New candidates welcome! Possible winners with respect to the addition of new candidates☆

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ABSTRACT

In voting contexts, some new candidates may show up in the course of the process. In this case, we may want to determine which of the initial candidates are possible winners, given that a fixed number k of new candidates will be added. We give a computational study of this problem, focusing on scoring rules, and we provide a formal comparison with related problems such as control via adding candidates or cloning.

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1. Introduction

In many real-life collective decision making situations, the set of candidates (or alternatives) may vary while the voting process goes on, and may change at any time before the decision is final; some new candidates may join, whereas some others may withdraw. This, of course, does not apply to situations where the vote takes place in a very short period of time (such as, typically, political elections in most countries), and neither does the addition of new candidates during the process apply to situations where the law forbids new candidates to be introduced after the voting process has started (which, again, is the case for most political elections). However, there are many practical settings where this may happen, especially situations where votes are sent by email during an extended period of time. This is typically the case when making a decision about the date and time of a meeting. In the course of the process, we may learn that the room is taken at a given time slot, making this time slot no longer a candidate. The opposite case also occurs frequently; we thought the room was taken on a given date and then we learn that it has become available, making this time slot a new candidate.

The paper focuses on candidate addition only. More precisely, the class of situations we consider is the following. A set of voters have expressed their votes about a set of (initial) candidates. Then some new candidates declare their intention to participate in the election. The winner will ultimately be determined using some given voting rule and the voters' preferences over the set of all candidates. In this class of situations, an important question arises: *who among the initial candidates can still be a winner once the voters' preferences about all candidates are known*? This is important in particular if there is some interest to detect as soon as possible the candidates who are not possible winners: for instance, candidates for a job may have the opportunity to apply for different positions, and time slots may be released for other potential meetings.

This question is strongly related to several streams of work in the recent literature on computational social choice, especially the problem of determining whether the vote elicitation process can be terminated (Conitzer and Sandholm, 2002; Walsh, 2008); the possible winner problem, and more generally the problem of applying a voting rule to incomplete preferences (Konczak and Lang, 2005; Pini et al., 2007; Xia and Conitzer, 2011; Betzler and Dorn, 2009; Betzler et al., 2009) or uncertain preferences with probabilistic information (Hazon et al., 2009); swap bribery, encompassing the possible winner problem as a particular case (Elkind et al., 2009); voting with an unknown set of available candidates (Lu and Boutilier, 2010); the control of a voting rule by the chair via adding candidates; and resistance to cloning—we shall come back to the latter two problems in more detail in the related work section.

Clearly, considering situations where new voters are added is a specific case of voting under incomplete preferences, where incompleteness is of a very specific type: the set of candidates is partitioned in two groups (the initial and the new candidates), and the incomplete preferences consist of complete rankings on the initial candidates. This class of situations is, in a sense, dual

to a class of situations that has been considered more often, namely, when the set of *voters* is partitioned in two groups: those voters who have already voted, and those who have not expressed their votes yet. The latter class of situations, while being a subclass of voting under incomplete preferences, has been more specifically studied as a *coalitional manipulation problem* (Conitzer et al., 2007; Faliszewski and Procaccia, 2010), where the problem is to determine whether it is possible for the voters who have not voted yet to make a given candidate win. Varying sets of voters have also been studied in the context of compiling the votes of a subelectorate (Chevaleyre et al., 2009; Xia and Conitzer, 2010): there, one is interested in summarizing a set of initial votes, while still being able to compute the outcome once the remaining voters have expressed their votes.

The layout of the paper is as follows. In Section 2 we recall the necessary background on voting and we introduce some notation. In Section 3 we state the problem formally, by defining voting situations where candidates may be added after the votes over a subset of initial candidates have already been elicited. In the following sections we focus on specific voting rules and we study the problem from a computational point of view. In Section 4, we focus on the family of K-approval rules, including plurality and veto as specific subcases, and give a full dichotomy result for the complexity of the possible winner problem with respect to the addition of k new candidates; namely, we show that the problem is NP-complete as soon as $K \geq 3$ and $k \geq 3$, and polynomial if $K \leq 2$ or $k \leq 2$. In Section 5 we focus on the Borda rule and show that the problem is polynomial-time solvable regardless of the number of new candidates. We also exhibit a more general family of voting rules, including Borda, for which this result can be generalized. In Section 6 we show that the problem can be hard for some positional scoring rules even if only one new candidate is added. In Section 7 we discuss the relationship to the general possible winner problem, to the control of an election by the chair via adding candidates, and to candidate cloning. Section 8 summarizes the results and mentions further research directions.

2. Background and notation

Let C be a finite set of *candidates*, and N a finite set of *voters*. The number of voters is denoted by n, and the (total) number of candidates by m. A *C-vote* (called simply a vote when this is not ambiguous) is a linear order over C, denoted by $\succ$ or by V. We sometimes denote votes in the following way: $a \succ b \succ c$ is denoted by abc, etc. An n-voter C-profile is a collection $P = \langle V_1, \ldots, V_n \rangle$ of C-votes. Let $\mathcal{P}_C$ be the set of all C-votes and therefore $\mathcal{P}_C^n$ be the set of all n-voter C-profiles. We denote by $\mathcal{P}_C^*$ the set of all n-voter C-profiles for $n \geq 1$, i.e., $\mathcal{P}_C^* = \cup_{n \geq 1} \mathcal{P}_C^n$.

A voting rule on C is a function r from $\mathcal{P}_C^*$ to C. A voting correspondence is a function from $\mathcal{P}_C^*$ to $2^C \setminus \{\emptyset\}$. The most natural way of obtaining a voting rule from a voting correspondence is to break ties according to a fixed priority order on candidates. In this paper, we do not fix a priority order on candidates (one reason being that the complete set of candidates is not known to start with), which means that we consider voting correspondences rather than rules, and ask whether x is a *possible cowinner* for a given profile P. This is equivalent to asking whether there exists a priority order for which x is a possible winner, or else whether x is a possible winner for the most favorable priority order (with x having priority over all other candidates). This is justified in our context by the fact that specifying such a priority order is problematic when we don't know in advance the identities of the potential new candidates. With a slight abuse of notation we denote voting correspondences by r just as voting rules. Let $r(P)$ be the set of *cowinners* for profile P.

For $P \in \mathcal{P}_C^*$ and $x, x' \in C$, let $n(P, i, x)$ be the number of votes in P ranking x in position i, $\text{ntop}(P, x) = n(P, 1, x)$ the number of votes in P ranking x first, and $N_P(x, x')$ the number of votes in P ranking x above x'. Let $\vec{s} = \langle s_1, \ldots, s_m \rangle$ be a vector of integers such that $s_1 \geq \cdots \geq s_m$ and $s_1 > s_m$. The scoring rule $r_{\vec{s}}(P)$ induced by $\vec{s}$ elects the candidate(s) maximizing $S_{\vec{s}}(x, P) = \sum_{i=1}^{m} s_i \cdot n(P, i, x)$.

If K is a fixed integer then K-approval, r_K, is the scoring rule corresponding to the vector $\vec{s}_K = \langle 1, \ldots, 1, 0, \ldots, 0 \rangle$—with K 1's and $m - K$ 0's. The K-approval score $S_{\vec{s}_K}(x, P)$ of a candidate x is denoted more simply by $S_K(x, P)$: in other words, $S_K(x, P)$ is the number of voters in P who rank x in the first K positions, i.e., $S_K(x, P) = \sum_{i=1,\ldots,K} n(P, i, x)$. When $K = 1$, we get the *plurality* rule r_P, and when $K = m - 1$ we get the *veto* (or *antiplurality*) rule. The *Borda* rule r_B is the scoring rule corresponding to the vector $\langle m - 1, m - 2, \ldots, 0 \rangle$.

We now define formally situations where new candidates are added.

Definition 1. A *voting situation with a varying set of candidates* is a 4-tuple $\Sigma = \langle N, X, P_X, k \rangle$ where N is a set of voters (with $|N| = n$), X a set of candidates, $P_X = \langle V_1, \ldots, V_n \rangle$ an n-voter X-profile, and k is a positive integer, encoded in unary.

X denotes the set of initial candidates, P_X the initial profile, and k the number of new candidates. Nothing is known a priori about the voters' preferences over the new candidates, henceforth their identity is irrelevant and only their number counts. The assumption that k is encoded in unary ensures that the number of new candidates is polynomial in the size of the input. Most of our results would still hold if the number of new candidates is exponentially large in the size of the input, but for the sake of simplicity, and also because in practice k will be small anyway, we prefer to exclude this possibility.

Because the number of candidates is not the same before and after the new candidates come in, we have to consider families of voting rules (for a varying number of candidates) rather than voting rules for a fixed number of candidates. While it is true that for many usual voting rules there is an obvious way of defining them for a varying number of candidates, this is not the case for all of them, especially scoring rules. Still, some natural scoring rules, including plurality, veto, more generally K-approval, as well as Borda, are naturally defined for any number of candidates. We shall therefore consider *families of voting rules*, parameterized by the number of candidates (r^m). We slightly abuse notation and denote these families of voting rules by r, and consequently often write $r(P)$ instead of $r^m(P)$. The complexity results we give in this paper make use of such families of voting rules, where the number of candidates is variable.

If P is a C-profile and $C' \subseteq C$, then the projection of P on C', denoted by $P^{\downarrow C'}$, is obtained by deleting all candidates in $C \setminus C'$ in each of the votes of P, and leaving unchanged the ranking on the candidates of C'. For instance, if $P = \langle abcd, dcab \rangle$, then $P^{\downarrow\{a,b\}} = \langle ab, ab \rangle$ and $P^{\downarrow\{a,b,c\}} = \langle abc, cab \rangle$. In all situations, the set of initial candidates is denoted by $X = \{x_1, \ldots, x_p\} \cup \{x^*\}$, the set of the k new candidates is denoted by $Y = \{y_1, \ldots, y_k\}$. If P_X is an X-profile and P' an $X \cup Y$-profile, then we say that P' extends P_X if the projection of P' on X is exactly P_X. For instance, let $X = \{x_1, x_2, x_3\} \cup \{x^*\}, Y = \{y_1, y_2\}$; the profile $P' = \langle x_1 y_1 x^* x_2 y_2 x_3, y_1 y_2 x_1 x_2 x_3 x^*, x_3 x_2 y_2 x^* y_1 x_1 \rangle$ extends the X-profile $P_X = \langle x_1 x^* x_2 x_3, x_1 x_2 x_3 x^*, x_3 x_2 x^* x_1 \rangle$.

3. Possible winners when new candidates are added

We recall from Konczak and Lang (2005) that given a collection $\langle P_1, \ldots, P_n \rangle$ of partial strict orders on C representing some incomplete information about the votes, a candidate x^* is a possible winner if there is a profile $\langle T_1, \ldots, T_n \rangle$ where each T_i is a ranking on C extending P_i in which x^* wins. Reformulated for the case where P_i is a ranking of the initial candidates (those in X), we get the following definition:

Definition 2. Given a voting situation $\Sigma = \langle N, X, P_X, k \rangle$, and a collection r of voting rules, we say that $x^* \in X$ is a *possible cowinner with respect to Σ and r* if there is a $(X \cup Y)$-profile P' extending P_X such that $x^* \in r(P')$, where $Y = \{y_1, \ldots, y_k\}$ is a set of k new candidates.

Note that we do not have Y in the input, because it would be redundant with k: it is enough to know the *number* of new candidates. Note also that all new candidates $\{y_1, \ldots, y_k\}$ have to appear in the extended votes composing P'.

Also, we do not consider the problem of deciding whether a new candidate y_j is a possible cowinner, because it is trivial. Indeed, as soon as the voting correspondence satisfies the extremely weak property that a candidate ranked first by all voters is always a cowinner (which is obviously satisfied by all common voting rules), any new candidate is a possible cowinner.

We now define formally the problems we study in this paper.

Definition 3. Given a collection r of voting rules, the POSSIBLE COWINNER PROBLEM WITH NEW CANDIDATES (or PcWNC) for r is defined as follows:

Input   A voting situation $\Sigma = \langle N, X, P, k \rangle$ and a candidate $x^* \in X$.

Question Is x^* a possible cowinner with respect to Σ and r?

Also, the subproblem of PcWNC where the number k of new candidates is fixed will be denoted by PcWNC(k).

We can also define the notion of *necessary cowinner* with respect to Σ and r : $x^* \in X$ is a *necessary cowinner with respect to Σ, Y, and r* if for every $(X \cup Y)$-profile P' extending P_X we have $x^* \in r(P)$. However, the study of necessary cowinners in this particular setting will almost never lead to any significant results. There may be necessary cowinners among the initial candidates, but this will happen rarely (and this case will be discussed for a few specific voting rules in the corresponding parts of the paper).

Now we are in position to consider specific voting rules.

4. K-approval

As a warm-up we start by considering the plurality rule.

4.1. Plurality

Let us start with an example: suppose $X = \{a, b, c\}$, $n = 13$, and the plurality scores in P_X are $a \mapsto 6, b \mapsto 4, c \mapsto 3$. There is only one new candidate (y). We have:

1. a is a possible cowinner (a will win in particular if the top candidate of every voter remains the same);
2. b is a possible cowinner: to see this, suppose that 2 voters who had ranked a first now rank y first; the new scores are $a \mapsto 4, b \mapsto 4, c \mapsto 3, y \mapsto 2$;
3. c is not a possible cowinner: to reduce the scores of a (resp. b) to that of c, we need at least 3 (resp. 1) voters who had ranked a (resp. b) first to now rank y first; but this then means that y gets at least 4 votes, while c has only 3.

More generally, we have the following result:

Proposition 1. *Let P_X be an n-voter profile on X, and $x^* \in X$. The candidate x^* is a possible cowinner for P_X and plurality with respect to the addition of k new candidates if and only if*

$$\mathrm{ntop}(P_X, x^*) \geq \frac{1}{k} \cdot \sum_{x_i \in X} \max(0, \mathrm{ntop}(P_X, x_i) - \mathrm{ntop}(P_X, x^*)).$$

Proof. Suppose first that the inequality holds. We build the following $(X \cup Y)$-profile P' extending P_X:

1. for every candidate x_i such that $\mathrm{ntop}(P_X, x_i) > \mathrm{ntop}(P_X, x^*)$ we simply take $\mathrm{ntop}(P_X, x_i) - \mathrm{ntop}(P_X, x^*)$ arbitrary votes ranking

x_i on top and place one of the y_j's on top of the vote (and the other y_j's anywhere), subject to the condition that no y_j is placed on top of a vote more than $\mathrm{ntop}(P_X, x^*)$ times. (This is possible because the inequality is satisfied.)

2. in all other votes (those not considered at step 1), place all y_j's anywhere except on top.

We obtain a profile P' extending P_X. First, we have $\mathrm{ntop}(P', x^*) = \mathrm{ntop}(P_X, x^*)$, because in all the votes in P_X where x^* is on top, the new top candidate in the corresponding vote in P' is still x^* (cf. step 2), and all the votes in P_X where x^* was not on top obviously cannot have x^* on top in the corresponding vote in P'. Second, let $x_i \neq x^*$. If $\mathrm{ntop}(P_X, x_i) \leq \mathrm{ntop}(P_X, x^*)$ then $\mathrm{ntop}(P', x_i) = \mathrm{ntop}(P_X, x_i)$; and if $\mathrm{ntop}(P_X, x_i) > \mathrm{ntop}(P_X, x^*)$ then we have $\mathrm{ntop}(P', x_i) = \mathrm{ntop}(P_X, x_i) - (\mathrm{ntop}(P_X, x_i) - \mathrm{ntop}(P_X, x^*)) = \mathrm{ntop}(P_X, x^*)$. Therefore, x^* is a cowinner for plurality in P'.

Conversely, if the inequality is not satisfied, in order for x^* to become a cowinner in P', the other x_i's must lose globally an amount of $\sum_{x_i \in X} \max(0, \mathrm{ntop}(P_X, x_i) - \mathrm{ntop}(P_X, x^*))$ votes; and since $\sum_{x_i \in X} \max(0, \mathrm{ntop}(P_X, x_i) - \mathrm{ntop}(P_X, x^*)) > k \cdot \mathrm{ntop}(P_X, x^*)$, for at least one of the y_j's we will have $\mathrm{ntop}(P', y_j) > \mathrm{ntop}(P', x^*)$; therefore x^* cannot be a cowinner for plurality in P'. □

We do not need to pay much attention to the veto rule, since the characterization of possible cowinners is trivial. Indeed, by placing any of the new candidates below x^* in every vote of P_X where x^* is ranked at the bottom position, we obtain a vote P' where no one vetoes x^*, so any candidate is a possible cowinner.

As a corollary, computing possible cowinners for the rules of plurality (and veto) with respect to candidate addition can be computed in polynomial time (which we already knew, since possible cowinners for plurality and veto can be computed in polynomial time Betzler and Dorn, 2009).

4.2. K-approval, one new candidate

We start with the case where a single candidate is added. Recall that we denote by $S_K(x_j, P_X)$ the score of x_j for P_X and K-approval (*i.e.* the number of voters who rank x_j among their top K candidates); and by $n(P_X, K, x_j)$ the number of voters who rank x_j exactly in position K.

Proposition 2. *Let K be an positive integer, P_X be an n-voter profile on X, and $x^* \in X$. The candidate x^* is a possible cowinner for P_X and K-approval with respect to the addition of one new candidate if and only if the following two conditions hold:*

1. *for every $x_i \neq x^*$, if $S_K(x_i, P_X) > S_K(x^*, P_X)$ then $n(P_X, K, x_i) \geq S_K(x_i, P_X) - S_K(x^*, P_X)$.*
2. *$S_K(x^*, P_X) \geq \sum_{x_i \in X} \max(0, S_K(x_i, P_X) - S_K(x^*, P_X))$.*

Proof. Assume conditions (1) and (2) are satisfied. Then, we build the following $(X \cup \{y\})$-profile extending P_X:

(i) for every x_i such that $S_K(x_i, P_X) > S_K(x^*, P_X)$, we take $S_K(x_i, P_X) - S_K(x^*, P_X)$ arbitrary votes who rank x_i in position K in P_X and place y on top (condition (1) ensures that we can find enough such votes).

(ii) In all other votes (those not considered at step (i)), place y in the bottom position.

We obtain a profile P' extending P_X. First, we have $S_K(x^*, P') = S_K(x^*, P_X)$, because (a) all votes in P_X ranking x^* in position K are extended in such a way that y is placed in the bottom position, therefore x^* gets a point in each of these votes if and only if it got a point in P_X, and (b) in all the other votes (those where x^* is not ranked in position K in P_X), x^* certainly gets a point in P' if and only if they got a point in P_X. This holds both in the case where y was added at the top or the bottom of the vote. Second, for every x_i such that $S_K(x_i, P_X) > S_K(x^*, P_X)$, x_i loses exactly $S_K(x_i, P_X) - S_K(x^*, P_X)$ points when P_X is extended into P', therefore

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}	v_{19}
x^*	x_1	x_2	x_3	x_1	x_1	x_1	x_2	x_2	x_2	x_2	x_2	x_3	x_3	x_3	x_3	x_3	x_3	x_4
x_1	x^*	x^*	x^*	x_4	x_4	x_5	x_1	x_3	x_4	x_5	x_5	x_1	x_2	x_4	x_4	x_5	x_6	x_6
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Box I.

$S_K(x_i, P') = S_K(x_i, P_X) - S_K(x_i, P_X) + S_K(x^*, P_X) = S_K(x^*, P_X)$. Third, $S_K(y, P') = \sum_{x_i \in X} \max(0, S_K(x_i, P_X) - S_K(x^*, P_X)) \leq S_K(x^*, P_X)$ – because of (2) – hence $S_K(y, P') \leq S_K(x^*, P')$. Therefore, x^* is a cowinner for K-approval in P'.

Now, assume condition (1) is not satisfied, that is, there is an x_i such that $S_K(x_i, P_X) > S_K(x^*, P_X)$ and such that $n(P_X, K, x_i) < S_K(x_i, P_X) - S_K(x^*, P_X)$. There is no way of having x_i lose more than $S_K(x_i, P_X)$ points, therefore x^* will never catch up with x_i's advantage and is therefore not a possible cowinner. Finally, assume condition (2) is not satisfied, which means that we have $\sum_{x_i \in X} \max(0, S_K(x_i, P_X) - S_K(x^*, P_X)) > S_K(x^*, P_X)$. Then, in order for x^* to reach the score of x_i's we must add y in one of the top K positions in a number of votes exceeding $S_K(x^*, P_X)$, therefore $S_K(y, P') > S_K(x^*, P_X) \geq S_K(x^*, P')$, and therefore x^* is not a possible cowinner. □

Therefore, computing possible cowinners for K-approval with respect to the addition of *one* candidate can be done in polynomial time.

4.3. 2-approval, any (fixed) number of new candidates

For each profile P and each candidate x', we simply write $s(x', P)$ for the score of x' in P under r_2, that is, $s(x', P) = S_2(x', P)$, i.e. the number of times that x' is ranked within the top two positions in P.

Let $P_X = \langle V_1, \ldots, V_n \rangle$ be an initial profile and $Y = \{y_1, \ldots, y_k\}$ the set of new candidates. Let $x^* \in X$. We want to know whether x^* is a possible cowinner for 2-approval and P_X. Let us partition P_X into P_1, P_2 and P_3, where P_1 consists of the votes in which x^* is ranked in the top position, P_2 consists of the votes in which x^* is ranked in the second position and P_3 consists of the votes in which x^* is *not* ranked within the top two positions. Let P be an extension of P_X to $X \cup Y$. For each candidate $x' \in X$, we define the following three subsets of P:

- HP(P, x') is the set of votes in P where x' is ranked in the second position and neither x^* nor any new candidate is ranked in the top position (HP stands for "high priority").
- MP(P, x') is the set of votes in P where x^* or any new candidate is ranked in the top position and x' is ranked in the second position (MP stands for "medium priority").
- LP(P, x') is the set of votes in P where x' is ranked in the top position and some $x'' \in X \setminus \{x^*\}$ is ranked in the second position (LP stands for "low priority").

These definitions also apply to P_X; our definitions then simplify into: HP(P_X, x') is the set of votes in P_X where x' is ranked second and x^* is not ranked first; MP(P_X, x') is the set of votes in P_X where x^* is ranked first and x' is ranked second; LP(P_X, x') is the set of votes in P_X where x' is ranked first and x^* is not ranked second. These definitions are summarized in Fig. 1. Finally, for $x \in X \cup Y$, let $\Delta(P, x) = S_2(x, P) - S_2(x^*, P)$.

Let us compute these sets on a concrete example, which will be reused throughout the section.

	top candidate belongs to	2nd candidate belongs to
HP(P, x')	$X \setminus \{x^*\}$	$\{x'\}$
MP(P, x')	$Y \cup \{x^*\}$	$\{x'\}$
LP(P, x')	$\{x'\}$	$X \setminus \{x^*\}$

Fig. 1. A vote $V \in P$ belongs respectively to the sets HP(.), MP(.), LP(.) if its top two candidates belong to the respective sets.

Example 1. Let $X = \{x^*, x_1, \ldots, x_6\}$ and consider the following profile P_X consisting of 19 votes (we only mention the first two candidates in each vote) see Box I:

We have $P_1 = \{v_1\}$, $P_2 = \{v_2, v_3, v_4\}$ and $P_3 = \{v_5, \ldots, v_{19}\}$. This is summarized together with the priority classification in the following table:

	HP	MP	LP	$\Delta(P_X, x_i)$
x_1	v_8, v_{13}	v_1	v_5, v_6, v_7	3
x_2	v_{14}		$v_8, v_9, v_{10}, v_{11}, v_{12}$	3
x_3	v_9		$v_{13}, v_{14}, v_{15},$ v_{16}, v_{17}, v_{18}	4
x_4	$v_5, v_6, v_{10},$ v_{15}, v_{16}		v_{19}	2
x_5	$v_7, v_{11}, v_{12}, v_{17}$			0
x_6	v_{18}, v_{19}			−2

If P^* is an extension of P_X to $X \cup Y$ then we write $P^* = \langle V_1^*, \ldots, V_n^* \rangle$, where V_i^* is the vote over $X \cup Y$ extending V_i. We now establish a useful property of the extensions of P_X for which x^* is a cowinner. Without loss of generality, we assume that in every vote V_i^*, every new candidate y_j is ranked either in the first two positions, or below all candidates of X.

Proposition 3. *If there exists an extension P of P_X such that $x^* \in r_2(P)$, then there exists an extension P^* of P_X such that $x^* \in r_2(P^*)$, and satisfying the following conditions:*

1. *For each $V_i \in P_X$, if x^* is ranked within the top two positions in V_i, then x^* is also ranked within the top two positions in V_i^*.*
2. *For each $V_i^* \in P^*$, if the top candidate of V_i^* is not in Y then the second-ranked candidate of V_i^* is not in Y either.*
3. *For each $x' \in X \setminus \{x^*\}$ and each $V_i \in$ MP(P_X, x') $\cup$ LP(P_X, x'), if x' is not ranked within the top two positions in V_i^*, then for each $V_j \in$ HP(P_X, x'), x' is not ranked within the top two positions in V_j^*.*

Proof. We consider in turn the different conditions:

1. This is because if there exists $V' \in P$ such that x^* is not in the top two positions whereas x^* is in the top two positions in its original vote $V \in P_X$, then we can simply move all of candidates in Y ranked higher than x^* to the bottom positions. Let V^* denote the vote obtained this way. By replacing V' with V^*, we increase the score of x^* by 1, and the score of each other candidate by no more than 1, which means that x^* is still a cowinner.
2. If there exists $V' \in P$ such that $x' \in X$ is ranked in the top position and $y \in Y$ is ranked in the second position, then we simply obtain V^* by switching y and x'.

3. The condition states that for each candidate x', whenever we want to reduce its score, we should first try to reduce it by putting a new candidate $y \in Y$ on top of some vote in $V \in$ HP(P_X, x'). This is because by putting y on top of some vote in HP(P_X, x'), we may use only one extra candidate $y' \in Y$ to reduce by one unit the score of the candidate ranked at the top position of V. Formally, suppose there exist $V_1 \in$ HP(P_X, x') and $V_2 \in$ MP$(P_X, x') \cup$ LP(P_X, x') such that x' is within the top two positions of V_1' (the extension of V_1) but not within the top two positions of V_2' (the extension of V_2). Let $y \in Y$ be any candidate ranked within the top two positions of V_2'. Let V_2^* denote the vote obtained from V_2' by moving y to the bottom, and let V_1^* denote the vote obtained from V_1' by moving y to the top position. Next, we replace V_1' and V_2' by V_1^* and V_2^*, respectively. It follows that the score of each candidate does not change, which means that x^* is still a cowinner. We repeat this procedure until statement (3) is satisfied for every $x' \in X \setminus \{x^*\}$. Since after each iteration there is at least one additional vote that will never be modified again, this procedure ends in $O(|P_X|)$ times. $\quad\square$

Proposition 3 simply tells us that when looking for an extension that makes x^* a cowinner, it suffices to restrict our attention to the extensions that satisfy conditions (1)–(3). Moreover, using (1) of Proposition 3, we deduce that $s(x^*, P^*) = s(x^*, P_X)$. Hence, for votes $V \in P_2$ (the votes in which x^* is ranked in the second position), we can assume that the new candidates of Y are put in bottom positions in P^*.

Define $X^\bullet$ as the set of all candidates in X such that $\Delta(P_X, x_i) > 0$. Our objective is to reduce all score differences to 0 for $x \in X^\bullet$, while keeping the score differences of each new candidate non-positive. (We do not have to care about the candidates in $X \setminus X^\bullet$.)

The intuition underlying our algorithm is that when trying to reduce $\Delta(P, x_i)$ on the current profile P, we first try to use the votes in HP(P_X, x_i), then the votes in MP(P_X, x_i), and finally the votes in LP(P_X, x_i). This is because putting some candidates from Y in the top positions in the votes of HP(P_X, x_i) not only reduces $\Delta(P, x_i)$ by one unit, but also creates an opportunity to "pay" one extra candidate from Y to reduce $\Delta(P, x_j)$ by one unit, where x_j is the candidate ranked on top of this vote. For the votes in MP(P_X, x_i), we can only reduce $\Delta(P_X, x_i)$ by one unit without any other benefit. For the votes in LP(P_X, x_i) we will have to use two candidates from Y to bring down $\Delta(P, x_i)$ by one unit; however, if we already put some $y \in Y$ in the top position in order to reduce $\Delta(P, x_j)$, where x_j is the candidate ranked in the second position in the original vote, then we only need to pay one extra candidate in Y to reduce $\Delta(P, x_i)$ by one unit. Therefore, the major issue consists of finding the most efficient way to choose the votes in HP(P_X, x_i) to reduce $\Delta(P, x_i)$, when $\Delta(P, x_i) \leq |$HP$(P_X, x_i)|$. We will solve this problem by reducing it to a max-flow problem.

The algorithm is composed of a main function *CheckCowinner(.)* which comes together with two sub-functions *AddNewAlternativeOnTop(.)* and *BuildMaxFlowGraph(.)* that we detail first.

Algorithm 1: AddNewAlternativeOnTop(P, V, Y)

1 $y_i \leftarrow \arg\min_j \{\Delta(P, y_j) : y_j \in Y\}$ // take lowest index i when tie-breaking
2 add y_i on top of V and update P
3 return P

The procedure *AddNewAlternativeOnTop* simply picks new candidates to be put on top of votes, and updates subsequently the profile. Note that in this procedure, candidates from Y to be added on top of the votes are those with the lowest score (or the lowest index, in case of ties). This results in choosing new candidates in a cyclic order $y_1 \rightarrow y_2 \cdots \rightarrow y_{|Y|} \rightarrow y_1 \cdots$.

As for the function *BuildMaxFlowGraph*(P, x^*, X_1, X_2), it builds the weighted directed graph $G = \langle W, E \rangle$ defined as follows:
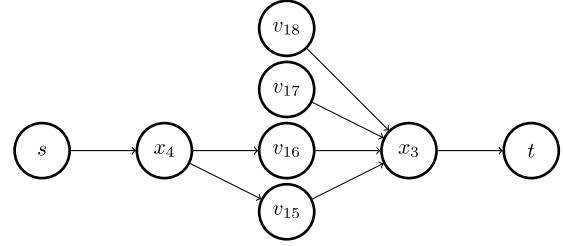

Fig. 2. The flow graph returned by *BuildMaxFlowGraph*$(\widetilde{P}, x^*, \{x_4\}, \{x_3\})$.

- $W = \{s, t\} \cup X_1 \cup X_2 \cup \bigcup_{x_i \in X_2}$ LP(P, x_i);
- E contains the following weighted edges:
  - for each $x \in X_1$, an edge (s, x) with weight $\Delta(P, x)$;
  - for each $x \in X_2$ and each $V \in$ LP(P, x): an edge (V, x) with weight 1; plus, if the candidate x' in second position in V is in X_1, an edge (x', V) with weight 1;
  - for each $x \in X_2$, an edge (x, t) with weight $\Delta(P, x)$.

We refer the reader to Fig. 2 for an illustration. (Once this graph is constructed, any standard function to compute a flow ϕ of maximal value can of course be used.) We are now in a position to detail the main function *CheckCowinner(.)*.

Algorithm 2: CheckCowinner(P_X, x^*, Y)

1 $P \leftarrow P_X$
2 $T \leftarrow 0$ // number of calls AddNewAlternativeOnTop
3 $X_1 \leftarrow \{x_i \in X^\bullet : |HP(P_X, x_i)| > \Delta(P_X, x_i)\}$
4 $X_2 \leftarrow \{x_i \in X^\bullet : |HP(P_X, x_i)| \leq \Delta(P_X, x_i)\}$
5 $REM \leftarrow \emptyset$
6 **for** $x_i \in X_2$ **do**
7   **for** $V \in$ HP(P, x_i) **do**
8 $P \leftarrow$ AddNewAlternativeOnTop(P, V, Y)
9 $T \leftarrow T + 1$
10 **for** $x_i \in X_2$ **do**
11   **for** $V \in$ MP(P, x_i) **do**
12   **if** $\Delta(P, x_i) > 0$ **then**
13 $P \leftarrow$ AddNewAlternativeOnTop(P, V, Y)
14 $T \leftarrow T + 1$
15   **else**
16 $REM \leftarrow REM \cup \{x_i\}$
17 $X_2 \leftarrow X_2 \setminus REM$
18 **if** $\exists y \in Y$ such that $\Delta(P, y) > 0$ **then**
19   return *false*
20 $G \leftarrow$ BuildMaxFlowGraph(P, x^*, X_1, X_2)
21 $\phi \leftarrow$ ComputeMaxFlow(G, s, t)
22 **if**
$$\phi \geq \sum_{i \leq m-1} \Delta(P, x_i) + \sum_{x_i \in X_2} \Delta(P, x_i) - (|Y| \cdot s(x^*, P_X) - T)$$
then
23   return *true*
24 return *false*

Proposition 4. *Given a profile P_X on X, a candidate $x^* \in X$ and a set of new candidates Y, a call to algorithm CheckCowinner(P_X, x^*, Y) returns in polynomial time the answer* true *if and only if there exists an extension of P_X in which x^* is a cowinner.*

Proof. Algorithm 2 starts by partitioning $X^\bullet$ into X_1 and X_2: an alternative $x \in X^\bullet$ is in X_1 if $|$HP$(P_X, x)| > \Delta(P_X, x)$ and in X_2 if $|$HP$(P_X, x)| \leq \Delta(P_X, x)$.

Let $x \in X_2$. Then by item (3) of Proposition 3, for each vote in $V \in$ HP(P, x), we can safely put one candidate from Y in the

top position of V; this is done in the first phase of Algorithm 2, lines 6–9. Note that after adding a new candidate on top of a vote $V \in HP(P, x)$ and after updating P, the modified vote will no longer belong to $HP(P, x)$. Instead, it will now belong to $MP(P, x')$ for some other candidate x'.

When Phase 1 is over, the score of $x \in X_2$ may still need to be lowered down, which can be done next by using votes from $MP(P_X, x)$. This is what Phase 2 does, from line 10 to line 16. There are three possibilities:

1. $|HP(P_X, x)| = \Delta(P_X, x)$. In this case, the votes in $HP(P_X, x)$ are sufficient to make x^* catch up x: after Phase 1, we have $\Delta(P_X, x) = 0$ and Phase 2 is void; we are done with x.
2. $|HP(P_X, x)| < \Delta(P_X, x)$ and $|HP(P_X, x)| + |MP(P_X, x)| \geq \Delta(P_X, x)$: in this case, to make x^* catch up x, it is enough to take $\Delta(P_X, x) - |HP(P_X, x)|$ arbitrary votes in $MP(P_X, x)$ and add one new candidate on top of them; this is what Phase 2 does, and after that we are done with x.
3. $|HP(P_X, x)| + |MP(P_X, x)| < \Delta(P_X, x)$: in this case, because of Proposition 3, we know that it is safe to add one new candidate on top of *all* votes of $MP(P_X, x)$; this is what Phase 2 does; after that, we still need to lower down the score of x, which will require to add new candidates on top of votes of $LP(P_X, x)$.

If at this point a newly added candidate has a score higher than x^*, then x^* cannot win, and we can stop the program (line 19).

For readability, let us denote by $\widetilde{P}$ the profile obtained after Phases 1 and 2. For each $x \in X_2$ satisfying condition 3, the only way to reduce $\Delta(\widetilde{P}, x)$ is to put two candidates of Y within the top two positions in a vote of $LP(\widetilde{P}, x)$, because in Phases 1 and 2 we have used up all the votes in $HP(P, x)$ and $MP(P, x)$. Now, reducing $\Delta(\widetilde{P}, x)$ by one unit will cost us two candidates in Y, but meanwhile, $\Delta(\widetilde{P}, x')$ is also reduced by one unit, where x' is the candidate ranked in the second position in V. We must have $x' \in X_1$. We note that $\bigcup_{x \in X_2} LP(P_X, x) \subseteq \bigcup_{x' \in X_1} HP(P_X, x')$. Choosing optimally the votes in $LP(P_X, x)$ for each $x \in X_2$ can be done by solving an integral max-flow instance which is build by algorithm *BuildMaxFlowGraph* (note that in case where either X_1 or X_2 is empty, we just assume that the flow has a null value).

Let us show that x^* is a possible cowinner if and only if the value of the flow from s to t is at least $\sum_{i \leq m-1} \Delta(\widetilde{P}, x_i) + \sum_{x_i \in X_2} \Delta(\widetilde{P}, x_i) - (|Y| \cdot s(x^*, P_X) - T)$. Observe that the flow does not necessarily bring all $\Delta(\widetilde{P}, x_i)$ to 0, therefore we sometimes need a postprocessing consisting of adding further new candidates on top of some votes (see steps 2 and 3 below).

Suppose the above max-flow instance has a solution whose value is at least

$$\sum_{i \leq m-1} \Delta(\widetilde{P}, x_i) + \sum_{x_i \in X_2} \Delta(\widetilde{P}, x_i) - (|Y| \cdot s(x^*, P_X) - T).$$

We next show how to solve our cowinner problem from the solution to this flow problem. Because the instance is integral, there must exist an integral solution. We arbitrarily choose one integral solution ϕ (as returned by *ComputeMaxFlox*), which assigns to each edge (x_i, x_j) an integer $\phi(x_i, x_j)$ which represents the value of the flow on this edge. Here, we give a procedure which produces an extension P of P_X where x^* is a cowinner:

1. For each $x_i \in X_2$ and each $V \in LP(\widetilde{P}, x_i)$, if there is a flow from x_i to x_j via V, then we obtain V^* from V by putting two candidates from Y in the top positions (that is, both $\Delta(\widetilde{P}, x_i)$ and $\Delta(\widetilde{P}, x_j)$ are reduced by 1, which comes at the cost of using candidates in Y twice). It is possible since $|Y| \geq 2$.
2. For each $x_i \in X_2$, if $\phi(x_i, t) < \Delta(\widetilde{P}, x_i)$, then we arbitrarily choose $\Delta(\widetilde{P}, x_i) - \phi(x_i, t)$ votes $V \in LP(\widetilde{P}, x_i)$ among those which haven't been selected in the previous step, and obtain V^* by putting two candidates from Y in the top two positions (again, we will specify how to choose the two candidates from Y later). It is possible since $|Y| \geq 2$.

3. For each $x_j \in X_1$, if $\phi(s, x_j) < \Delta(\widetilde{P}, x_i)$, then we arbitrarily choose $\Delta(\widetilde{P}, x_i) - \phi(s, x_j)$ votes $V \in HP(\widetilde{P}, x_j)$ such that V^* is not defined above (in step 1 or step 2), and we obtain V^* by putting exactly one candidate from Y in the top position of V. This is possible because, by construction, $|HP(\widetilde{P}, x_j)| = |HP(P, x_j)| \geq \Delta(P, x_i) \geq \Delta(\widetilde{P}, x_i)$ for $x_j \in X_1$.
4. For each V^*, if a candidate $y \in Y$ is not selected for one of the first two positions, then it is ranked at the bottom position.

In the above procedure (similarly to what is done in Algorithm 1), priority is given to candidates from Y with the lowest score (or the lowest index, in case of ties) when it comes to choose those to be added on top of the votes.

Let us now determine the number of times that new candidates from Y are inserted on top of the votes. Recall that until line 20 of the algorithm, we have used the candidates from Y exactly T times. Now consider the four-step procedure described above. Observe that to reduce by one unit the score deficit with respect to one candidate, steps 1 and 3 require one occurrence of a candidate of Y (step 1 uses two occurrences but reduces the score deficit with respect to two candidates), while step 2 requires *two* occurrences. Thus, for each $i \leq m - 1$, we have to use $\Delta(\widetilde{P}, x_i)$ times the candidates from Y, plus the additional occurrences required in step 2. More precisely, step 2 requires, for each $x_i \in X_2$, $\Delta(\widetilde{P}, x_i) - \phi(x_i, t)$ additional occurrences of new candidates in the completed votes. Therefore, the total number of times that the candidates of Y are ranked either in first or second position (denoted s_Y for readability), is such that:

$$s_Y \leq \sum_{i \leq m-1} \Delta(\widetilde{P}, x_i) + \left(\sum_{x_i \in X_2} \Delta(\widetilde{P}, x_i) - \sum_{x_i \in X_2} \phi(x_i, t) \right) \tag{1}$$

$$= \sum_{i \leq m-1} \Delta(\widetilde{P}, x_i) + \left(\sum_{x_i \in X_2} \Delta(\widetilde{P}, x_i) - \phi \right) \tag{2}$$

But we also have:

$$\phi \geq \sum_{i \leq m-1} \Delta(\widetilde{P}, x_i) + \sum_{x_i \in X_2} \Delta(\widetilde{P}, x_i) - (|Y| \cdot s(x^*, P_X) - T). \tag{3}$$

By combining (2) and (3), we thus get:

$$s_Y \leq |Y| \cdot s(x^*, P_X) - T$$
$$\leq |Y| \cdot s(x^*, P_X).$$

That is, our algorithm will put candidates from Y in the top two positions in the extension no more than $|Y| \cdot s(x^*, P_X)$ times. Because the addition of new candidates is done in a cyclic order, each new candidate will eventually appear at most $s(x^*, P_X)$ in the top two positions of the votes. Thus, the score of these new candidates will not exceed that of x^*. It follows that x^* is a cowinner in P^*, since for all other candidates $x_i \in X$, we have $\Delta(P^*, x_i) \leq 0$.

Next, we show that if x^* is a possible cowinner, then the value of a max-flow is at least

$$\sum_{i \leq m-1} \Delta(\widetilde{P}, x_i) + \sum_{x_i \in X_2} \Delta(\widetilde{P}, x_i) - (|Y| \cdot s(x^*, P_X) - T).$$

Due to Proposition 3, each extension profile P^* of P_X where x^* becomes a cowinner to the problem instance can be converted to a profile $\widetilde{P}$ as in the steps before line 20 in the algorithm. Now, for each $x_i \in X_2$, let l_i denote the number of votes $V \in LP(\widetilde{P}, x_i)$ such that in its extension V^*, the top two positions are the candidates of Y. We must have that $l_i \geq \Delta(\widetilde{P}, x_i)$. For every $x_i \in X_2$, we arbitrarily choose $l_i - \Delta(\widetilde{P}, x_i)$ such votes, and move the first ranked candidate to the bottom position. For each $x_j \in X_1$, let l_j denote the number of votes $V \in HP(\widetilde{P}, x_j) \cup MP(\widetilde{P}, x_j)$ such that in its extension V^*, a candidate from Y is ranked in the top position. We must have that

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}	v_{19}
	•	x_1	x_2	x_3	x_1	x_1	x_1	•	•	x_2	x_2	x_2	•	•	x_3	x_3	x_3	x_3	x_4
	x^*	x^*	x^*	x^*	x_4	x_4	x_5	•	•	x_4	x_5	x_5	•	•	x_4	x_4	x_5	x_6	x_6
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Box II.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}	v_{19}
	$\mathbf{y_1}$	x_1	x_2	x_3	$\mathbf{y_2}$	x_1	x_1	$\mathbf{y_1}$	$\mathbf{y_2}$	x_2	x_2	x_2	$\mathbf{y_1}$	$\mathbf{y_2}$	x_3	$\mathbf{y_1}$	x_3	x_3	x_4
	x^*	x^*	x^*	x^*	x_1	x_4	x_5	$\mathbf{y_3}$	$\mathbf{y_3}$	x_4	x_5	x_5	$\mathbf{y_3}$	$\mathbf{y_3}$	x_4	$\mathbf{y_2}$	x_5	x_6	x_6
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Box III.

$l_j \geq \Delta(\widetilde{P}, x_j)$. For every $x_j \in X_1$, we arbitrarily choose $l_j - \Delta(\widetilde{P}, x_j)$ such votes, and move the first ranked candidate to the bottom position.

Now, let there be a flow from $x_j \in X_1$ to $x_i \in X_2$ via V if $V \in \mathrm{LP}(\widetilde{P}, x_i)$ and the top two positions in V^* are both in Y. This defines a flow whose value is at least $\sum_{x_i \in X_2} \Delta(\widetilde{P}, x_i) - \sum_{x_j \in X_1}(l_j - \Delta(\widetilde{P}, x_j))$. Because the score of each candidate of Y is no more than $s(x^*, P_X)$, we know that $|Y| \cdot s(x^*, P_X) - T \geq \sum_{i \leq m-1} l_i$. Actually, $|Y| \cdot s(x^*, P_X)$ is the maximum score that the whole set of new candidates of Y can reach in such a way that x^* is a cowinner. In the partial profile $\widetilde{P}$ (line 20 of Algorithm *CheckCowinner* (P_X, x^*, Y)), the global score of Y is T. Finally, since $\sum_{i \leq m-1} l_i + T$ corresponds to the global score that Y has in profile P^* (where x^* becomes a cowinner), we get $|Y| \cdot s(x^*, P_X) \geq \sum_{i \leq m-1} l_i + T$.

Hence, $|Y|s(x^*, P_X) - T \geq \sum_{i \leq m-1} l_i \geq \sum_{x_i \in X_2} \Delta(\widetilde{P}, x_i) + \sum_{x_j \in X_1} l_j$, or equivalently, $-\sum_{x_j \in X_1} l_j \geq \sum_{x_i \in X_2} \Delta(\widetilde{P}, x_i) - (|Y| \cdot s(x^*, P_X) - T)$. Hence, we get:

$$\phi \geq \sum_{x_i \in X_2} \Delta(\widetilde{P}, x_i) - \sum_{x_j \in X_1}(l_j - \Delta(\widetilde{P}, x_j))$$
$$= \sum_{i \leq m-1} \Delta(\widetilde{P}, x_i) - \sum_{x_j \in X_1} l_j$$
$$\geq \sum_{i \leq m-1} \Delta(\widetilde{P}, x_i) + \sum_{x_i \in X_2} \Delta(\widetilde{P}, x_i) - (|Y|s(x^*, P_X) - T).$$

Thus, we have shown that x^* is a possible cowinner if and only if the value of the flow from s to t is at least $\sum_{i \leq m-1} \Delta(\widetilde{P}, x_i) + \sum_{x_i \in X_2} \Delta(\widetilde{P}, x_i) - (|Y| \cdot s(x^*, P_X) - T)$. This concludes the proof. □

Corollary 1. *Deciding whether x^* is a possible cowinner for 2-approval with respect to the addition of new candidates is in* P.

To better understand Algorithm 1, we will now run it step by step on the example introduced previously.

Example 2. Consider the profile described in Example 1. We assume the number of new candidates is $k = 3$. First, the initial scores of the candidates are $s(x^*, P_X) = 4, s(x_1, P_X) = 7, s(x_2, P_X) = 7, s(x_3, P_X) = 8, s(x_4, P_X) = 6, s(x_5, P_X) = 4$ and $s(x_6, P_X) = 2$. The candidates whose score exceeds that of x^* are x_1, x_2, x_3 and x_4, with the score differences $\Delta(P, x_1) = 3, \Delta(P, x_2) = 3, \Delta(P, x_3) = 4$ and $\Delta(P, x_4) = 2$. At the first phase, we check if there are candidates x_i for which $|\mathrm{HP}(P_X, x_i)| \leq \Delta(P_X, x_i)$. This is the case for x_1, x_2 and x_3, thus we put one new candidate on top of v_8, v_9, v_{13} and v_{14}. The updated table is as follows:

	HP	MP	LP	$\Delta(P, x_i)$
x_1		v_1	v_5, v_6, v_7	1
x_2		v'_8, v'_9	v_{10}, v_{11}, v_{12}	2
x_3		v'_{13}, v'_{14}	$v_{15}, v_{16}, v_{17}, v_{18}$	3
x_4	$v_5, v_6, v_{10}, v_{15}, v_{16}$		v_{19}	2

Here, v'_i refers to the vote v_i to which new candidates have been added.

At the second phase, $X_2 = \{x_1, x_2, x_3\}$ and $X_1 = \{x_4\}$ (we do not worry about x_5 and x_6 for which nothing special has to be done). We put one new candidate on top of v_1, v'_8, v'_9, v'_{13} and v'_{14}, and we are done with x_1 and x_2 (since $\Delta(P, x_1) = 0$ and $\Delta(P, x_2) = 0$). The profile is now $\widetilde{P}$ and the updated table is

	HP	MP	LP	$\Delta(\widetilde{P}, x_i)$
x_3			$v_{15}, v_{16}, v_{17}, v_{18}$	1
x_4	$v_5, v_6, v_{10}, v_{15}, v_{16}$		v_{19}	2

So far we have used the new candidates 9 times, and $s(x^*, \widetilde{P}) = 4$, therefore if we have less than three new candidates we stop (x^* is not a possible cowinner) otherwise we continue. Now the situation is as follows and we have to solve the corresponding maxflow problem (we omit the value of edges when it equals 1) see Box II.

The maximum flow has value 1 and is obtained for instance by having a flow 1 for instance through the edges $s \rightarrow x_4, x_4 \rightarrow v_{16}, v_{16} \rightarrow x_3, x_3 \rightarrow t$ (going through v_{15} is an equally good option). Therefore we place two new candidates on top of v_{16}, which has the effect of making the score of x_3 and x_4 decrease by one unit each. We still have to make the score of x_4 decrease by one unit, and for this we must place one new candidates on top of any of the votes v_5, v_6, v_{10}, v_{15} (say v_5). In total we will have used the new candidates 12 times, therefore, c is a possible cowinner if and only if the number of new candidates is at least 3. A possible extension (with 3 new candidates) is given in Box III:

4.4. K-approval, two new candidates

Let $X = \{x^*\} \cup \{x_1, \ldots, x_p\}$ be the set of (initial) candidates, x^* being the candidate that we want to make a cowinner, $Y = \{y_1, y_2\}$ the two new candidates, and $P_X = \langle V_1, \ldots, V_n \rangle$ the initial profile, where each V_i is a sequence of K candidates in X. We first introduce the following notation:

- For each $x \in X$, $U^{P_X}(x)$ is the number of votes in P_X whose candidates ranked $K - 1$ and K are respectively x and x^*, and $T^{P_X}(x) = S_{K-2}(x, P_X) + U^{P_X}(x)$. (Recall that $S_{K-2}(x, P_X)$ is the number of voters in P_X who rank x in the first $K - 2$ positions.)

We establish the following lemma.

Lemma 1. *For each $x \in X$, there exists a completion Q of P_X by adding two candidates such that $S_K(x, Q) \leq S_K(x^*, Q)$ if and only if $T^{P_X}(x) \leq S_K(x^*, P_X)$.*

Proof. Assume $T^{P_X}(x) > S_K(x^*, P_X)$, and let Q be a completion of P_X by adding two candidates in which x^* is a cowinner. Let us partition P_X into P_1, P_2 and P_3, as follows: every vote in P_1 is such that the candidates ranked $K - 1$ and K are respectively x and x^*; P_2 contains all votes ranking x in the first $K - 2$ positions; and P_3 contains all other votes in P_X. Let Q_1, Q_2 and Q_3 be the corresponding votes in Q, and let α be the number of votes in Q_1 where the two new candidates have been placed in the first K positions, thus eliminating both x and x^* from the K first positions; clearly, we have $S_K(x, Q_1) = S_K(x, P_1) - \alpha = U^{P_X}(x) - \alpha$ and $S_K(x^*, Q_1) \leq S_K(x^*, P_1) - \alpha$ (the inequality can be strict, in case there are some votes in Q_1 where only one new candidate was placed in the first K positions). Now, regardless of the position of the two new candidates, we have $S_K(x, Q_2) = S_{K-2}(x, P_2)$. We get $S_K(x, Q) = S_K(x, Q_1) + S_K(x, Q_2) + S_K(x, Q_3) \geq U^{P_X}(x) - \alpha + S_{K-2}(x, P_2) = T^{P_X}(x) - \alpha$, whereas $S_K(x^*, Q) \leq S_K(x^*, P_X) - \alpha$. The initial assumption $T^{P_X}(x) > S_K(x^*, P_X)$ implies $T^{P_X}(x) - \alpha > S_K(x^*, P_X) - \alpha$, therefore $S_K(x, Q) > S_K(x^*, Q)$.

Conversely, assume $T^{P_X}(x) \leq S_K(x^*, P_X)$, and let us build Q as follows: we introduce one new candidate on top of each vote of P_X that ranks x in position K, and two new candidates on top of each vote of P_X that ranks x in position $K - 1$ and $x' \neq x^*$ in position K. It is easy to check that $S_K(x^*, Q) = S_K(x^*, P_X)$. Now, the only votes of Q where x remains among the first K position are those of Q_1 and of Q_2, therefore $S_K(x, Q) = T^{P_X}(x) \leq S_K(x^*, P_X) = S_K(x^*, Q)$. □

Proposition 5. *Deciding whether x^* is a possible cowinner for K-approval with respect to the addition of 2 new candidates is in P.*

Proof. A consequence of Lemma 1 is that if $T^{P_X}(x) > S_K(x^*, P_X)$ for some x, then x^* cannot be a possible cowinner in P_X under 2-approval with 2 new candidates; and obviously, checking whether $T^{P_X}(x) > S_K(x^*, P_X)$ holds for some x can be done in polynomial time. Therefore, *from now on, we assume that $T^{P_X}(x) \leq S_K(x^*, P_X)$ holds for every $x \in X$*—assuming this will not change the complexity of the problem.

We now give a polynomial reduction from the possible cowinner problem for K-approval and 2 new candidates to the possible cowinner problem for 2-approval and 2 new candidates, which we already know to be polynomial. Let $\langle N, X, P_X, 2 \rangle$ be an instance of the possible cowinner problem for K-approval with respect to the addition of 2 new candidates. We build an instance $\langle N', X', R_{X'}, 2 \rangle$ of the possible cowinner problem for 2-approval with respect to the addition of 2 new candidates in the following way. The profile P_X is translated into the following profile $R = R_{X'}$:

- the set of candidates is $X' = X \cup \{z_j, 1 \leq j \leq \sum_{x \in X \setminus \{x^*\}} S_{K-2}(x, P_X)\} \cup \{z'_j, 1 \leq j \leq S_{K-2}(x^*, P_X)\}$, where all z_j and z'_j are fresh candidates;
- for every vote V_i in P_X, we have in R a vote W_i including the candidates ranked in positions $K - 1$ and K of V_i, and then the remaining candidates in any order. We denote by R_1 be the resulting set of votes;
- for every $x \in X \setminus \{x^*\}$, we have $S_{K-2}(x, P_X)$ votes xz_j, and then the remaining candidates in any order. We denote by R_2 the resulting set of votes;
- similarly, we have $S_{K-2}(x^*, P_X)$ votes z'_jx^*, and then the remaining candidates in any order. We denote by R_3 the resulting set of votes.

We note that if $x \in X$ then $S_K(x, P_X) = S_2(x, R)$, and for every fresh candidate z, $S_2(z, R) = 1$. Without loss of generality we assume $S_K(x^*, P_X) \geq 1$ (otherwise we know for sure that x^* cannot be a possible cowinner).

We decompose the rest of the proof into two lemmas.

Lemma 2. *If x^* is a possible cowinner for K-approval with 2 new candidates in P_X, then it is a possible cowinner for 2-approval with 2 new candidates in R.*

Proof. Suppose that x^* is a possible cowinner for K-approval with 2 new candidates $Y = \{y_1, y_2\}$ in P_X and let $P' = \langle V'_1, \ldots, V'_n \rangle$ be an extension of P_X with two new candidates where x^* is a cowinner. Let us use these two new candidates in the same way in R: every time a new candidate is used for being placed on top of V_i, it is also used for being placed on top of W_i. Let R' be the resulting profile. All candidates in X have the same scores in P_X and in R, they also will have the same scores in P' and R'; as for the fresh candidates z_j, z'_j, $S_2(z_j, R') = S_2(z'_j, R') = 1 \leq S_2(x^*, R')$; therefore, x^* is a cowinner in R' and a possible cowinner for 2-approval with 2 new candidates in R. □

Lemma 3. *If x^* is a possible cowinner for K-approval with 2 new candidates in R, then it is a possible cowinner for 2-approval in P_X.*

Proof. Suppose that x^* is a possible cowinner for 2-approval with 2 new candidates $Y = \{y_1, y_2\}$ in R, and let R' be a completion of R where x^* is a cowinner for 2-approval. Let us write $R' = R'_1 \cup R'_2 \cup R'_3$, where R'_1 (resp. R'_2, R'_3) consists of the completions of the votes in R_1 (resp. R_2, R_3). By a slight abuse of language we denote by R_1, R'_1 etc. only the part of the votes in R_1, R'_1 etc. consisting of the top two candidates only.

We first claim that we can assume without loss of generality that $R'_2 = R_2$ and $R'_3 = R_3$ that is, the only votes in R' where some new candidates have been placed on one of the top two positions are in R'_1. Suppose this is not the case; then we are in one of the following four situations: (1) there is a vote in R'_2 of the form y_jx_i, where $y_j \in Y$ and $x_i \in X$, or (2) there is a vote in R'_2 of the form y_1y_2 or y_2y_1, or (3) there is a vote in R'_3 of the form $y_iz'_j$ or (4) there is a vote in R'_3 of the form y_1y_2 or y_2y_1. Consider first cases (1), (3) and (4). Take one of these votes in R'_2 (case (1)) or in R'_3 (cases (3) or (4)) and replace it by the original vote xz_j in R_2 (case 1) or in z'_jx^* in R_3 (cases (3) or (4)). Let R'' be the profile obtained. We have $S_2(x^*, R'') \geq S_2(x^*, R') \geq 1$, for every $x_i \in X$, $S_2(x_i, R'') = S_2(x_i, R')$, for every z_j, $S_2(z_j, R'') \leq 1$, and for every z'_j, $S_2(z'_j, R'') \leq 1$.

Therefore, when transforming R' into R'', the score of x^* does not decrease whereas the score of all other candidates does not increase; because x^* is a cowinner in R', it is still a cowinner in R''. Lastly, R'' is also an extension of R.

By induction, if we perform this operation for each occurrence of cases (1), (3) or (4), we end up with a profile R'', which is an extension of R for which situations (1), (3) and (4) do not occur, and such that x^* is a cowinner for 2-approval in R''. Let $R'' = R''_1 \cup R''_2 \cup R''_3 = R''_1 \cup R'_2 \cup R_3$.

Now, consider case (2). Let x_iz_j be one of the votes in R_2 corresponding to a vote y_1y_2 (or y_2y_1) in R''_2. Apply the following procedure in this order:

1. Assume that x_i does not appear in R''_1 except in votes of the form x_ix^*, and let R''' be the profile obtained from R'' by replacing the vote y_1y_2 in R''_2 by the original vote x_iz_j in R_2. Then $S_2(x_i, R''') = S_2(x_i, R''_1) + S_2(x_i, R''_2) \leq S_2(x_i, R''_1) + S_2(x_i, R_2)$. Now, $S_2(x_i, R''') \leq S_2(x_i, R'') = U^{P_X}(x_i)$ and $S_2(x_i, R_2) = S_{K-2}(x_i, P_X)$, therefore $S_2(x_i, R''') \leq U^{P_X}(x_i) + S_{K-2}(x_i, P_X) = T^{P_X}(x_i) \leq S_K(x^*, P_X) = S_2(x^*, R) = S_2(x^*, R''')$. Therefore, x^* is also a cowinner in R'''.

2. Now, assume that x_i appears in at least one vote of R_1'' of the form $x^* x_i$, $x_i x_j$ or $x_j x_i$. If this is a vote of the form $x_i x_j$, we replace $y_1 y_2$ in R_2'' by the original vote $x_i z_j$ in R_2 and the vote $x_i x_j$ by a vote $y_1 y_2$. If this is a vote of the form $x^* x_i$ or $x_j x_i$, we replace $y_1 y_2$ in R_2'' by the original vote $x_i z_j$ in R_2 and the vote $x^* x_i$ (resp. $x_j x_i$) by $y_1 x^*$ (resp. $y_1 x_j$). In all three cases, the score of all candidates remain the same after the transformation, except the score of y_2 and z_j, which can only decrease, therefore x^* is still a cowinner after the transformation.

We perform this procedure on x_i iteratively until all the votes $y_1 y_2$ (or $y_2 y_1$) in R_2''' have been replaced by the original votes $x_i z_j$ in R_2. After doing this sequentially on all candidates of X such that case (2) occurs, we end up with a profile R'''' of P such that $R_2'''' = R_2$ and $R_3'''' = R_3$ and x^* is a cowinner in R''''. This proves the claim.

Now, let R' be a completion of R where x^* is a cowinner for 2-approval, where $R_2' = R_2$ and $R_3' = R_3$. From R' we build the following extension P' of P: for every vote $W_i \in R_1$,

- if W_i' is of the form $x^* x'$ then $V_i' = W_i'$;
- if W_i' is of the form $y_i x$ then V_i' is obtained from V_i by placing y_i on top;
- if W_i' is of the form $y_1 y_2$ (or $y_2 y_1$) then V_i' is obtained from V_i by placing $\{y_1, y_2\}$ on top.

The scores of all candidates are the same in P' and in R', therefore x^* is a cowinner in P' if only if it is a cowinner in R'. Therefore, it is a cowinner in P', which means that x^* is a possible cowinner for 2-approval in P. □

We can now end the proof of Proposition 5: from Lemmas 2 and 3 we conclude that deciding whether x^* is a possible cowinner for K-approval with respect to the addition of two candidates can be polynomially reduced to a problem of a deciding whether x^* is a possible cowinner for 2-approval, which we know is in P. □

4.5. 3-approval, 3 new candidates

We will now see that the problems addressed in previous subsections constitute the frontier of what can be solved in polynomial-time for K-approval rules. In the rest of this paper, the hardness proofs will use reductions from the 3-dimensional matching (3-DM) problem.

Definition 4. An instance of 3-DM consists of a subset $\mathcal{C} = \{e_1, \ldots, e_m\} \subseteq A \times B \times C$ of triples, where A, B, C are 3 pairwise disjoint sets of size n' with $A = \{a_1, \ldots, a_{n'}\}$, $B = \{b_1, \ldots, b_{n'}\}$ and $C = \{c_1, \ldots, c_{n'}\}$. For $z \in A \cup B \cup C$, $d(z)$ denotes the number of occurrences of z in $\mathcal{C}$, that is the number of triples of $\mathcal{C}$ which contain z. A matching is a subset $M \subseteq \mathcal{C}$ such that no two elements in M agree on any coordinate. The 3-DM problem consists of answering this question: does there exist a perfect matching M on $\mathcal{C}$, that is, a matching of size n'?

The 3-DM problem is known to be NP-complete (problem [SP1] page 221 in Garey and Johnson (1979)), even with the restriction where $\forall z \in A \cup B \cup C$, $d(z) \in \{2, 3\}$ (that is, no element of $A \cup B \cup C$ occurs in more than 3 triples, and each element of $A \cup B \cup C$ appears in at least 2 triples).

Proposition 6. *Deciding if x^* is a possible cowinner for 3-approval with respect to the addition of 3 new candidates, is an NP-complete problem.*

Proof. This problem is clearly in NP. The hardness proof is based on a reduction from 3-DM (see Definition 4).

Let $I = (\mathcal{C}, A \times B \times C)$ be an instance of 3-DM with $n' \geq 3$ and $\forall z \in A \cup B \cup C$, $d(z) \in \{2, 3\}$. From I, we build an instance of the PcWNC problem as follows. The set X of candidates contains

x^*, $X_1 = \{x_i', y_i', z_i' : 1 \leq i \leq n'\}$ where x_i', y_i', z_i' correspond to elements of $A \cup B \cup C$ and a set X_2 of dummy candidates. We now describe the votes informally; their formal definition will follow. The set N of voters contains $N_1 = \{v^e : e \in \mathcal{C}\}$ and a set N_2 of dummy voters. For each voter, we only indicate her first three candidates. Thus, the vote of v^e is (x_i', y_j', z_k') where $e = (a_i, b_j, c_k) \in \mathcal{C}$. The preference of dummy voters are such that:

(i) the scores of the candidates in X satisfy $\forall x \in X_1$, $S_3(x, P_X) = n' + 1$, $S_3(x^*, P_X) = n'$ and $\forall x \in X_2$, $S_3(x, P_X) = 1$;
(ii) the vote of any voter of N_2 contains at most one candidate from $\{x_i', y_i', z_i' : 1 \leq i \leq n'\}$ in the first three positions, and if it contains one, then it is in top position.

Formally, the instance $\langle N, X, P_X, 3, x^* \rangle$ of the possible cowinner problem for 3-approval and 3 new candidates is described as follows: the set of voters is $N = N_1 \cup N_2$ where $N_1 = \{v^e : e \in \mathcal{C}\}$ and $N_2 = N_A \cup N_B \cup N_C \cup N_{x^*}$, the set of candidates is $X = X_1 \cup X_2$. For the candidates in X, we have x^* together with:

- $X_1 = X' \cup Y' \cup Z'$ where $X' = \{x_1', \ldots, x_{n'}'\}$, $Y' = \{y_1', \ldots, y_{n'}'\}$ and $Z' = \{z_1', \ldots, z_{n'}'\}$.
- $X_2 = \{x_i^* : 1 \leq i \leq 2n'\} \cup$
  $\{x_i^j : 1 \leq i \leq n', 1 \leq j \leq 2(n' - d(x_i) + 1)\} \cup$
  $\{y_i^j : 1 \leq i \leq n', 1 \leq j \leq 2(n' - d(y_i) + 1)\} \cup$
  $\{z_i^j : 1 \leq i \leq n', 1 \leq j \leq 2(n' - d(z_i) + 1)\}$.
  Note that $n' - d(x_i) + 1 \geq 1$ since $d(z) \leq 3 \leq n'$.

For each voter $v_i \in N$, we only indicate her first three candidates (in order of preference). The set of all X-votes $\mathcal{P}_X$ of the voters in N is as follows:

- $N_A = \{v_{i,j}^A : 1 \leq i \leq n', 0 \leq j \leq (n' - d(x_i))\}$. The vote of $v_{i,j}^A$ is $(x_i', x_i^{2j+1}, x_i^{2j+2})$.
- $N_B = \{v_{i,j}^B : 1 \leq i \leq n', 0 \leq j \leq (n' - d(y_i))\}$. The vote of $v_{i,j}^B$ is $(y_i', y_i^{2j+1}, y_i^{2j+2})$.
- $N_C = \{v_{i,j}^C : 1 \leq i \leq n', 0 \leq j \leq (n' - d(z_i))\}$. The vote of $v_{i,j}^C$ is $(z_i', z_i^{2j+1}, z_i^{2j+2})$.
- $N_1 = \{v^e : e \in \mathcal{C}\}$. The vote of v^e is (x_i', y_j', z_k') where $e = (a_i, b_j, c_k) \in \mathcal{C}$.
- $N_{x^*} = \{v_j^{x^*} : 0 \leq j \leq n' - 1\}$. The vote of $v_j^{x^*}$ is $(x^*, x_{2j+1}^*, x_{2j+2}^*)$.

We claim that I admits a perfect matching $M \subseteq \mathcal{C}$ if and only if x^* becomes a possible cowinner by adding three new candidates.

Let $Y = \{y_1, y_2, y_3\}$ be the new candidates added. Since we cannot increase the score of x^*, we must decrease by one point the scores of candidates of $X' \cup Y' \cup Z'$. Let us focus on candidates in X'. In order to reduce the score of x_i', we must modify the votes of voters in N_1 or in N_A. By construction, each such voter must put y_1, y_2, y_3 in the first three positions (since in N_A or from (ii), candidates of X' are put in top position when they appear in the first three positions) and then, the score of each y_i increases by 1 at each time. Since there are n' candidates in X', we deduce $S_3(y_i, P) \geq n'$ for every $i = 1, 2, 3$. On the other hand, if x^* becomes a cowinner, $S_3(y_i, P) \leq S_3(x^*, P) \leq S_3(x^*, P_X) = n'$ from (i). Thus, $S_3(y_i, P) = n'$ for every $i = 1, 2, 3$ and there are exactly n' voters N' which put y_1, y_2, y_3 in the first three positions (for the remaining voters of $N \setminus N'$, y_i is ranked in position at least 4 for every $i = 1, 2, 3$).

We claim that $N' \subseteq N_1$. Otherwise, at least one voter of N_A put y_1, y_2, y_3 in the first three positions. There remains at most $n' - 1$ voters of N' to decrease by 1 the score of candidates in Y'. It is impossible because $|Y'| = n'$ and, from (ii) and by construction of N_1, each candidate of Y' appears at most once in the first three positions for all voters. Finally, since the score of candidates in $Y' \cup Z'$ must also decrease by 1, we deduce that x^* is a possible cowinner iff $M = \{e \in \mathcal{C} : y_1, y_2, y_3$ are in the first three positions for voter $v_e\}$ is a perfect matching of $\mathcal{C}$. □

4.6. General case

We finalize the study of the possible cowinner problem for K-approval with respect to candidate addition by showing that the problem is hard in any other case. For this we proceed in two steps: we first prove that for each $k \geq 3$, the problem PcWNC(k) for 3-approval is NP-complete (Lemma 4). Next we prove that if the problem PcWNC(k) for K-approval is NP-complete then it is also the case for the problem PcWNC(k) for $(K + 1)$-approval (Lemma 5).

Lemma 4. *For all $k \geq 3$, the problem* PcWNC(k) *for 3-approval can be reduced in polynomial-time to the problem* PcWNC($k + 1$) *for 3-approval.*

Proof. Let $\langle N, X, P_X, k, x^* \rangle$, where $P = P_X = \langle V_1, \ldots, V_n \rangle$, be an instance of PcWNC(k) for 3-approval. Assume $S_3(x^*, P) \geq 1$ (otherwise, the problem is trivial). Consider the following instance $\langle N', X', Q_{X'}, k + 1, x^* \rangle$ of the PcWNC($k + 1$) for 3-approval:

- the set of candidates is $X' = X \cup \{z\} \cup \{t_i^1, t_i^2 \mid 1 \leq i \leq 2S_3(x^*, P)\}$;
- there are $n + 2S_3(x^*, P)$ votes:
  - for every vote V_j in P we have a vote W_j in Q whose first three candidates are the same as in V_j and in the same order, and the other candidates are in an arbitrary order.
  - for every $i = 1, \ldots, 2S_3(x^*, P)$, we have a vote U_i in which the first 3 candidates are t_i^1, t_i^2, z, the remaining candidates being ranked arbitrarily.

Assume x^* is a possible cowinner for $P = P_X$ (w.r.t. the addition of k new candidates) and let P' be an extension of P where x^* is a cowinner. Let $Y = \{y_1, \ldots, y_k\}$ denote the new candidates for the instance $\langle N, X, P_X, k \rangle$, and $Y' = \{y_1, \ldots, y_{k+1}\}$ the new candidates for the instance $\langle N, X', Q_{X'}, k + 1 \rangle$. Consider the following extension Q' of $Q = Q_{X'}$: for every vote V_j' of P' we have a vote W_j' in Q' whose 3 first candidates are the same as in V_j' (and the remaining ones in an arbitrary order); and for every vote U_i such that $1 \leq i \leq S_3(x^*, P)$ we have a vote U_i' whose first 3 candidates are y_{k+1}, t_i^1, t_i^2 and for every vote U_i such that $S_3(x^*, P) + 1 \leq i \leq 2S_3(x^*, P)$, we have a vote U_i' whose first 3 candidates are t_i^1, t_i^2, z. It is easy to check that Q' is an extension of Q. The scores of all candidates in $X \cup Y$ are the same in P' and Q', while the score of each t_i^1, t_i^2 is 1, the scores of z and of y_{k+1} are $S_3(x^*, P)$; therefore x^* is a cowinner in Q' and a possible cowinner in Q.

Conversely, assume x^* is a possible cowinner in $Q = Q_{X'}$ and let Q' be an extension of Q in which x^* is a cowinner. We are now going to reason abut the number of occurrences of the new candidates $y_1, \ldots, y_{k+1}$ in the first three positions of the votes of Q'. For the sake of notation, for any vote V we denote $S_3(Y', V) = \sum_{y \in Y'} S_3(y, V)$: in words, $S_3(Y', V)$ is the number of new candidates in the first three positions of V. Similarly, if R is a profile, we denote $S_3(Y', R) = \sum_{y \in Y'} S_3(y, R)$.

Without loss of generality, we assume that $S_3(x^*, Q') = S_3(x^*, Q) = S_3(x^*, P)$, since under 3-approval it is never beneficial to decrease the score of x^* to make it a possible cowinner. We have $S_3(z, Q') \leq S_3(x^*, Q') = S_3(x^*, P)$ and $S_3(z, Q) = 2S_3(x^*, P)$, therefore a new candidate must be put above z in at least $S_3(x^*, P)$ votes U_i'; therefore,

$$\sum_{j=1}^{2S_3(x^*, P)} S_3(Y', U_i) \geq S_3(x^*, P). \tag{4}$$

Now, $S_3(Y', Q') = \sum_{i=1}^{n} S_3(Y', W_i) + \sum_{j=1}^{2S_3(x^*, P)} S_3(Y', U_i)$, which together with (4) entails

$$\sum_{i=1}^{n} S_3(Y', W_i) \leq S_3(Y', Q') - S_3(x^*, P). \tag{5}$$

Now, x^* is a cowinner in Q', therefore, for all $y_j \in Y'$ we have $S_3(y_j, Q') \leq S_3(x^*, Q') = S_3(x^*, P)$, from which we get

$$S_3(Y', Q') \leq (k + 1)S_3(x^*, P). \tag{6}$$

From (5) and (6) we get

$$\sum_{i=1}^{n} S_3(Y', W_i) \leq kS_3(x^*, P). \tag{7}$$

Now, consider the extension P' of P built from the restriction of Q' to $\{W_1', \ldots, W_n'\}$ by changing the candidates in Y placed in the first three positions *in such a way that each candidate appears at most in* $S_3(x^*, P)$ *votes*, which is made possible by (7). We have:

- $S_3(x^*, P') = S_3(x^*, P')$;
- for each $y \in Y$, $S_3(y, P') \leq S_3(x^*, P) = S_3(x^*, P')$;
- for each $x \in X \setminus \{x^*\}$, $S_3(x, P') = S_3(x, Q')$; because x^* is a possible cowinner in Q', we have $S_3(x, Q') \leq S_3(x^*, Q') = S_3(x^*, P)$, therefore, $S_3(x, P') \leq S_3(x^*, P) = S_3(x^*, P')$.

From this we conclude that x^* is a possible cowinner in P'. □

Lemma 5. *The problem* PcWNC($k + 1$) *for K-approval can be reduced in polynomial-time to the problem* PcWNC(k) *for $(K + 1)$-approval.*

Proof. Let $\langle N, X, P_X, k, x^* \rangle$ where $P_X = \langle V_1, \ldots, V_n \rangle$ be an instance of PcWNC(k) for K-approval. Consider the following instance $\langle N', X', R_{X'}, k, x^* \rangle$ of the PcWNC(k) for $(K + 1)$-approval:

- the set of candidates is $X' = X \cup \{t_i \mid 1 \leq i \leq n\}$;
- for every vote V_i in P we have a vote W_i in R whose top candidate is t_i and the candidates ranked in position 2 to $K + 1$ are the candidates ranked in positions 1 to K in V_i, the remaining candidates being ranked arbitrarily.

Assume x^* is a possible cowinner for $P = P_X$ and let $P' = \langle V_1', \ldots, V_n' \rangle$ be an extension of P where x^* is a cowinner. Denote by $y_1, \ldots, y_k$ the new candidates. Consider the extension $R' = \langle W_1', \ldots, W_n' \rangle$ of $R = R_{X'}$ where W_i' ranks t_i first and then the candidates ranked in the first K positions in V_i'. For every $x \in X$ we have $S_{K+1}(x, R') = S_K(x, P')$; for every $i = 1, \ldots, k$ we have $S_{K+1}(y_i, R') = S_K(y_i, P')$; and for every $j = 1, \ldots, n$, we have $S_{K+1}(t_j, R') = 1$. Therefore x^* is a possible cowinner in R' and a possible cowinner in R.

Conversely, assume x^* is a possible cowinner in $R = R_{X'}$ and let $R' = \langle W_1', \ldots, W_n' \rangle$ be a completion of R in which it is a possible cowinner. Since none of the t_i threatens x^*, without loss of generality we assume t_i still appears in the first $K + 1$ positions of W_i'—otherwise, change W_i' by moving t_i to the top of W_i'. Consider now the extension $P' = \langle V_1', \ldots, V_n' \rangle$ of $P = P_X$ where V_i' is obtained from W_i' by removing all the t's. Since exactly one t_i appears in the first $K + 1$ positions of W_i', the K candidates approved in V_i' are exactly the $K + 1$ candidates approved in W_i' minus t_i. From this we conclude that for every $x \in X$ we have $S_{K+1}(x, P') = S_K(x, R')$ and for every $i = 1, \ldots, k$ we have $S_{K+1}(y_i, P') = S_K(y_i, R')$. Therefore x^* is a possible cowinner in P' and a possible cowinner in P. □

Proposition 7. *Deciding whether a candidate is a possible cowinner for K-approval with respect to the addition of k new candidates is NP-complete for each (K, k) such that $K \geq 3$ and $k \geq 3$.*

Proof. Since deciding whether x^* is a possible cowinner for 3-approval with respect to the addition of 3 new candidates is NP-hard, using inductively the reductions of Lemmas 4 and 5 shows that NP-hardness propagates to every $(K, k) \geq (3, 3)$. Hence, the problem PcWNC(k) for K-approval is NP-complete for any fixed pair of values $K \geq 3$ and $k \geq 3$. □

We summarize the results obtained in this section by the following table:

	$k = 1$	$k = 2$	$k \geq 3$
Plurality	P (Proposition 1)	P (Proposition 1)	P (Proposition 1)
2-approval	P (Proposition 2)	P (Corollary 1)	P (Corollary 1)
K-approval, $K \geq 3$	P (Proposition 2)	P (Proposition 5)	NP-complete (Proposition 7)

Observe that it would also be possible to address the PcWNC(k) problem (for $K \geq 3$ and $k \geq 3$) by working out a direct polynomial reduction from 3-DM, as done in Proposition 6. This would however result in a much less readable proof. One further interest of the proposed reduction is to show how it is possible to "neutralize" the (extended) power induced by adding more candidates by also adding one more (dummy) candidate in the initial instance. Intuitively, by setting the score of dummy candidate t to $2S_K(x^*, P)$, a single new candidate y_i will have to be "consumed" to ensure that t does not win. More generally, the same proof holds even if K and k depend on the instance (i.e. are not constant). If we allow $f(n)$ new candidates (where f is polynomially bounded function) instead of k a constant, the hardness result also holds (in the proof of Lemma 4, we duplicate each vote V $f(n)$ times by adding candidates z_i for $i = 1, \ldots, f(n)$ instead of z and we add dummy voters and candidates). Formally, we replace the construction given in Lemma 4 by:

- the set of candidates is $X' = X \cup \{z_1, \ldots, z_{f(n)}\} \cup \{t_{i,\ell}^1, t_{i,\ell}^2 \mid 1 \leq i \leq 2S_3(x^*, P), 1 \leq \ell \leq f(n)\}$;
- there are $n + 2f(n)S_3(x^*, P)$ votes:
  - for every vote V_j in P we have a vote W_j in Q whose first three candidates are the same as in V_j and in the same order, and the other candidates are in an arbitrary order.
  - for every $i = 1, \ldots, 2S_3(x^*, P)$ and $\ell = 1, \ldots, f(n)$, we have a vote $U_{i,\ell}$ in which the first 3 candidates are $t_{i,\ell}^1, t_{i,\ell}^2, z_\ell$, the remaining candidates being ranked arbitrarily.

Finally, $Y' = \{y_1, \ldots, y_{k+f(n)}\}$ are the new candidates.

Thus, using above construction, Lemma 4 and Proposition 6, we obtain that for any ε such that $0 < \varepsilon < 1$, PcWNC($f(n)$) for 3-approval is an NP-complete problem where $f(n) = \Theta(|N|^{1-\varepsilon}) = \Theta(|X|^{1-\varepsilon})$ (by setting $f(n) = |N|^r$ in the above construction where r is a constant arbitrarily large). On the other hand, PcWNC($f(n)$) for K-approval is a problem which can be solved in polynomial time when $f(n) = K \cdot |N|$, i.e., when the number of new candidates is K times the number of voters.

Note that some candidates (other than the new candidates) can be *necessary* cowinners with K-approval. Specifically, each candidate x_i such that $S_{K-k}(P_X, x_i) = n$ is a necessary cowinner, since she is approved by all voters and there are not enough new candidates to push her (in at least one vote) out of the set of approved candidates.

5. Borda

Let us now consider the Borda rule (r_B). Characterizing possible Borda cowinners when adding candidates is easy due to the following lemma:

Lemma 6. *Let P_X be an X-profile where $X = \{x^*\} \cup \{x_1, \ldots, x_p\}$ and let $Y = \{y_1, \ldots, y_k\}$ be a set of k new candidates. Let $r_{\bar{s}}$ be a scoring rule for $p + k$ candidates[1] defined by the vector $\bar{s} = \langle s_1, \ldots, s_p, \ldots, s_{p+k} \rangle$ such that $(s_i - s_{i+1}) \leq (s_{i+1} - s_{i+2})$ for all i. $x^* \in X$ is a possible cowinner for P_X w.r.t. the addition of k newcandidates for the scoring rule $r_{\bar{s}}(P)$ iff $x^* \in r(P)$ where P is the profile on $X \cup Y$ obtained from P_X by putting $y_1, \ldots, y_k$ right below x^* (in arbitrary order) in every vote of P_X.*

Proof. We show that it is never strictly better to put the new candidates anywhere but right below x in the new profile. Let P be an extension of P_X in which x^* is a cowinner, and assume there is a vote $V \in P$ and a new candidate y such that either (i) $y \succ_v x^*$ or (ii) there exists at least one candidate x' such that $x^* \succ_v x' \succ_v y$.

If we are in case (i), let us move y right below x^*; let V' be the resulting vote, and P' the resulting profile. Obviously, $S_{\bar{s}}(y, P') \leq S_{\bar{s}}(y, P)$ and $S_{\bar{s}}(x^*, P') \geq S_{\bar{s}}(x^*, P)$, therefore $S_{\bar{s}}(x^*, P') \geq S_{\bar{s}}(y, P')$. For each candidate z such that $y \succ_v z \succ_v x^*$, let i be the rank of z in v and $j > i$ be the rank of x^* in v. Then $(S_{\bar{s}}(z, P') - S_{\bar{s}}(x^*, P')) - (S_{\bar{s}}(z, P) - S_{\bar{s}}(x^*, P)) = (s_{i-1} - s_{j-1}) - (s_i - s_j) = (s_{i-1} - s_i) - (s_{j-1} - s_j) \leq 0$, therefore $S_{\bar{s}}(x^*, P') \geq S_{\bar{s}}(z, P')$. The scores of all other candidates are left unchanged, therefore x^* is still a cowinner in P'. By applying this process iteratively for all new candidates and in all votes until (i) no longer holds, we obtain a profile Q in which x^* is a cowinner, and such that x^* is ranked above all new candidates in every vote.

Now, if (ii) holds for some new candidate y and some vote V of Q, then we move y upwards, right below x^*; let V' be the resulting vote and Q' the resulting profile. The score of y improves, but since y is still ranked above all new candidates in every vote of Q', we have $S_{\bar{s}}(x^*, Q') \geq S_{\bar{s}}(y, Q')$. For each candidate $z \in X \cup Y \setminus \{x^*, y\}$ such that $x^* \succ_v z \succ_v y$ in vote V, z moves down one position in Q', therefore $S_{\bar{s}}(z, Q') \leq S_{\bar{s}}(x', Q) \leq S_{\bar{s}}(x^*, Q) = S_{\bar{s}}(x^*, Q')$. The scores of all other candidates do not change, therefore x^* is still a cowinner in Q'. By applying this process iteratively and in all votes, until (ii) no longer holds, we obtain a profile in which x^* is a cowinner and neither (i) nor (ii) holds.

We conclude that x^* is a possible cowinner for a profile if and only if it is a cowinner in an extension of the profile where all new candidates have been placed right below x^*. □

In words, Lemma 6 applies to the rules where the difference of scores between successive ranks can only become smaller or remain constant as we come closer to the highest ranks. This condition is satisfied by Borda (but not by plurality), by veto, and by rules such as "lexicographic veto", where the scoring vector is $\langle M^p, M^p - M, M^p - M^2, \ldots, M^p - M^{p-1}, 0 \rangle$ where $M > n$.

The following result then easily follows:

Proposition 8. *Let P_X be an X-profile where $X = \{x^*\} \cup \{x_1, \ldots, x_p\}$ and let $Y = \{y_1, \ldots, y_k\}$ be a set of k new candidates. A candidate x^* is a possible cowinner for Borda with respect to the addition of k new candidates if and only if*

$$k \geq \max_{z \in X \setminus \{x^*\}} \frac{S_B(z, P_X) - S_B(x^*, P_X)}{N_{P_X}(x^*, z)}.$$

Proof. By Lemma 6, x^* a possible cowinner if and only if it is a cowinner in the $X \cup \{y_1, \ldots, y_k\}$-completion P of P_X where $y_1, \ldots, y_k$ are placed right below x^*, that is, if and only if $S_B(x^*, P) = S_B(x^*, P_X) + kn$. Now, for each vote, all candidates in $X \setminus \{x^*\}$ ranked above x^* get k additional points in the extended

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[1] In this lemma we do not have to deal with profiles with less than $p + k$ candidates, therefore it is not necessary to mention how r_s is derived for fewer candidates than $p + k$.

vote, while those ranked below x^* keep the same score. Hence, for every $z \in X \setminus \{x^*\}$ we have $S_B(z, P) = S_B(z, P_X) + k(n - N_{P_X}(x^*, z))$, therefore, x^* is a cowinner in P if and only if $S_B(x^*, P_X) + kn \geq S_B(z, P_X) + k(n - N_{P_X}(x^*, z))$, which is equivalent to $k \geq [S_B(z, P_X) - S_B(x^*, P_X)]/N_{P_X}(x^*, z)$. (We recall that $N_{P_X}(x^*, z)$ stands for the number of votes in P_X ranking x^* above z.) □

In words, checking whether x^* is a possible cowinner boils down to checking, for each other candidate z, whether there are enough votes where x^* is preferred to z to compensate for the scoredifference with this candidate. This means that possible cowinners with respect to adding any number of new candidates can be computed in polynomial time for Borda, and more generally for any rule satisfying the conditions of Lemma 6. Note that computing possible winners for Borda is NP-hard (Xia and Conitzer, 2011), therefore, the restriction of the problem to candidate addition induces a complexity reduction.

Example 3. Take $X = \{a, b, c, d\}$, $n = 4$, and $P_X = \langle bacd, bacd, bacd, dacb \rangle$. The Borda scores in P_X are $S_B(a, P_X) = 8$, $S_B(b, P_X) = 9$, $S_B(c, P_X) = 3$, and $S_B(d, P_X) = 4$, while $N(a, b) = 1$, $N(a, c) = 4$, $N(a, d) = 3$, $N(b, c) = 3$, $N(b, d) = 3$, $N(c, d) = 3$, and for all x, y, $N(x, y) = 4 - N(y, x)$. Let $\delta(x, z) = S_B(z, P_X) - S_B(x, P_X)/N_{P_X}(x, z)$. The following matrix gives the values of $\delta(x, z)$ for the possible pairs of distinct candidates (for the sake of readability, non-positive values are denoted by ≤ 0).

$\delta(x, z)$	a	b	c	d	max
a	–	1	≤ 0	≤ 0	1
b	≤ 0	–	≤ 0	≤ 0	≤ 0
c	$+\infty$	5	–	≤ 0	$+\infty$
d	5	6	1	–	6

Applying Proposition 8, b is a possible cowinner whatever the value of k, a is a possible cowinner if and only if $k \geq 1$, d is a possible cowinner if and only if $k \geq 6$, c is not a possible cowinner whatever the value of k.[2] Note that for $k \geq 6$, d is a possible cowinner whereas c is not, although c has a higher Borda score than d in P_X.

6. Hardness with a single new candidate

Even though we have seen that the possible cowinner problem can be NP-hard for some scoring rules, NP-hardness required the addition of several new candidates. We now show that there exists a scoring rule for which the possible cowinner problem is NP-hard with respect to the addition of *one* new candidate.

The scoring rule we use is very simple: it allows each voter to approve exactly 3 candidates, and offers 3 different levels of approval (assigning respectively 3, 2, 1 points to the three preferred candidates). Let r_Δ be the scoring rule defined by the vector $\vec{s} = \langle 3, 2, 1, 0, \ldots, 0 \rangle$ with $m - 3$ 0's completing the vector.

Proposition 9. *Deciding if x^* is a possible cowinner for r_Δ with respect to the addition of one candidate is NP-complete.*

Proof. This problem is clearly in NP. The hardness proof is quite similar to that of Proposition 6. Let $I = (\mathcal{C}, A \times B \times C)$ be an instance of 3-DM with $n' \geq 5$ and $\forall z \in A \cup B \cup C$, $d(z) \in \{2, 3\}$. From I, we build an instance $\langle N, X, P_X, 1, x^* \rangle$ of the PcWNC problem as follows. The set X of candidates contains x^*, $X_1 = \{x_i', y_i', z_i' : 1 \leq i \leq n'\}$ where x_i', y_i', z_i' correspond to elements of A, B and C

respectively and a set X_2 of dummy candidates. The set N of voters contains $N_1 = \{v^e : e \in \mathcal{C}\}$ and a set N_2 of dummy voters. For each voter $v_i \in N$, we only indicate the vote for the first three candidates. So, the vote $V_i = (t_1, t_2, t_3)$ means that candidate t_i receives $4 - i$ points. The vote V_e of voter v^e is (x_i', y_j', z_k') where $e = (a_i, b_j, c_k) \in \mathcal{C}$. The preferences of dummy voters are such that (a) the score of the candidates in X satisfies $\forall x \in X_1, S_{\bar{s}}(x, P_X) = 3n' + 1$, $S_{\bar{s}}(x^*, P_X) = 3n'$ and $\forall x \in X_2, S_{\bar{s}}(x, P_X) \leq 3$ and (b) each voter in N_2 ranks at most one candidate of $\{x_i', y_i', z_i' : 1 \leq i \leq n'\}$ in the first three positions, and if he ranks one in second position, then x^* occurs in third position.

Formally, the instance of the PcWNC problem is built as follows. The set of voters is $N = N_1 \cup N_2$ where $N_1 = \{v^e : e \in \mathcal{C}\}$ and $N_2 = N_A \cup N_B \cup N_C \cup N_{x^*}$, the set of candidates is $X = X_1 \cup X_2 \cup \{x^*\}$ where $X_1 = X' \cup Y' \cup Z'$ with $X' = \{x_1', \ldots, x_{n'}'\}$, $Y' = \{y_1', \ldots, y_{n'}'\}$, $Z' = \{z_1', \ldots, z_{n'}'\}$ and $X_2 = X_A \cup X_B \cup X_C \cup X_{x^*}$. These sets are defined as follows:

- $X_A = \{x_i^j : 1 \leq i \leq n', 1 \leq j \leq 2(n' - d(a_i))\}$.
- $X_B = \{y_i^j : 1 \leq i \leq n', 1 \leq j \leq 2(3n' - 2d(b_i) + 1)\}$.
- $X_C = \{z_i^j : 1 \leq i \leq n', 1 \leq j \leq 2(3n' - d(c_i) + 1)\}$.
- $X_{x^*} = \{x_i^* : i = 1 \leq i \leq 2n'\}$.

The set of all X-votes $\mathcal{P}_X$ is given by:

- $N_A = \{v_{i,j}^A : 1 \leq i \leq n', 0 \leq j \leq (n' - d(a_i) - 2)\} \cup \{v_i^{A,j} : 1 \leq i \leq n', j = 1, 2\}$. The vote $V_{i,j}^A$ of $v_{i,j}^A$ is $V_{i,j}^A = (x_i', x_i^{2j+1}, x_i^{2j+2})$. Note that $n' - d(a_i) - 2 \geq 0$. The vote of $v_i^{A,j}$ is $V_i^{A,j} = (x_i^{2(n'-d(a_i)-1)+j}, x_i', x^*)$.
- $N_B = \{v_{i,j}^B : 1 \leq i \leq n', 0 \leq j \leq 3n' - 2d(b_i)\}$. The vote of $v_{i,j}^B$ is $V_{i,j}^B = (y_i^{2j+1}, y_i^{2j+2}, y_i')$.
- $N_C = \{v_{i,j}^C : 1 \leq i \leq n', 0 \leq j \leq 3n' - d(c_i)\}$. The vote of $v_{i,j}^C$ is $V_{i,j}^C = (z_i^{2j+1}, z_i^{2j+2}, z_i')$.
- $N_{x^*} = \{v_j^{x^*} : 0 \leq j \leq n' - 1\}$. The vote of $v_j^{x^*}$ is $V_j^{x^*} = (v_{2j+1}^{x^*}, v_{2j+2}^{x^*}, x^*)$. Note that $n' - 1 \geq 0$.
- $N_1 = \{v^e : e \in \mathcal{C}\}$. The vote of v^e is $V_e = (x_i', y_j', z_k')$ where $e = (a_i, b_j, c_k) \in \mathcal{C}$.

We claim that I admits a perfect matching $M \subseteq \mathcal{C}$ if and only if x^* becomes a possible cowinner by adding a new candidate y_1. Observe that the scores of the candidates in X satisfy:

(i) $\forall x \in X_1, S_{\bar{s}}(x, P_X) = 3n' + 1$.
(ii) $S_{\bar{s}}(x^*, P_X) = 3n'$.
(iii) $\forall x \in X_2, S_{\bar{s}}(x, P_X) \leq 3$.

Items (i), (ii) and (iii) correspond to the conditions (a) and (b) described previously. For instance, each candidate x_i' from X_1 gets respectively 3, 3, and 2 points from the votes V_e, $V_{i,j}^A$, and $V_i^{A,j}$, summing up to $3d(a_i) + 3(n' - d(a_i) - 1) + 2 = 3n' + 1$. The reader can easily check that the conditions also hold for all other candidates.

Let y_1 be the new candidate. By construction of this scoring rule, we must decrease the score of candidates in X which dominate the score of x^*, that is the candidates of X_1 using (i) and (iii).

Let P' be a $X \cup \{y_1\}$-profile such that x^* is a cowinner. Let us focus on candidates in X'. In order to reduce the score of x_i' by 1, we must modify the preference for at least one voter v^e or $v_{i,j}^A$ or $v_i^{A,j}$. If we modify it for some voter in $v_i^{A,j}$, then the score of x_i' (with respect to $v_i^{A,j}$) decreases by one if and only if the score of x^* (with respect to $v_i^{A,j}$) also decreases by one. In conclusion, we must modify the preference of x_i' for at least one voter v^e or $v_{i,j}^A$. By construction, each such voter must put y_1 in top position and then, the score of y_1 increases by 3 at each time. Since there are

n' candidates in X', we deduce $S_{\vec{s'}}(y_1, P') \geq 3n'$. From the above remark, we also get $S_{\vec{s'}}(x^*, P') \leq S_{\vec{s'}}(x^*, P_X) = 3n'$. Thus for each $i \in \{1, \ldots, n'\}$, exactly one voter among those of v^e or $v^A_{i,j}$ must put candidate y_1 in top position. Finally, if it is one voter $v^A_{i,j}$, then we deduce $S_{\vec{s'}}(y_1, P') > 3n$ because the score of $Y' \cup Z'$ must also decrease, which is not possible since y_1 will then win.

Following a line of reasoning similar to the one developed in the proof of Proposition 6, we conclude that for each $i \in \{1, \ldots, n'\}$, exactly one voter among those of v^e must put candidate y_1 in top position. Since the score of $Y' \cup Z'$ must also decrease by 1, we deduce that x^* is a possible cowinner if and only if $M = \{e \in \mathcal{C} : y_1 \text{ is in top position in vote } V_e\}$ and is a perfect matching of $\mathcal{C}$ (for the remaining voters, y_1 is put in last position). $\quad\square$

This rule shows that it may be difficult to identify possible cowinners with a single additional candidate. Giving a characterization of all rules possessing this property is an open problem.

7. Related work

7.1. The possible winner problem

The possible winner problem was introduced in Konczak and Lang (2005): given an incomplete profile $P = \langle V_1, \ldots, V_n \rangle$ where each V_i is a partial order over the set of candidates X, x is a possible winner for P given a voting rule r if there exists a complete extension $P' = \langle V'_1, \ldots, V'_n \rangle$ of P, where each V'_i is a linear order on X extending V_i, such that $r(P') = x$. Possible winners are defined in a similar way for a voting correspondence C, in which case we say that x is a possible *cowinner* if there exists an extension P' of P such that $x \in C(P')$. Clearly, the possible winner problem defined in this paper is a restriction of the general possible winner problem to the following set of incomplete profiles:

(Restr) there exists $X' \subseteq X$ such that for every i,

V_i is a linear order on X'.

As an immediate corollary, the complexity of the possible (co)winner problem with respect to candidate addition is at most as difficult as that of the general problem. This raises the question whether (Restr) leads to a complexity reduction for the scoring rules we have considered here.

The possible (co)winner problem for scoring rules has received a significant amount of attention in the last years. Xia and Conitzer (2011) proved that the problem was NP-complete for the Borda rule, and more generally for scoring rules whose scoring vector contains four consecutive, equally decreasing values, followed by another strictly decreasing value. Betzler and Dorn (2009) went further by showing that NP-completeness holds more generally for all pure[3] scoring rules, except plurality, veto, and scoring rules whose vector s^m is of the form $s^m = \langle 2, 1, \ldots, 1, 0 \rangle$ for large enough values of m. The issue was finally closed by Baumeister and Rothe (2010), who showed that the problem for $s^m = \langle 2, 1, \ldots, 1, 0 \rangle$ is NP-complete as well. These results compare to ours in the following way: all our NP-hardness results strengthen the known NP-hardness results for the general possible winner problem, while our polynomiality results show a complexity reduction induced by (Restr).

Two recent papers give results about the PcWNC problem for other voting rules. Xia et al. (2011) give results about the possible (co)winner with new candidates for other voting rules;

they showed that PWNC and PcWNC are NP-complete for Bucklin and maximin, that PcWNC is NP-complete for Copeland$_0$, and they give several results for approval voting, depending on how the extension a vote is defined. Baumeister et al. (2011) generalize our Proposition 9 by showing that the PcWNC problem is NP-complete for any pure scoring rule of the form $\langle \alpha_1, \alpha_2, 1; 0, \ldots, 0 \rangle$; they also give NP-completeness results for plurality and 2-approval when voters are weighted.

Results about the PcWNC known so far (except our Proposition 9 and its generalization by Baumeister et al., 2011) are summarized in the following table. For the sake of completeness, we also mention the complexity of the other prominent subproblem of the possible cowinner problem, namely unweighted coalitional manipulation.

	General problem	Candidate addition	Manipulation
Plurality and veto	P	P(Proposition 1)	P
Borda	NP-complete (Xia and Conitzer, 2011)	P(Proposition 8)	NP-complete (Betzler et al., 2011; Davies et al., 2011)
2-approval	NP-complete (Betzler and Dorn, 2009)	P (Corollary 1)	P
K-approval, $K \geq 3$	NP-complete (Betzler and Dorn, 2009)	NP-complete (Proposition 6)	P
Bucklin	NP-complete (Xia and Conitzer, 2011)	NP-complete (Xia et al., 2011)	P (Xia et al., 2009)
Maximin	NP-complete (Xia and Conitzer, 2011)	NP-complete (Xia et al., 2011)	NP-complete (Xia et al., 2009)
Copeland$_0$	NP-complete (Xia and Conitzer, 2011)	NP-complete (Xia et al., 2011)	NP-complete (Faliszewski et al., 2008)

Another interesting line of work is the *parameterized* complexity of the possible winner problem for scoring rules, which has been investigated in Betzler et al. (2009). Among other results, they show that for all scoring rules, the problem is fixed-parameter tractable with respect to the number of candidates (in particular, when the number of candidates is bounded by a constant, the problem becomes polynomial-time solvable). This polynomiality result clearly holds in the possible winner problem with respect to candidate addition, with some caution: the number of candidates here is the *total* number of candidates (the initial ones plus the new ones); this result has practical impact in some situations mentioned in the introduction, such as finding a date for a meeting, where the number of candidates is typically low.

We end this subsection by mentioning other works on the possible winner problem and its variants and subproblems, that are less directly connected to our results. The possible winner problem has also been studied from the probabilistic point of view by Bachrach et al. (2010), where the aim is to count the number of extensions in which a given candidate is the winner. Such a probabilistic analysis is highly relevant in candidate-adding situations: given P_X, a number k of new candidates, and a prior probability distribution on votes, computing the probability that a given candidate $x \in X$ will be the winner, or that one of the initial (resp. new) candidates will be the winner, is extremely interesting.[4]

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[3] A (family of) scoring rules $(r_m)_{m \geq 1}$ is pure if for each m, the scoring vector for $m + 1$ candidates is obtained from the scoring vector for m candidates by inserting an additional score at an arbitrary position. All interesting families of scoring rules are pure; this is in particular the case for K-approval and Borda.

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[4] Note that if the voting rule is insensitive to the identity of candidates (*i.e.* neutral), then although the prior probability that one of the k new candidates will

7.2. Control via adding candidates

The possible winners with respect to the addition of candidates is highly reminiscent of constructive control by the chair via adding candidates—this problem first appeared in Bartholdi et al. (1992) and was later studied in more depth for many voting rules, see *e.g.*, Hemaspaandra et al. (2007) and Faliszewski et al. (2009). However, even if a voting situation where new candidates are added looks similar to an instance of constructive control by adding candidates, these problems differ significantly. In control via adding candidates, the input consists of a set of candidates X, a set of "spoiler" candidates Y, and a full profile $P_{X \cup Y}$: the chair knows how the voters would vote on the new candidates; the problem is to determine whether a given candidate x^* can be made a winner by adding at most $k \leq |Y|$ candidates from Y. In the possible winner problem with respect to candidate addition, we have to take into account all possible ways for voters to rank the new candidates. In spite of their significant differences, there is a straightforward connection between these problems: if an instance $\langle N, X, P_{X \cup Y}, x^*, k \rangle$ of control via adding candidates is positive, then x^* is a possible winner in P_X with respect to the addition of k new candidates (the voting rule being the same in both problems).

Bartholdi et al. (1992) noted that a voting rule is immune to control by adding candidates as soon as it satisfies the Weak Axiom of Revealed Preference (WARP), which requires that the winner among a set of candidates W to be the winner among every subset of candidates to which he belongs (Plott, 1976); formally: *for any $Z \subseteq W$, if $r(P_W) \in Z$ then $r(P_Z) = r(P_W)$*. This property can be used in a similar way for the possible winner problem with respect to candidate addition. Obviously, if the voting rule r satisfies WARP, then any possible winner from X is a winner for the current profile P_X.[5] Unfortunately, this social-choice theoretic property is very strong: Dutta et al. (2001) show that a voting rule satisfies this property (there, it is called *candidate stability*) and unanimity if and only if it is dictatorial.

7.3. Cloning

Finally, the possible winner problem via candidate addition is closely related to manipulation by candidate cloning. Independence of clones was first studied in Tideman (1987), further studied in Laffond et al. (1996) and Laslier (2000), and a variant of this property was recently considered from the computational point of view in Elkind et al. (2010). The main difference between x being a possible winner with respect to candidate addition and the existence of a candidate cloning strategy so that x or one of its clones becomes the winner, as in Elkind et al. (2010), is that candidate cloning requires a candidate and its clones to be contiguous in all votes. In other terms, whereas our problem considers the introduction of genuinely new candidates, cloning merely introduces copies of existing ones.

The complexity of this problem is considered by Elkind et al. (2010) for several voting rules. Although the proposed model allows for the possibility of having a bounded number of new clones (via a notion of cost), most of their results focus on the case of unboundedly many clones. Therefore, to be able to compare their results with ours, we should first say something about the variant of the possible winner problem with respect to candidate addition, *when the number of new candidates is not known beforehand and can be arbitrarily large*. The definitions of voting situations and possible winners are straightforward adaptations of Definitions 1 and 2: a voting situation is now a triple $\Sigma = \langle N, X, P_X \rangle$ and x^* is a possible cowinner with respect to Σ and r if there exists an integer k and a set Y of cardinality k such that there is a $(X \cup Y)$-profile P extending P_X such that $x^* \in r(P)$. We now give a necessary and sufficient condition for a candidate to be a possible winner, for a class of scoring rules including the Borda rule.

Proposition 10. *Let $\mathscr{S}$ be a collection of scoring vectors (s^m), $m \geq 1$, such that*

- *for every p, (s_j^m), $1 \leq j \leq m$ is strictly decreasing;*
- *for all $j, j' \in \mathbb{N}$, (1) $\lim_{m \to \infty} \frac{s_j^m - s_{j'}^m}{s_1} = 0$ and (2) $\lim_{m \to \infty} \frac{s_j^m - s_{m-j'}^m}{s_1} = 1$.*

Then, x^* is a possible winner w.r.t. $\langle N, X, P_X, +\infty \rangle$ if and only if it is undominated[6] in P_X.

Proof. First, suppose x^* is undominated in P_X. For any candidate $x_i \neq x^*$, define Δ_i^v as the difference between the score of x^* and the score of x_i, divided by s_1, in the vote v. As in the construction of Lemma 1, put k new candidates right below x^* in every vote, and let P' be the resulting profile. As the value of k grows, for any vote v ranking candidate x^* below x_i, the value of Δ_i^v will tend towards 0 (by condition 1). Also, condition 2 ensures that for each vote v ranking x^* above x_i, the value of Δ_i^v tends towards 1. Because x^* is undominated, such votes always exist for every candidate $x_i \neq x^*$. Therefore, when k grows, $\sum_{v \in P'} \Delta_i^v$ tends towards the number of votes ranking x^* above x_i, which is at least 1. This implies that the score of x^* will be eventually larger than the score of x_i, and this is true for every $x_i \neq x^*$, therefore x^* will eventually become the winner as k grows. Conversely, suppose x^* is dominated by some candidate x_i. Because the scores (s_j^m), $1 \leq j \leq m$ are strictly decreasing, the score of x will always remain strictly below the score of x_i in the completion of the profile, hence x^* is not a possible cowinner. □

Clearly, this large class of voting rules includes Borda, since it satisfies the conditions of Proposition 10. However, it does not include plurality, and more generally K-approval, which violate condition (1). Still, a very simple condition can be stated for K-approval: a candidate is a possible winner as soon as it is approved at least once.

Proposition 11. *When r is K-approval, x^* is a possible winner w.r.t. the addition of an unbounded number of new candidates if and only if $S_K(x^*, P_X) \geq 1$.*

Proof. The condition is obviously necessary. Suppose the condition holds on a given profile. We extend this profile by taking a set of new candidates y_{ij} where $1 \leq i \leq n$ and $1 \leq j \leq K$. Consider the i-th vote: if x^* is ranked in one the top k positions, put all new candidates at the bottom of the vote. Otherwise, introduce the new candidates $\{y_{i1}, \ldots, y_{iK}\}$ at the top of the vote, and all other new candidates at the bottom. The score of the new candidates is at most 1, while that of $x_i \neq x^*$ is at most that of x^* (which is unchanged). □

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be a cowinner under the impartial culture assumption is at least $\frac{k}{|X|+k}$, this is no longer the case when P_X is known: for instance, let us use plurality and consider the profile $P_X = \langle ab, ab, ab \rangle$, and let the number of new candidates be one. For a third candidate to be a cowinner, he either needs to be placed first in all three votes (which occurs with probability $\frac{1}{27}$), or to be placed first in two votes, but not in the third vote (which occurs with probability $\frac{6}{27}$); therefore the probability that the new candidate is a cowinner in the completed profile is only $\frac{7}{27}$.

[5] In order for the converse to hold, we must add one more condition, such as *consensus* (a Pareto-dominated candidate cannot be elected). Then, if the winner for the current profile P_X is x, by ranking all new candidates at the bottom of all votes, none of them can be the winner in $P_{X \cup Y}$, and by WARP, no candidate $x' \in X \setminus \{x\}$ can either, therefore x is a possible winner for P_X with respect to candidate addition.

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[6] We recall that candidate x dominates candidate x' if every voter ranks x above x', and that a candidate is undominated if no other candidate dominates it.

Note that for $K \geq 2$ this condition does *not* imply that the candidate is undominated (nor vice-versa). It does obviously when $K = 1$, *i.e.*, for plurality.

Let us see now how the above results relate to those in Elkind et al. (2010). We first note that in the case of the Borda rule we have the same condition. Indeed one sees intuitively that Lemma 6 tells us that for some voting rules (including Borda), introducing new candidates in a contiguous manner, as with cloning, is the best thing to do. For plurality, again the condition is similar in both cases. However, for K-approval as soon as $K > 1$, the problem becomes hard in the cloning setting whereas it is easy in our setting with an unbounded number of new candidates.

8. Conclusion

In this paper we have considered voting situations where new candidates may show up during the process. This problem increasingly occurs in our societies, as many votes now take place online (through dedicated platforms, or simply by email exchange) during an extended period of time.

We have identified the computational complexity of computing possible winners for some scoring rules. Some of them allow polynomial algorithms for the problem (*e.g.* plurality, 2-approval, Borda, veto) regardless of the (fixed) number of new candidates showing up. For the rules of the K-approval family, when $K \geq 3$, the problem remains polynomial only if the number of new candidates is at most 2. Finally, we have exhibited a simple rule where the problem is hard for a single new candidate.

The results address the problem of making some designated candidate a cowinner, which is similar to x being unique winner under the assumption of the most favorable tie-breaking. In the other extreme case (if we want x to be a strict winner, *i.e.*, to win regardless of the tie-breaking rule), the results are easily adapted: for instance, the inequalities in Propositions 1 and 8 become strict. For K-approval, the first condition of Proposition 2 becomes strict but the second one should now read $S_K(P_X, x) \geq \sum_{x_i \in X} \max(0, S_K(P_X, x_i) - S_K(P_X, x) + 1)$. As for veto, all other initial candidates need to be vetoed at least once. The hardness proofs can also be readily adapted to the unique winner setting. A more general treatment would require cumbersome expressions, and is also somewhat problematic since the identities of the new candidates are not known anyway (making it difficult to specify easily a tie-breaking rule on these candidates).

As for future work, a first direction to follow would be to try to obtain more general results for scoring rules, as those obtained by Betzler and Dorn (2009) for the general version of the possible winner problem. Extending the study to other families of voting rules, such as rules based on the majority graph, is also worth investigating.

Of course, identifying possible winners is not the end of the story. In practice, as mentioned earlier, one may for instance also be interested in a refinement of this notion: knowing how likely it is that a given candidate will win. Another interesting issue consists of designing elicitation protocols when the preferences about the 'old' candidates are already known. In this case, a trade-off occurs between the storage cost and communication cost, since keeping track of more information is likely to help reduce the burden of elicitation.

Acknowledgments

References

Bachrach, Y., Betzler, N., Faliszewski, P., 2010. Probabilistic possible-winner determination. In: Proceedings of AAAI-10, pp. 697–702.

Bartholdi, J., Tovey, C., Trick, M., 1992. How hard is it to control an election? Social Choice and Welfare 16 (8–9), 27–40.

Baumeister, D., Roos, M., Rothe, J., 2011. Computational complexity of two variants of the possible winner problem. In: Proceedings of AAMAS-11, pp. 853–860.

Baumeister, D., Rothe, J., 2010. Taking the final step to a full dichotomy of the possible winner problem in pure scoring rules. In: Proceedings of ECAI 2010, pp. 1019–1020.

Betzler, N., Dorn, B., 2009. Towards a dichotomy of finding possible winners in elections based on scoring rules. In: Proceedings of MFCS 2009. In: Lecture Notes in Computer Science, vol. 5734. Springer, pp. 124–136.

Betzler, N., Hemmann, S., Niedermeier, R., 2009. A multivariate complexity analysis of determining possible winners given incomplete votes. In: Proceedings of IJCAI-09, pp. 53–58.

Betzler, N., Niedermeier, R., Woeginger, G., 2011. Unweighted coalitional manipulation under the Borda rule is NP-hard. In: Proceedings of IJCAI-11, pp. 55–60.

Chevaleyre, Y., Lang, J., Maudet, N., Ravilly-Abadie, G., 2009. Compiling the votes of a subelectorate. In: Proceedings of IJCAI-09, pp. 97–102.

Chevaleyre, Y., Lang, J., Maudet, N., Monnot, J., 2010. Possible Winners when New Candidates are Added: the case of scoring rules. In: Proceedings of AAAI-2010, pp. 762–767.

Conitzer, V., Sandholm, T., 2002. Vote elicitation: complexity and strategy-proofness. In: Proceedings of AAAI-02, pp. 392–397.

Conitzer, V., Sandholm, T., Lang, J., 2007. When are elections with few candidates hard to manipulate? Journal of the ACM 54 (3), 1–33.

Davies, J., Katsirelos, G., Narodytska, N., Walsh, T., 2011. Complexity of and algorithms for Borda manipulation. In: Proceedings of AAAI-11, pp. 657–662.

Dutta, B., Jackson, M., Le Breton, M., 2001. Strategic candidacy and voting procedures. Econometrica 69 (4), 1013–1037.

Elkind, E., Faliszewski, P., Slinko, A.M., 2009. Swap bribery. In: Proceedings of SAGT, pp. 299–310.

Elkind, E., Faliszewski, P., Slinko, A., 2010. Cloning in elections. In: Proceedings of AAAI-10, pp. 768–773.

Faliszewski, P., Hemaspaandra, E., Hemaspaandra, L., 2009. Multimode control attacks on elections. In: Proceedings of IJCAI-09, pp. 128–133.

Faliszewski, P., Hemaspaandra, E., Schnoor, H., 2008. Copeland voting: ties matter. In: AAMAS-08, No. 2, pp. 983–990.

Faliszewski, P., Procaccia, A., 2010. The AI war on manipulation: are we winning? AI Magazine 31 (4), 53–64.

Garey, M., Johnson, D., 1979. Computers and Intractability. A Guide to the Theory of NP-Completeness. Freeman.

Hazon, N., Aumann, Y., Kraus, S., Wooldridge, M., 2009. Evaluation of election outcomes under uncertainty. In: Proceedings of AAMAS-08, pp. 959–966.

Hemaspaandra, E., Hemaspaandra, L., Rothe, J., 2007. Anyone but him: the complexity of precluding an alternative. Artificial Intelligence 171 (5–6), 255–285.

Konczak, K., Lang, J., 2005. Voting procedures with incomplete preferences. In: Proceedings of the IJCAI-05 Multidisciplinary Workshop on Advances in Preference Handling.

Laffond, G., Lainé, J., Laslier, J.-F., 1996. Composition-consistent tournament solutions and social choice functions. Social Choice and Welfare 13 (1), 75–93.

Laslier, J.-F., 2000. Aggregation of preferences with a variable set of alternatives. Social Choice and Welfare 17 (2), 269–282.

Lu, T., Boutilier, C., 2010. The unavailable candidate model. A decision–theoretic view of social choice. In: ACM Conference on Electronic Commerce 2010, pp. 263–274.

Pini, M.S., Rossi, F., Venable, K. Brent, Walsh, T., 2007. Incompleteness and incomparability in preference aggregation. In: Proceedings of IJCAI'07, pp. 1464–1469.

Plott, C.R., 1976. Axiomatic social choice theory: an overview and interpretation. American Journal of Political Science 20, 511–596.

Tideman, T., 1987. Independence of clones as a criterion for voting rules. Social Choice and Welfare 4 (3), 185–206.

Walsh, T., 2008. Complexity of terminating preference elicitation. In: Proceedings of AAMAS-08, pp. 967–974.

Xia, L., Conitzer, V., 2010. Compilation complexity of common voting rules. In: Proceedings of AAAI-10, pp. 915–920.

Xia, L., Conitzer, V., 2011. Determining possible and necessary winners given partial orders. Journal of Artificial Intelligence Research (JAIR) 41, 25–67.

Xia, L., Lang, J., Monnot, J., 2011. Possible winners hen new alternatives join: new results coming up! In: Proceedings of AAMAS-11, pp. 829–836.

Xia, L., Zuckerman, M., Procaccia, A., Conitzer, V., Rosenschein, J., 2009. Complexity of unweighted coalitional manipulation under some common voting rules. In: Proceedings of IJCAI-09, pp. 348–353.