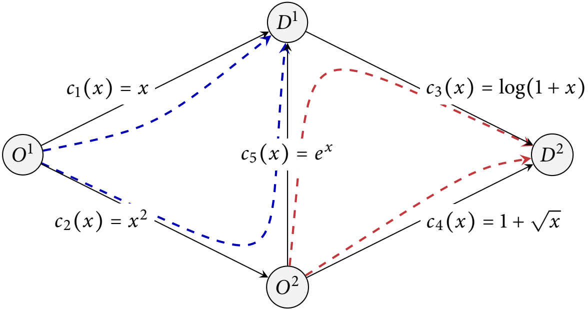

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- Light traffic, benchmark $c(x) = 1$: network is tight ✓

Asymptotic analysis

Theorem (Colini-Baldeschi, Cominetti, M & Scarsini, 2018)

Let Γ_M be a network with total traffic inflow M. If the network is tight in light $(\omega = 0)$ and/or heavy $(\omega = \infty)$ traffic, then

$$\lim_{M\to\omega} \mathrm{PoA}(\Gamma_M) = 1$$

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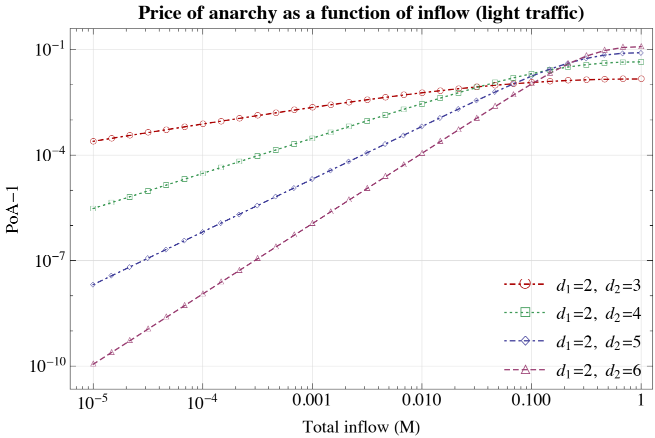
Corollary

If the network's cost functions are polynomials, $\operatorname{PoA}(\Gamma_M) \to 1$ as $M \to \omega \in \{0, \infty\}$.

In networks with polynomial costs, the gap between fairness and efficiency disappears under both light and heavy traffic

Convergence rate

Two-link Pigou network with cost functions $c_1(x_1) = x_1^{d_1}$, $c_2(x_2) = x_2^{d_2}$



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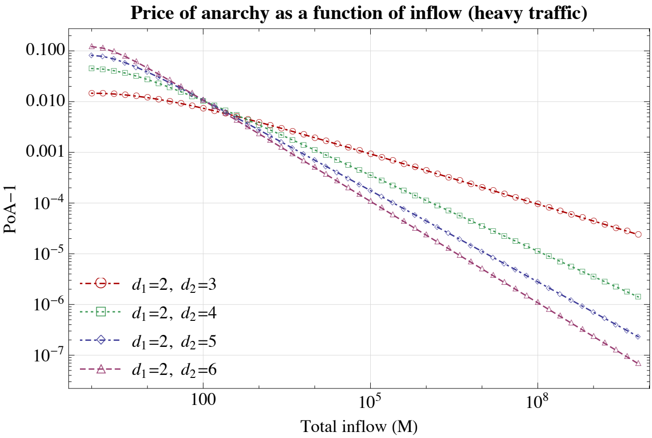

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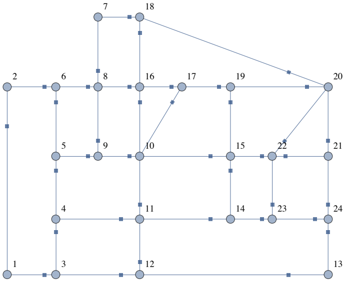
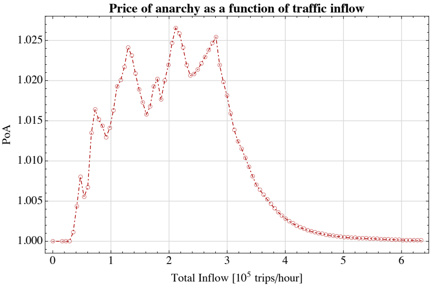
Sioux Falls: a case study

The price of anarchy in Sioux Falls:



(e) The Sioux Falls road network



(f) The price of anarchy in Sioux Falls.

Average traffic inflow $M_{\text{avg}} \approx 3.6 \times 10^5$ trips/hour (LeBlanc et al., 1975)

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Algorithms and dynamics

Towards optimality / equilibrium

Two (related) optimization problems:

1. Total latency minimization:

$$\begin{aligned} \text{minimize} \quad & C(f) = \sum_{e \in \mathcal{E}} x_e c_e(x_e) \\ \text{subject to} \quad & x_e = \sum_{p \ni e} f_p, \; f \in \mathcal{F} \end{aligned} \tag{LM}$$

2. Fairness (Wardrop equilibrium):

$$\begin{aligned} \text{minimize} \quad & L(f) = \sum_{e \in \mathcal{E}} C_e(x_e) \\ \text{subject to} \quad & x_e = \sum_{p \ni e} f_p, \; f \in \mathcal{F} \end{aligned} \tag{WE}$$

where $C_e(x_e) = \int_0^{x_e} c_e(w)\,dw$ is the *primitive* of c_e

How can either problem be solved in a scalable and efficient manner?

Challenges

Generic problem formulation (single O/D pair for simplicity):

$$\begin{aligned} \text{minimize} \quad & g(f) = \sum_{e \in \mathcal{E}} g_e(x_e) \\ \text{subject to} \quad & f \in \mathcal{F} \end{aligned} \qquad \text{(Opt)}$$

where:

- $x_e = \sum_{p \ni e} f_p$ [edge-route duality]
- $\mathcal{F} = M \cdot \Delta(\mathcal{P}) = \{f \in \mathbb{R}_+^{\mathcal{P}} : \sum_{p \in \mathcal{P}} f_p = M\}$ [simplicial structure]

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Challenges:

- Information: cost functions a priori unknown
- Dimensionality: exponential number of paths
- Control plane: dynamic/distributed flow control

Flow/load gradients

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Follow the negative gradient of g:

$$v = -\nabla g$$

By edge-route duality:

$$v_p(f) = -\frac{\partial g}{\partial f_p} = -\sum_{e \in \mathcal{E}} g_e'(x_e) \frac{\partial x_e}{\partial f_p} = -\sum_{e \in p} g_e'(x_e) =: \sum_{e \in p} v_e(x_e)$$

To get route flow gradient ← sum edge load gradients along route

Gradient information

Assume algorithmic scheme generates at $n = 1, 2, \ldots$

- Flow profile $f_n = (f_{p,n})_{p \in \mathcal{P}}$
- Load profile $x_n = (x_{e,n})_{e \in \mathcal{E}}$, $x_{e,n} = \sum_{e \in p} f_{p,n}$

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When called at x_n, $n = 1, 2, \ldots$, assume gradients estimated up to some (random) error:

$$\hat{v}_{e,n} = v_e(x_{e,n}) + U_{e,n+1}$$

with the following hypotheses for the *error process* U:

(H1) **Zero-mean:** $\mathbb{E}[U_{n+1} \,|\, \mathcal{F}_n] = 0$

(H2) **Finite variance:** $\mathbb{E}[\|U_{n+1}\|^2 \,|\, \mathcal{F}_n] \leq \sigma^2$

Gradient descent

Projected gradient descent:

$$f_{n+1} = \Pi_{\mathcal{F}}(f_n + \gamma_n \hat{v}_n) \tag{GD}$$

where

$$\Pi_{\mathcal{F}}(f) = \arg\min_{f' \in \mathcal{F}} \| f' - f \|$$

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Theorem (folk)

Suppose that $|g_e'| \leq G$. *If* (GD) *is run with* $\gamma_n \propto 1/\sqrt{Gn}$ *and returns*

$$\bar{f}_n = \frac{\sum_{k=1}^n \gamma_k f_k}{\sum_{k=1}^n \gamma_k},$$

then

$$\mathbb{E}[g(\bar{f}_n)] \leq \min g + \mathcal{O}(G\sqrt{|\mathcal{P}|/n})$$

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- Can be improved to convergence in high probability ✓

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- **Exponential dependence on the size of the graph because of $|\mathcal{P}|$** ✗
- **Projection step has complexity $\Theta(|\mathcal{P}| \log|\mathcal{P}|)$** ✗
- **Need to store $\Theta(|\mathcal{P}|)$ variables** ✗

Exponentiated gradient descent

An idea from **reinforcement learning** (Vovk, Littlestone & Warmuth, ...):

- Keep a score for each path, based on its performance so far
- Allocate traffic proportionally to the exponential of this score

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Key insight: score by aggregating (negative) gradient steps

Exponentiated gradient descent (EGD)

Require: step-size sequence $\gamma_k > 0$

1: set $y_p \leftarrow 0$ for each route $p \in \mathcal{P}$ # initialization
2: **for** $k = 1, 2, \dots, n$ **do**
3: assign traffic $f_p \propto \exp(y_p)$ # exponential weights
4: set $y \leftarrow y + \gamma_k \hat{v}_k$ # score update
5: **end for**
6: **return** $\bar{f}_n = \sum_{k=1}^n \gamma_k f_k \Big/ \sum_{k=1}^n \gamma_k$

The good

Theorem (M, Paschos, Vigneri, 2018)

Suppose that $|g'_e| \leq G$. *If EGD is run with* $\gamma_n \propto 1/\sqrt{Gn}$, *then*

$$\mathbb{E}[g(f_n)] \leq \min g + \mathcal{O}(G\sqrt{\log|\mathcal{P}|/n})$$

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- Value convergence despite imperfect feedback ✓
- Can be improved to convergence in high probability ✓
- Rate in n cannot be improved (but not too slow in practice) ✓
- **Normalization step has complexity $\Theta(|\mathcal{P}|)$** ✗
- **Still need to store $\Theta(|\mathcal{P}|)$ variables** ✗

Distribution in the control plane

Can we distribute the algorithm at the node level?

- Given: an O/D pair (O, D)
- Each node $v \in \mathcal{V}$ has a subset of edges e_v that can be used to reach D
- No backtracking: acyclic routing (multi-)graph $\mathcal{G} = (\mathcal{V}, \bigcup_{v \in \mathcal{V}} e_v)$
- Each node controls traffic allocation over $\mathcal{E}_v$, i.e., a vector

$$z = (z_e)_{e \in \mathcal{E}_v} \in \Delta(\mathcal{E}_v)$$

- Small dimensionality per control node – **but how to implement EGD?**

The role of weight propagation

Key steps in EGD:

- Update scores: $y_e \leftarrow y_e + \gamma \hat{v}_e$ ✓
- Traffic allocation: ??? ✗

Straightforward choice of weights:

$$z_e = \frac{\exp(y_e)}{\sum_{e' \in \mathcal{E}_v} \exp(y_{e'})}$$

OK in terms of dimension; **complete failure in terms of optimization**

Backpedaling

Key insight: must not be blind to what is happening down the road!

0. **Require:** edge score vector $y = (y_e)_{e \in \mathcal{E}}$

   Initialize: latent weight variables w_v for each $v \in \mathcal{V}$, w_e for each $e \in \mathcal{E}$.
   Set $w_D = 0$ at destination; backpropagate w_D through all edges linking to D.

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1. **Weigh and wait:** When node v receives weight information from connecting node v' via edge $e \in \mathcal{E}_v$, set

$$w_e = y_e + w_{v'}$$

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1. **Weigh and wait:** When node v receives weight information from connecting node v' via edge $e \in \mathcal{E}_v$, set

$$w_e = y_e + w_{v'}$$

2. **Sum and send:** If node v has received an update via all outgoing edges $\mathcal{E}_v$, set

$$w_v = \log \sum_{e \in \mathcal{E}_v} \exp(w_e)$$

   and push w_v back through all edges linking to v

Exponential weights and backpedaling

Proposition

Let $y \in \mathbb{R}^{\mathcal{E}}$ be an edge score vector and suppose each node $v \in \mathcal{V}$ allocates traffic following the exponential rule

$$z_e = \frac{\exp(w_e)}{\exp(w_v)} \quad \text{for all } e \in \mathcal{E}_v,$$

with w_e and w_v defined via backpedaling. Then, the total traffic flowing through route $p \in \mathcal{P}$ is

$$f_p = \frac{\exp(y_p)}{\sum_{q \in \mathcal{P}} \exp(y_q)}$$

where $y_p = \sum_{e \in p} y_e$ denotes the corresponding path score.

Exponential node weights with backpedaling induce exponential path weights!

Distributed EGD

Theorem (Gaujal, Héliou, M, 2018)

Suppose that $|g'_e| \leq G$. *If EGD is run at the node level with backpedaling and a step-size* $\gamma_n \propto 1/\sqrt{Gn}$, *then*

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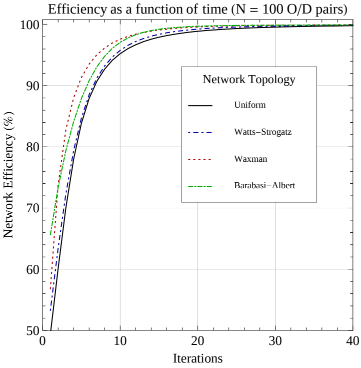
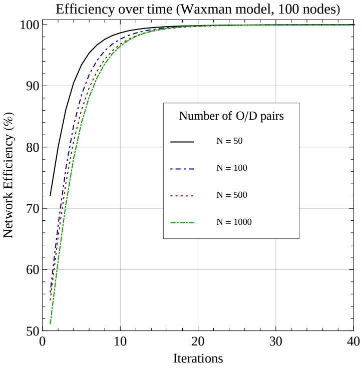
- Value convergence despite imperfect feedback ✓
- Can be improved to convergence in high probability ✓
- **Linear dependence on the size of the graph through $|\mathcal{P}|$** ✓
- **Update step has $\mathcal{O}(|\mathcal{E}_v|)$ complexity per node** ✓
- **Only need to store $\mathcal{O}(|\mathcal{E}_v|)$ variables per node** ✓

Convergence rate

Distributed EGD in randomly generated networks





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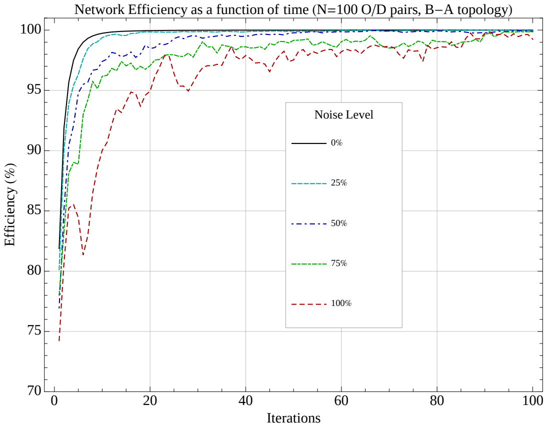

Conclusions and perspectives

"Traffic congestion is caused by vehicles, not by people in themselves"

— Jane Jacobs, *The Death and Life of Great American Cities*

- The price of anarchy disappears in light and heavy traffic, independently of the network topology and even with multiple O/D pairs
- Exponential weights + backpedaling allow fast, distributed optimization
- **Size of the network:** not a curse, but a blessing in disguise

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Open questions

- Capacitated networks (M/M/1, M/G/1, etc.)?
- What if there is **no** gradient feedback whatsoever?
- What about **atomic** routing games?