Variable forgetting in preference relations over propositional domains

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Abstract. Representing (and reasoning about) preference relations over combinatorial domains is computationally expensive. For many problems involving such preferences, it is relevant to simplify them by projecting them on a subset of variables. We investigate several possible definitions, focusing without loss of generality on propositional (binary) variables.

1 Introduction and background

Decision-making problems are concerned with managing agents' preferences. When the set of alternatives is small, preferences can be represented explicitly, by simply ranking alternatives, and the associated tasks such as optilization or aggregation are computationally easy. However, in many real-world applications, domains have a combinatorial structure. In that case, managing agents' preferences can be an enormous computational burden. This has led to the study of compact preference representation languages.

For some problems it might be relevant to process preference relations, so as to simplify it and make it more compact, even if this results in a loss of information. Especially, it may be helpful to *project* a preference relation on a subset of the variables. This way of summarizing a preference relation is relevant in particular when some variables are more important than others, or when some variables should be assigned prior to others. Consider for instance a group decision making scenario. Rather than aggregating the whole preference relations before finding out an optimal assignment of variables, which generally is computationally intractable, it may be a good idea to focus on "primary" variables first, project the preference relation on those variables, aggregate them, decide on the values to be assigned to those variables, and only then consider secondary variables.

Projection operations have not been considered much as far as preference relations are concerned, but there is a huge amount of work about projecting (or *marginalizing*) probability distributions (especially when they are represented by Bayesian networks), and more generally valuation functions [6, 3], as well as sets of constraints, and propositional formulas (cf. the forgetting operation [5]). In this paper we define similar projection operations for ordinal preference relations. For the sake of simplicity, we focus on combinatorial domains formed from propositional (i.e., binary) variables.

Let V be a finite set of propositional variables. For any subset X of V, an X-*alternative* is an element of 2^X, that is, an assignment of a binary truth value to each one of the variables in X. X-alternatives are denoted by $\vec{x}$, $\vec{x}\,'$ etc. If X and Y are disjoint subsets of V then the concatenation of $\vec{x} \in 2^X$ and $\vec{y} \in 2^Y$ is the $X \cup Y$-alternative, denoted by $\vec{x}\vec{y}$, assigning values to variables of X (resp. Y) as $\vec{x}$ (resp. $\vec{y}$) does.

A V-*preference relation* R is a reflexive and transitive relation over 2^V. The *strict preference* $>_R$ associated with R is the strict order defined by $\vec{v} >_R \vec{v}\,'$ iff $R(\vec{v}, \vec{v}\,')$ and not $R(\vec{v}\,', \vec{v})$. The *indifference relation* $\sim_R$ associated with R is the equivalence relation defined by $\vec{v} \sim_R \vec{v}\,'$ iff $R(\vec{v}, \vec{v}\,')$ and $R(\vec{v}\,', \vec{v})$. If neither $R(\vec{v}, \vec{v}\,')$ nor $R(\vec{v}\,', \vec{v})$ then $\vec{v}$ and $\vec{v}\,'$ are *incomparable* w.r.t. R, denoted by $\vec{v}Q\vec{v}\,'$. For the sake of notation, when we specify a preference relation explicitly, we omit pairs coming from reflexivity and transitivity. R^* denotes the transitive closure of a relation R over 2^V.

For any V-preference relation R and any partition $\{X, Y, Z\}$ of V, X is preferentially independent from Y given Z w.r.t. R iff for all $\vec{x}, \vec{x}\,' \in 2^X$, all $\vec{y}, \vec{y}\,' \in 2^Y$ and all $\vec{z} \in 2^Z$, $R(\vec{x}\vec{y}\vec{z}, \vec{x}\,'\vec{y}\vec{z})$ implies $R(\vec{x}\vec{y}\,'\vec{z}, \vec{x}\,'\vec{y}\,'\vec{z})$. If $Z = \emptyset$ then we say that X is preferentially independent from $V \setminus X$ w.r.t. R.

2 Lower and upper projections

Definition 1 (lower and upper projections) *Let R be a V-preference relation and $X \subseteq V$. Let $Y = V \setminus X$;*

- $R_L^{\downarrow X}$, *called the* lower projection *of R on X, is the binary relation over X defined as follows: $R_L^{\downarrow X}(\vec{x}, \vec{x}\,')$ holds iff $R(\vec{x}\vec{y}, \vec{x}\,'\vec{y})$ holds for all $\vec{y} \in 2^Y$;*
- $R_U^{\downarrow X}$, *called the* upper projection *of R on X, is the transitive closure of the relation R' over X defined by:*
  $R'(\vec{x}, \vec{x}\,')$ holds iff $R(\vec{x}\vec{y}, \vec{x}\,'\vec{y})$ holds for some $\vec{y} \in 2^Y$.

Note that, when R is complete, $R_U^{\downarrow X}$ is obviously complete as well but $R_L^{\downarrow X}$ may fail to be complete. other properties are:

Proposition 1 *For any R, R', X, Y:*
1. *$R_L^{\downarrow X}$ and $R_U^{\downarrow X}$ are X-preference relations;*
2. *if R is complete then $R_U^{\downarrow X}$ is complete;*
3. *if $R \subseteq R'$ then $R_L^{\downarrow X} \subseteq (R')_L^{\downarrow X}$ and $R_U^{\downarrow X} \subseteq (R')_U^{\downarrow X}$;*
4. *$(R \cap R')_L^{\downarrow X} = R_L^{\downarrow X} \cap (R')_L^{\downarrow X}$; $(R \cap R')_U^{\downarrow X} \subseteq R_U^{\downarrow X} \cap (R')_U^{\downarrow X}$;*
5. *$((R \cup R')^*)_U^{\downarrow X} = (R_U^{\downarrow X} \cup (R')_U^{\downarrow X})^*$;*
   $(R \cup R')_L^{\downarrow X} \supseteq (R_L^{\downarrow X} \cup (R')_L^{\downarrow X})^*$;
6. *$(R_U^{\downarrow X})_U^{\downarrow Y} = (R_U^{\downarrow Y})_U^{\downarrow X}$; $(R_L^{\downarrow X})_L^{\downarrow Y} = (R_L^{\downarrow Y})_L^{\downarrow X}$.*

Proposition 2 *For any V-preference relation R and any $X \subseteq V$, $R_L^{\downarrow X} = R_U^{\downarrow X}$ iff X is preferentially independent from $V \setminus X$ w.r.t. R.*

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3 Optimistic and pessimistic projections

Definition 2 (optimistic/pessimistic projections) *Let R be a V-preference relation and $X \subseteq V$. Let $Y = V \setminus X$;*

- $R^{\downarrow X}_{StrongOpt}$, *the* strong optimistic projection *of R on X, is defined by:* $R^{\downarrow X}_{StrongOpt}(\vec{x}, \vec{x}\,')$ *iff* $\exists \vec{y}\ \forall \vec{y}\,'$, $R(\vec{x}\vec{y}, \vec{x}\,'\vec{y}\,')$;
- $R^{\downarrow X}_{WeakOpt}$, *the* weak optimistic projection *of R on X, is defined by:* $R^{\downarrow X}_{WeakOpt}(\vec{x}, \vec{x}\,')$ *iff* $\forall \vec{y}\,'\ \exists \vec{y}\ R(\vec{x}\vec{y}, \vec{x}\,'\vec{y}\,')$;
- $R^{\downarrow X}_{StrongPess}$, *the* strong pessimistic projection *of R on X, is defined by:* $R^{\downarrow X}_{StrongPess}(\vec{x}, \vec{x}\,')$ *iff* $\exists \vec{y}\,'\ \forall \vec{y}$, $R(\vec{x}\vec{y}, \vec{x}\,'\vec{y}\,')$;
- $R^{\downarrow X}_{WeakPess}$, *the* weak pessimistic projection *of R on X, is defined by:* $R^{\downarrow X}_{WeakPess}(\vec{x}, \vec{x}\,')$ *iff* $\forall \vec{y}\ \exists \vec{y}\,'\ R(\vec{x}\vec{y}, \vec{x}\,'\vec{y}\,')$.

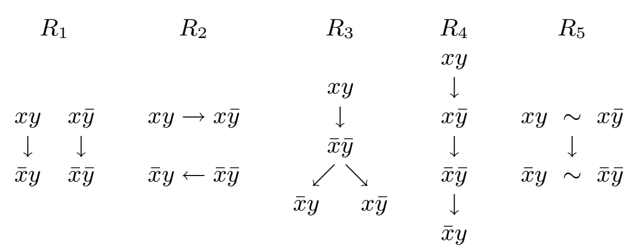
The optimistic projections focus on finding some possibility to have $\vec{x}$ dominating $\vec{x}\,'$ whatever the context for $\vec{x}\,'$. The pessimistic projections focus on finding some possibility to have $\vec{x}\,'$ dominated by $\vec{x}$ whatever the context for $\vec{x}$.

When R is complete, $R^{\downarrow X}_{StrongOpt}$ and $R^{\downarrow X}_{WeakOpt}$ coincide, as well as $R^{\downarrow X}_{StrongPess}$ and $R^{\downarrow X}_{WeakPess}$, and all four are complete. In this case, $R^{\downarrow X}_{StrongOpt}(\vec{x}, \vec{x}\,')$ (and equivalently $R^{\downarrow X}_{WeakOpt}(\vec{x}, \vec{x}\,')$) iff the best alternatives extending $\vec{x}$ are at least as good as the best alternatives extending $\vec{x}\,'$, whereas $R^{\downarrow X}_{StrongPess}(\vec{x}, \vec{x}\,')$ (and equivalently $R^{\downarrow X}_{WeakPess}(\vec{x}, \vec{x}\,')$) if and only if the worst alternatives extending $\vec{x}$ are at least as good as the worst alternatives extending $\vec{x}\,'$[3].

Proposition 3 *We have the following inclusions.*

- $R^{\downarrow X}_{L} \subseteq R^{\downarrow X}_{StrongOpt} \subseteq R^{\downarrow X}_{WeakOpt} \subseteq R^{\downarrow X}_{U}$;
- $R^{\downarrow X}_{L} \subseteq R^{\downarrow X}_{StrongPess} \subseteq R^{\downarrow X}_{WeakPess} \subseteq R^{\downarrow X}_{U}$.

4 Examples



When $R = R_1$: all projections on x coincide and are equal to the preference relation $x > \bar{x}$. All projections of R_1 on y coincide and are equal to the preference relation $yQ\bar{y}$ in which y and $\bar{y}$ are incomparable.

When $R = R_2$: all projections on x are equal to $xQ\bar{x}$. $R^{\downarrow\{y\}}_{L}$ as well as $R^{\downarrow\{y\}}_{StrongOpt}$ and $R^{\downarrow\{y\}}_{StrongPess}$ are equal to $yQ\bar{y}$, while $R^{\downarrow\{y\}}_{U}$ as well as $R^{\downarrow\{y\}}_{WeakOpt}$ and $R^{\downarrow\{y\}}_{WeakPess}$ are equal to $y \sim \bar{y}$.

When $R = R_3$: $R^{\downarrow\{x\}}_{L}$ is equal to $xQ\bar{x}$; $R^{\downarrow\{x\}}_{U}$ is equal to $x \sim \bar{x}$; $R^{\downarrow\{x\}}_{StrongOpt}$ and $R^{\downarrow\{x\}}_{WeakOpt}$ are equal to $x > \bar{x}$; $R^{\downarrow\{x\}}_{StrongPess}$, is equal to $xQ\bar{x}$, while $R^{\downarrow\{x\}}_{WeakPess}$ is equal to $x \sim \bar{x}$. Things are symmetric for the projections on y.

When $R = R_4$: all projections on x are equal to $x > \bar{x}$. $R^{\downarrow\{y\}}_{L}$ is equal to $yQ\bar{y}$; $R^{\downarrow\{y\}}_{U}$ is equal to $y \sim \bar{y}$; the optimistic projections on y (which coincide because R is complete) are equal to $y > \bar{y}$; the pessimistic projections on y (which coincide, again because R is complete) are equal to $\bar{y} > y$.

Lastly, for R_5, all projections on x are equal to $x > \bar{x}$ and all projections on y are equal to $y \sim \bar{y}$.

[3] These criteria are reminiscent of those used in qualitative decision theory (see e.g. [1, 2] – with the slightly different interpretation that X-alternatives represent possible decisions and elements of $(V \setminus X)$-alternatives represent possible states of the world.

5 Connection to propositional logic

Let L_V be the propositional language built up from V. if $\varphi \in L_V$, $Var(\varphi)$ denotes the set of propositional variables occurring in φ. We make use of the next two notions from [4] where $\varphi \in L_V$ and $X \subseteq V$: the *strongest necessary condition* of φ on X, denoted by $\exists(V \setminus X).\varphi$, is the strongest formula ψ of L_V such that $Var(\psi) \subseteq X$ and $\varphi \models \psi$; the *weakest sufficient condition* of φ on X, denoted by $\forall(V \setminus X).\varphi$), is the weakest formula ψ of L_V such that $Var(\psi) \subseteq X$ and $\psi \models \varphi$. $\exists(V \setminus X).\varphi$ is usually known as the *forgetting* of $V \setminus X$ in φ.

A V-preference relation is *bipartite* iff there exists $G \subseteq 2^V$ such that for all $\vec{v}$, $\vec{v}\,' \in 2^V$, then $R(\vec{v}, \vec{v}\,')$ holds iff $\vec{v} \in G$ or $\vec{v}\,' \in 2^V \setminus G$; the *characteristic formula* θ_R of a bipartite V-preference relation R is the propositional formula – unique up to logical equivalence – whose set of models is exactly G (in symbols, $Mod(\theta_R) = G$). Note that if R is bipartite, it is complete and then strong and weak notions coincide.

Proposition 4 *Let R be a bipartite preference relation whose characteristic formula is θ_R. Let $X \subseteq V$ and $Y = V \setminus X$. Then*

- $R^{\downarrow X}_{WeakOpt} = R^{\downarrow X}_{StrongOpt}$ *is the bipartite relation whose characteristic formula is* $\exists(V \setminus X).\theta_R$.
- $R^{\downarrow X}_{WeakPess} = R^{\downarrow X}_{StrongPess}$ *is the bipartite relation whose characteristic formula is* $\forall(V \setminus X).\theta_R$.

6 Conclusion and perspectives

This paper is meant to pave the way towards simplifying and decomposing preference relations over combinatorial structures. It is still a preliminary work and raises many questions. One of the most salient issues that we did not investigate is about computing the various notions of projection when the initial preference relation is represented in a *compact representation language* The long version of the paper also includes a section on the various possible notions of *independence* of a preference relation from a set of variables.

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