Effectivity functions and efficient coalitions in Boolean games

Elise Bonzon
Université Paris Descartes, LIPADE, F-75270 Paris Cédex 06
elise.bonzon@parisdescartes.fr

Marie-Christine Lagasquie-Schiex
Université de Toulouse, UPS, IRIT, F-31062 Toulouse Cédex 9
lagasq@irit.fr

Jérôme Lang
Université Paris Dauphine, LAMSADE, F-75775 Paris Cédex 16
lang@lamsade.dauphine.fr

Abstract. Boolean games are a logical setting for representing strategic games in a succinct way, taking advantage of the expressive power and conciseness of propositional logic. A Boolean game consists of a set of players, each of which controls a set of propositional variables and has a specific goal expressed by a propositional formula. We show here that Boolean games are a very simple setting, yet sophisticated enough, for analysing the formation of coalitions. Due to the fact that players have dichotomous preferences, the following notion emerges naturally: a coalition in a Boolean game is efficient if it has the power to guarantee that all goals of the members of the coalition are satisfied. We study the properties of efficient coalitions.

Keywords: Game theory, propositional logic, coalitions

1. Introduction

Boolean games (Harrenstein et al., 2001; Harrenstein, 2004; Dunne and van der Hoek, 2004; Bonzon et al., 2009) are a logical setting for representing strategic games in a succinct way, taking advantage of the expressive power and conciseness of propositional logic. Informally, a Boolean game consists of a set of players, each of which controls a set of propositional variables and has a goal expressed by a propositional formula.

Boolean games are games with both a *structural specificity* and a *preferential specificity*. The structural specificity expresses a restriction on strategy profiles: *a player's (pure) strategy is a truth assignment of the variables she controls*. The preferential specificity expresses a restriction on the player's preferences: a player in a Boolean game has a *dichotomous* preference relation, that is, either her goal is satisfied or it is not, and this goal is represented succinctly by a propositional formula. The preferential specificity can be easily relaxed, and there are a number of extensions to Boolean games that allow players to have nondichotomous preferences: in (Harrenstein, 2004) (last chapter), each agent has a *set* of goals; in (Bonzon et al., 2006; Bonzon et al., 2009), an agent's preferences is described by a CP-net or more generally a specification in some compact preference representation language; in (Dunne et al., 2008), each agent has a *quasi-dichotomous* utility function, namely, a dichotomous utility function induced by her goal, plus a negligible cost associated to her possible variable assignments (with the specific choice that making a variable false is costless). The structural specificity, on the other hand, is central to the framework, and relaxing it would probably make it depart largely from Boolean games.

Previous work on Boolean games has focused on representational issues, by giving logical characterizations of several solution concepts such as Nash equilibria, and by investigating the computational issues related to these solution concepts. Studying the power of coalitions, as well as the formation of coalitions, in Boolean games, has rarely been addressed (with the exception of (Dunne et al., 2008)). The goal of this paper is to address both issues. As these issues are, to a large extent, independent, this paper is composed of two almost independent parts.

The first part of the paper focuses on the power of coalitions in Boolean games. Equivalently, it amounts at studying the meaning of the structural specificity: how restrictive is it, and how can it be characterized? A natural way of answering this question is to study Boolean games from the point of view of *effectivity functions*, which model the power of coalitions of agents. More precisely, we would like to characterize the properties of effectivity functions that are implied by the structural specificity of Boolean games. Almost ten years ago, Pauly showed a correspondence between stategic games and a particular class of effectivity functions he named playable effectivity functions (it has been shown recently that this correspondence is not completely exact, but this has no impact on our work; see endnote 1). Now, Boolean Games do not cover all strategic games: their structural specificity is a true restriction, and therefore we expect that Boolean games correspond to a strict subset of playable effectivity functions. The contribution of this first part of the paper consists in characterizing this subset. Note that in this first part, the agents' preferences are irrelevant, and therefore the results are totally independent from the question whether the preferential specificity is assumed or not.

The second part of this paper also focuses on the power of coalitions, but from a different perspective, taking agents' preferences into account, and taking the preferential specificity for granted (even if some of our results still hold under the weaker assumption that preferences are quasi-dichotomous). Due to the dichotomous nature of agents' preferences, the following simple notion emerges naturally: a coalition in a Boolean game is *efficient* if it has the power to guarantee that all goals of the members of the coalition are satisfied. This notion is of primary importance, because it is expected that agents in a Boolean game will join such coalitions. Similarly as for related notions in cooperative game theory, the existence of an efficient coalition is not guaranteed, and deciding whether a Boolean game possesses an efficient coalition is an important issue. Efficient coalitions enjoy interesting structural properties, and are not easy to identify, especially because a subset or a superset of an efficient coalition may not be efficient, and likewise, the union or the intersection of efficient coalitions may not be efficient. We characterize efficient coalitions in terms of some topological properties and study the relation between efficiency and some solution concepts coming from cooperative game theory: the weak core (the standard notion of core) and the strong core (which is a sort of generalization of Pareto optimality). Then, the computational complexity of the membership and non-emptiness problems is identified for the three notions of efficiency, weak core and strong core. Finally, the last contribution regards the representation of a Boolean game in terms of dependency graphs: we show that dependency graphs and the relative notion of stable coalitions can be used as a correct (but not complete) method to find efficient coalitions. Completeness is restored in the special case where goals requires only one player to be satisfied.

We recall the Boolean game framework in Section 2. In Section 3 we study the specificity of the power of coalitions in Boolean games, as compared to static games in general. For this we show that the effectivity function in a Boolean game satisfies some specific properties, that fully characterize Boolean games. In Section 4, we define efficient coalitions in Boolean games, and focus first on their structural properties. We give an exact characterization of sets of coalitions that can be obtained as the set of efficient coalitions associated with a Boolean game, and we relate coalition efficiency to the well-known notion of core. In Section 5 we study efficient coalitions from a computational point of view. In Section 6, we address the role of dependencies between agents in the computation of efficient coalitions. Sections 7 and 8 discuss respectively related work and further research issues.

2. Boolean games

For any finite set $V = \{a, b, \ldots\}$ of propositional variables, L_V denotes the propositional language built up from V, the Boolean constants $\top$ and $\bot$, and the usual connectives. Formulas of L_V are denoted by φ, ψ etc. A *literal* is a variable x of V or the negation of a variable. A *term* is a consistent conjunction of literals. A *clause* is a disjunction of literals. If α is a term, then $Lit(\alpha)$ is the set of literals appearing in α. If $\varphi \in L_V$, then $Var(\varphi)$ denotes the set of propositional variables appearing in φ.

2^V is the set of the interpretations for V, with the usual convention that for $M \in 2^V$ and $x \in V$, M gives the value *true* to x if $x \in M$ and *false* otherwise. $\models$ denotes the consequence relation of classical propositional logic. Let $V' \subseteq V$. A V'-interpretation is a truth assignement to each variable of V', that is, an element of $2^{V'}$. V'-interpretations are denoted by listing all variables of V', with a $^{-}$ symbol when the variable is set to false: for instance, let $V' = \{a,b,d\}$, then the V'-interpretation $M = \{a,d\}$ assigning a and d to true and b to false is denoted by $a\bar{b}d$. If $Var(\varphi) \subseteq X$, then $Mod_X(\varphi)$ represents the set of X-interpretations satisfying φ.

If $\{V_1,\ldots,V_p\}$ is a partition of V and $\{M_1,\ldots,M_p\}$ are partial interpretations, where $M_i \in 2^{V_i}$, $(M_1,\ldots,M_p)$ denotes the interpretation $M_1 \cup \ldots \cup M_p$.

Let ψ be a propositional formula. A term α is an *implicant* of ψ if and only if $\alpha \models \psi$ holds. α is a *prime implicant* of ψ if and only if α is an implicant of ψ and for every implicant α' of ψ, if $\alpha \models \alpha'$ holds, then $\alpha' \models \alpha$ holds. $PI(\psi)$ denotes the set of all the prime implicants of ψ.

Given a set of propositional variables V, a Boolean game on V is an n-player game, where the actions available to each player consist in assigning a truth value to each variable in a given subset of V. The preferences of each player i are represented by a propositional formula φ_i formed upon the variables in V.

Without loss of generality, we can assume that V is finite. Indeed, only a finite set of variables occurs in the goals φ_i and the constraints γ_i, and the variables not occurring in them do not play any role and can safely be forgotten.

DEFINITION 1. *An n-***player Boolean game** *is a 5-tuple (N,V,π,Γ,Φ), where*

- $N = \{1,2,\ldots,n\}$ *is a set of players (also called agents);*
- V *is a set of propositional variables;*
- $\pi : N \mapsto 2^V$ *is a control assignment function mapping each player to the set of variables she controls;*
- $\Gamma = \{\gamma_1,\ldots,\gamma_n\}$ *is a set of constraints, where each γ_i is a satisfiable propositional formula of $L_{\pi(i)}$.*
- $\Phi = \{\varphi_1,\ldots,\varphi_n\}$ *is a set of goals, where each φ_i is a satisfiable formula of L_V.*

For ease of notation, the set of all the variables controlled by i is written π_i instead of $\pi(i)$. Each variable is controlled by one and only one agent, that is, $\{\pi_1,\ldots,\pi_n\}$ forms a partition of V. The role of constraints is to restrict the set of feasible strategies of each agent: agent i assigns each variable of π_i to a truth value, in such a way that the resulting assignment satisfies γ_i.

DEFINITION 2. *Let $G = (N, V, \pi, \Gamma, \Phi)$ be a Boolean game. A (pure)* **strategy** *for Player i in G is a π_i-interpretation satisfying γ_i. The set of strategies for Player i in G is $\Sigma_i = \{\sigma_i \in 2^{\pi_i} \mid \sigma_i \models \gamma_i\}$. A* **strategy profile** *σ for G is an n-tuple $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ where for all i, $\sigma_i \in \Sigma_i$. $\Sigma = \Sigma_1 \times \ldots \times \Sigma_n$ is the set of all strategy profiles.*

For each i, γ_i is a constraint restricting the possible strategy profiles for Player i.

Note that since $\{\pi_1, \ldots, \pi_n\}$ forms a partition of V, a strategy profile σ is an interpretation for V, i.e., $\sigma \in 2^V$. The following notations are usual in game theory. Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a strategy profile. For any nonempty set of players $I \subseteq N$, the projection of σ on I is defined by $\sigma_I = (\sigma_i)_{i \in I}$ and $\sigma_{-I} = \sigma_{N \setminus I}$. If $I = \{i\}$, we denote the projection of σ on $\{i\}$ by σ_i instead of $\sigma_{\{i\}}$; similarly, we note σ_{-i} instead of $\sigma_{-\{i\}}$. π_I denotes the set of the variables controlled by I, and $\pi_{-I} = \pi_{N \setminus I}$. The set of strategies for $I \subseteq N$ is $\Sigma_I = \times_{i \in I} \Sigma_i$, and the joint goal of coalition $I \subseteq N$ is $\Phi_I = \bigwedge_{i \in I} \varphi_i$.

If σ and σ' are two strategy profiles, (σ_{-I}, σ'_I) denotes the strategy profile obtained from σ by replacing σ_i with σ'_i for all $i \in I$. For the sake of notation, the set of all strategy profiles constructed from σ_C will be written $\{\sigma | \sigma \supseteq \sigma_C\}$ instead of $\{\sigma | \sigma = (\sigma_{-C}, \sigma_C), \forall \sigma_{-C}\}$.

The goal φ_i of player i is a compact representation of a dichotomous preference relation, or equivalently, of a binary utility function $u_i : \Sigma \to \{0, 1\}$ defined by $u_i(\sigma) = 0$ if $\sigma \models \neg\varphi_i$ and $u_i(\sigma) = 1$ if $\sigma \models \varphi_i$. σ is at least as good as σ' for i, denoted by $\sigma \succeq_i \sigma'$, if $u_i(\sigma) \geq u_i(\sigma')$, or equivalently, if $\sigma \models \neg\varphi_i$ implies $\sigma' \models \neg\varphi_i$; σ is strictly better than σ' for i, denoted by $\sigma \succ_i \sigma'$, if $u_i(\sigma) > u_i(\sigma')$, or, equivalently, $\sigma \models \varphi_i$ and $\sigma' \models \neg\varphi_i$.

As we said in the introduction, for the results of the first part of the paper, preferences do not play any role (and *a fortiori*, neither does their dichotomous nature). For this we introduce the notion of *pre-Boolean games*, which are preference-free Boolean games.

DEFINITION 3. *A* **pre-Boolean game** *is a 4-tuple (N, V, π, Γ), with N, V, π, Γ as in Definition 1.*

Thus, a Boolean game consists of a pre-Boolean game together with a description Φ of the player's (dichotomous) utilities.

Boolean games can easily be extended so as to allow for non-dichotomous preferences, represented in some compact language for preference representation (see (Harrenstein, 2004; Bonzon et al., 2006; Bonzon et al., 2009; Dunne et al., 2008)). Among these generalized Boolean games, an interesting subclass consists of Boolean games in which the players' utility functions are nearly dichotomous.

DEFINITION 4. *A* **quasi-dichotomous Boolean game** *is a 6-uple $(N, V, \pi, \Gamma, \Phi, \langle c_1, \ldots, c_n \rangle)$ where $(N, V, \pi, \Gamma, \Phi)$ is a Boolean game and for each Player*

i, c_i is a function mapping each strategy profile $\sigma = (\sigma_1, \dots, \sigma_n)$ to a cost such that $c_i(\sigma) < 1$. The utility function $u_i : \Sigma \to \{0, 1\}$ of player i defined by $u_i(\sigma) = -c_i(\sigma)$ if $\sigma \models \neg\varphi_i$ and $u_i(\sigma) = 1 - c_i(\sigma)$ if $\sigma \models \varphi_i$. Note that for any σ such that $\sigma \models \varphi_i$ and any σ' such that $\sigma' \models \neg\varphi_i$ we have $u_i(\sigma) > u_i(\sigma')$: whatever the cost function, an agent is always better off in a state that satisfies her goal than in a state that does not.
If G is a quasi-dichotomous Boolean game, the standard Boolean game G^* associated with G is obtained from G by simply ignoring the cost function c.

Obviously, any standard Boolean game corresponds to a quasi-dichotomous Boolean game, obtained by letting $c_i(\sigma) = 0$ for all i and for all σ.

Quasi-dichotomous Boolean games were introduced first in (Dunne et al., 2008), with the difference that the cost function c in (Dunne et al., 2008) depends only on the player's own action, that is, $c_i(\sigma) = c_i(\sigma_i)$, plus the additional assumption that each agent has a cost associated to each "positive" action (setting one of her controlled variables to true), and $c_i(\sigma_i)$ is the sum of the costs of all of her variables assigned to true.

In the definition above we did not specify how the cost function c_i is represented. Representing it explicitly, by listing all combinations of strategies together with their utility for each agent, would not fit the spirit of Boolean games, and would render somehow useless the compact representation of the goals. It is thus natural to assume that each c_i will be represented in some compact representation language, possibly making some further restriction, such as in (Dunne et al., 2008).

3. Coalitions and effectivity functions in Boolean games

Recall that the structural specificity of Boolean games is that individual strategies are truth assignments to a given set of propositional variables. We might wonder how restrictive this specificity is. In this section we study Boolean games from the point of view of effectivity functions. Effectivity functions have been developed in social choice to model the power of coalitions (Moulin, 1983; Abdou and Keiding, 1991; Pauly, 2001). Clearly, the definition of Σ_i as $Mod_{\pi_i}(\gamma_i)$ induces some constraints on the power of players and coalitions. Our aim is to give an exact characterization of effectivity functions induced by Boolean games.

Since in Boolean games the power of an agent i is independent from her goal φ_i, it suffices to consider pre-Boolean games when dealing with effectivity functions. As usual, N is the set of agents, a **coalition** C is a subset of N, and S is a generic set of states.

DEFINITION 5. *A* **coalitional effectivity function** *is a function* $\mathrm{Eff}\colon 2^N \to 2^{2^S}$ *satisfying* monotonicity*: for every coalition $C \subseteq N$, $X \in \mathrm{Eff}(C)$ implies $Y \in \mathrm{Eff}(C)$ whenever $X \subseteq Y \subseteq S$.*

The function Eff associates with every group of players the set of states, or outcomes, for which the group is effective. We usually interpret $X \in \mathrm{Eff}(C)$ as "the players in C have a joint strategy for bringing about an outcome in X".

A *strategic game* is usually defined as a tuple $\langle N, \Sigma, S, o\rangle$, where Σ is the set of strategy profiles for players in N, and $o\colon \times_{i\in N} \Sigma_i \to S$ is the *outcome function*. (Pauly, 2001) gives a more precise account for effectivity in strategic games by defining α-effectivity: a coalition $C \subseteq N$ is **α-effective** for $X \subseteq S$ if and only if the players in C have a joint strategy for bringing an outcome of X, *whatever the strategies of the other players are*.

DEFINITION 6. *A* **coalitional α-effectivity function** *for a non-empty strategic game G is a function* $\mathrm{Eff}^{\alpha}_{G}\colon 2^N \to 2^{2^S}$ *defined by: $X \in \mathrm{Eff}^{\alpha}_{G}(C)$ iff $\exists\sigma_C$ $\forall\sigma_{-C}$, $o(\sigma_C, \sigma_{-C}) \in X$.*

In a Boolean game, outcomes are identified with strategy profiles: $S = \Sigma$. A pre-Boolean game G then induces an α-effectivity function $\mathrm{Eff}^{\alpha}_{G}$ as follows:

DEFINITION 7. *Let $G = (N, V, \pi, \Gamma)$ be a pre-Boolean game. The* **coalitional α-effectivity function induced by** *G is the function* $\mathrm{Eff}^{\alpha}_{G} : 2^N \to 2^{2^{\Sigma}}$ *defined by: for any $X \subseteq \Sigma$ and any $C \subseteq N$, $X \in \mathrm{Eff}^{\alpha}_{G}(C)$ if there exists $\sigma_C \in \Sigma_C$ such that for any $\sigma_{-C} \in \Sigma_{-C}$, $(\sigma_C, \sigma_{-C}) \in X$.*

For the sake of notation, the α-effectivity function induced by a pre-Boolean game G will be denoted by Eff_G instead of $\mathrm{Eff}^{\alpha}_{G}$. Note that effectivity functions induced by pre-Boolean games can be equivalently expressed as mappings $\mathrm{Eff}_G : 2^N \to 2^{L_V}$ from coalitions to sets of logical formulas: $\varphi \in \mathrm{Eff}_G(I)$ if $Mod_{\pi_I}(\varphi) \in \mathrm{Eff}_G(I)$. This definition obviously implies syntax-independence, that is, if $\varphi \equiv \psi$ then $\varphi \in \mathrm{Eff}_G(I)$ iff $\psi \in \mathrm{Eff}_G(I)$.

This definition is a particular case of the α-effectivity function induced by a strategic game (see (Pauly, 2001), Chapter 2). Therefore, these functions satisfy the following properties (cf. (Pauly, 2001), Theorem 2.27):

1. $\forall C \subseteq N$, $\varnothing \notin \mathrm{Eff}_G(C)$;
2. $\forall C \subseteq N$, $\Sigma \in \mathrm{Eff}_G(C)$;
3. for all $X \subseteq \Sigma$, if $\overline{X} \notin \mathrm{Eff}_G(\varnothing)$ then $X \in \mathrm{Eff}_G(N)$;
4. Eff_G is superadditive, that is, if for all $C, C' \subseteq N$ such that $C \cap C' = \emptyset$, and $X, Y \subseteq \Sigma$, if $X \in \mathrm{Eff}_G(C)$ and $Y \in \mathrm{Eff}_G(C')$ then $X \cap Y \in \mathrm{Eff}_G(C \cup C')$.

An effectivity function satisfying these four properties is called **strongly playable**. Note that strong playability implies regularity and coalition-monotonicity ((Pauly, 2001), Lemma 2.26). Pauly (2001) proves a correspondence between strong playability of an effectivity function Eff and the existence of a strategic game G such that $\text{Eff}_G = \text{Eff}$.[1] However, pre-Boolean games are a specific case of strategic game forms, therefore we would like to have an exact characterization of those effectivity functions that correspond to a pre-Boolean game. We first have to define two additional properties. Define $At(C)$ as the minimal sets in $\text{Eff}(C)$, that is, $At(C) = \{X \in \text{Eff}(C) |$ there is no $Y \in \text{Eff}(C)$ such that $Y \subset X\}$. $At(C)$ is called the *nonmonotonic core* of C, and denoted by $\text{Eff}^{nc}(C)$, in (Goranko et al., 2011).

Atomicity: Eff satisfies *atomicity* if for every $C \subseteq N$, $At(C)$ forms a partition of S.

Decomposability: Eff satisfies *decomposability* if for every two disjoints subsets I, J of N and for every $X \subseteq S$, $X \in \text{Eff}(I \cup J)$ if and only if there exist $Y \in \text{Eff}(I)$ and $Z \in \text{Eff}(J)$ such that $X = Y \cap Z$.

Decomposability is a strong property that implies superadditivity. Note also that decomposability and atomicity are strongly related to the following properties in (Agotnes and Alechina, 2011):

(2) for any $C \neq \emptyset, \text{Eff}^{nc}(C) = \{\cap_{i \in C} X_i : X_i \in \text{Eff}^{nc}(i)\}$
(3) $X, Y \in \text{Eff}^{nc}(i)$ and $X \neq Y$ implies $X \cap Y = \emptyset$
(4) $X \in \text{Eff}^{nc}(j)$ and $x \in X$ implies $\exists Y \in \text{Eff}^{nc}(i), x \in Y$

where $\text{Eff}^{nc}(C)$ denotes the set of all inclusion-minimal sets in $\text{Eff}(C)$. Decomposability is equivalent to (2) whereas in the presence of decomposability and $\text{Eff}(N) = 2^S \setminus \{\emptyset\}$, atomicity is equivalent to the conjunction of (3) and (4). We give a short proof of these equivalences in Appendix.

In the rest of this section we prove the following characterization result: a coalitional α-effectivity function Eff satisfies strong playability, atomicity, decomposability and $\text{Eff}(N) = 2^S \setminus \varnothing$ if and only if there exists a pre-Boolean game $G = (N, V, \pi, \Gamma)$ and a bijective function $\mu : S \to Mod(\Gamma)$ such that for every $C \subseteq N$: $\text{Eff}_G(C) = \{\mu(X) | X \in \text{Eff}(C)\}$.

The proof of these two results go along a series of lemmas, which we establish first.[2] If G is a pre-Boolean game, the set of atoms for the effectivity functions Eff_G will be denoted by At_G.

LEMMA 1. *For any pre-Boolean game G, Eff_G satisfies strong playability, atomicity, decomposability, and $\text{Eff}_G(N) = 2^\Sigma \setminus \varnothing$.*

Proof: Eff_G is a (specific) α-effectivity function, therefore by Theorem 2.27 in (Pauly, 2001), Eff_G satisfies strong playability (and, *a fortiori*, superaddivity).

As for atomicity, remark first that $X \in At_G(C)$ if and only if X is the set of all π_C-interpretations satisfying $\gamma_C = \bigwedge_i \gamma_i$, which clearly implies that any two distinct subsets in $At_G(C)$ are disjoint. Then remark that $\bigcup_{\sigma_C \in \Sigma_C} \{\sigma | \sigma \supset \sigma_C\}$ $= \Sigma$. Therefore, $At_G(C)$ forms a partition of Σ.

As for decomposability, from left to right: let $X \in \mathrm{Eff}_G(I \cup J)$. Then there exists a joint strategy $\sigma_{I \cup J}$ such that if $W = \{\sigma \in \Sigma | \sigma \supseteq \sigma_{I \cup J}\}$, then $W \subseteq X$. Consider now $Y = \{\sigma \in \Sigma | \sigma \supseteq \sigma_I\}$ and $Z = \{\sigma \in \Sigma | \sigma \supseteq \sigma_J\}$. We have $Y \in \mathrm{Eff}_G(I)$, $Z \in \mathrm{Eff}_G(J)$ and $X = Y \cap Z$. From right to left: let $Y \in \mathrm{Eff}_G(I)$ and $Z \in \mathrm{Eff}_G(J)$, then by superaddivity, $Y \cap Z \in \mathrm{Eff}_G(I \cup J)$.

Lastly, let $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma$. If each player i plays σ_i then σ is obtained, therefore $\{\sigma\} \in \mathrm{Eff}_G(N)$. By monotonicity, every nonempty subset of Σ is in $\mathrm{Eff}_G(N)$ as well, therefore $\mathrm{Eff}_G(N) = 2^\Sigma \setminus \varnothing$. ■

LEMMA 2. *If there exists a pre-Boolean game $G = (N, V, \pi, \Gamma)$ and an bijective function $\mu : S \to Mod(\Gamma)$ such that for every $C \subseteq N$:* $\mathrm{Eff}_G(C) = \{\mu(X) | X \in \mathrm{Eff}(C)\}$*, then* Eff *satisfies strong playability, atomicity, decomposability and* $\mathrm{Eff}(N) = 2^S \setminus \varnothing$.

Proof: Eff_G satisfies these properties and μ is a bijection between S and $\mu(\Sigma) = Mod(\Gamma)$, therefore these properties transfer to Eff. ■

LEMMA 3. *Let G be a pre-Boolean game, Σ its set of strategy profiles and T_i be a minimal subset of* $\mathrm{Eff}_G(i)$. *Then $T_i = \{\sigma | \sigma \supseteq \sigma_i\}$ for all $\sigma_i \in \Sigma_i$.*

Proof: Player i can only enforce a subset of Σ_i, that is, $X \in \mathrm{Eff}_G(i)$if X contains $\Sigma_1 \times \ldots \times \Sigma_{i-1} \times \Sigma_i^* \times \Sigma_{i+1} \times \ldots \times \Sigma_n$ for some $\Sigma_i^* \subseteq \Sigma_i$. Therefore the minimal subsets of $\mathrm{Eff}_G(i)$ are exactly those of the form $\Sigma_1 \times \ldots \times \Sigma_{i-1} \times \{\sigma_i\} \times \Sigma_{i+1} \times \ldots \times \Sigma_n$, that is, of the form $\{\sigma | \sigma \supseteq \sigma_i\}$. ■

From now on, let Eff **be a coalitional effectivity function satisfying strong playability, atomicity, decomposability and** $\mathrm{Eff}(N) = 2^S \setminus \varnothing$**.**

Let $At(C)$ be the set of atoms for C associated with Eff. Due to decomposability, Eff is entirely determined by $\{At(i), i \in N\}$.

LEMMA 4. *For every $s \in S$ there exists a unique $(Z_1, \ldots, Z_n) \in At(1) \times \ldots \times At(n)$ such that $Z_1 \cap \ldots \cap Z_n = \{s\}$.*

Proof: Let $s \in S$. Because $\mathrm{Eff}(N) = 2^S \setminus \{\varnothing\}$, we have $\{s\} \in \mathrm{Eff}(N)$, and by decomposability, there exists $(T_1, \ldots, T_n)$ such that for every i, $T_i \in \mathrm{Eff}(i)$ and $T_1 \cap \ldots \cap T_n = \{s\}$. Let $i \in N$. By definition of $At(i)$, there exists $Z_i \in At(i)$ such that $s \in Z_i$ and $Z_i \subseteq T_i$. Suppose there exists $Z_i' \in At(i)$ such that $s \in Z_i'$

and $Z'_i \subseteq T_i$. $Z_i \cap Z'_i \neq \varnothing$, since s belongs to both Z_i and Z'_i. Therefore, by atomicity, $Z_i = Z'_i$, and this holds for every i. ■

Lemma 4 allows us to write $Z_i(s)$ for every s and$i$ to be the unique subset in $At(i)$ containing s. For any non-empty coalition C, let us write $Z_C(s) = \bigcap_{i \in C} Z_i(s)$.

Let us now build the Boolean game $G^* = G(\mathrm{Eff})$ as follows. The intuition of the construction is that by the atoms of player i correspond to her strategies, and in order to ensure that the number of i's strategies is equal to the number of atoms for i, we introduce a suitable number of variables controlled by i, add a constraint that limits the number of i's strategies to the number of atoms. (We also give a detailed example after the formal construction.)

- for every i, consider the following numbering of $At(i)$: let r_i be a bijective mapping from $At(i)$ to $\{0, 1, \ldots, |At(i)| - 1\}$. Then create $p_i = \lceil \log_2 |At(i)| \rceil$ propositional variables $x_i^1, \ldots, x_i^{p_i}$. Finally, let $V = \{x_i^j | i \in N, 1 \leq j \leq p_i\}$;
- for every i, let $\pi_i = \{x_i^1, \ldots, x_i^{p_i}\}$;
- for every i and every $j \leq p_i$, let $\varepsilon_{i,j}$ be the jth digit in the binary representation of i. Note that $\varepsilon_{i,p_i} = 1$ by definition of p_i. If x is a propositional variable then we use the following notation: $0.x = \neg x$ and $1.x = x$. Then define
$$\gamma_i = \bigwedge_{j \in \{2, \ldots, p_i\}, \varepsilon_{i,j} = 0} \left(\bigwedge_{1 \leq k \leq j-1} \varepsilon_{i,j}.x_i^k \rightarrow \neg x_i^j \right)$$
- finally, for each $s \in S$, let $\mu(s) \in 2^V$ defined by: $x_i^j \in \mu(s)$ if and only if the jth digit of the binary representation of $r_i(Z_i(s))$ is 1.

For every $i \in N$ and every $Z \in At(i)$, let $k = r_i(Z)$ and $\sigma_i(Z)$ the strategy of Player i in G^* corresponding to the binary representation of k using $\{x_i^1, \ldots, x_i^{p_i}\}$, x_i^1 being the most significant bit. For instance, if $p_i = 3$ and $r(Z_i) = 6$ then $\sigma_i(Z) = (x_i^1, x_i^2, \neg x_i^3)$.

We denote by Σ_{G^*} the set of strategy profiles (or, equivalently, states) of G^*. Strategies of Σ_{G^*} are denoted by σ_{G^*}. The set of atoms of Eff_{G^*} is denoted by $At_{G^*}(i)$.

Since the decomposition of states into atoms is unique (Lemma 4), two different states s and s' are mapped to two different valuations, *i.e.*, $s \neq s'$ implies $\mu(s) \neq \mu(s')$. Now, for any $Z \in At(i)$, constraint γ_i ensures that $\sigma_i(Z) \models \gamma_i$; this being true for all i, for any σ we have $\mu(\sigma) \models \gamma_1 \wedge \ldots \wedge \gamma_n$. Therefore, μ is a bijection between S and $\Sigma_{G^*} = Mod(\gamma_1 \wedge \ldots \wedge \gamma_n)$.

To understand it better, it is helpful to see how this construction works on an example. Let $N = \{1,2,3\}$, $S = \{1,2,3,4,5,6,7,8,9,A,B,C\}$, $At(1) = \{1234, 5678, 9ABC\}$, $At(2) = \{13579B, 2468AC\}$, $At(3) = \{12569C, 3478AB\}$ (curly brackets for subsets of Σ are omitted – 1234 means $\{1,2,3,4\}$ and so on). By decomposability, we obtain:

– $At(12) = \{13,24,57,68,9B,AC\}$,

– $At(13) = \{12,34,56,78,9C,AB\}$, and

– $At(23) = \{159,37B,26C,48A\}$.

$|At(1)| = 3$, therefore $p_1 = 2$. $|At(2)| = |At(3)| = 2$, therefore $p_2 = p_3 = 1$. Thus, $V = \{x_1^1, x_1^2, x_2^1, x_3^1\}$. Let $At(1) = \{Z_0, Z_1, Z_2\}$, that is, $r_1(1234) = 0$, $r_1(5678) = 1$ and $r_1(9ABC) = 2$. Likewise, $r_2(13579B) = 0$, $r_2(2468AC) = 1$, $r_3(12569C) = 0$ and $r_3(3478AB) = 1$. Consider $s = 6$. We have $s = 5678 \cap 2468AC \cap 12569C$, therefore $\mu(s) = (x_1^1, \neg x_1^2, x_2^1, \neg x_3^1)$. The constraints are $\gamma_1 = (x_1^1 \to \neg x_1^2)$, $\gamma_2 = \gamma_3 = \top$. Thus, $G^* = (N, V, \pi, \Gamma)$ where $N = \{1,2,3\}$, $V = \{x_1^1, x_1^2, x_2^1, x_3^1\}$, $\pi_1 = \{x_1^1, x_1^2\}$, $\pi_2 = \{x_2^1\}$, $\pi_3 = \{x_3^1\}$, $\gamma_1 = (x_1^1 \to \neg x_1^2)$ and $\gamma_2 = \gamma_3 = \top$.

LEMMA 5. *For every $i \in N$ and $Z \in At(i)$: $\mu(Z) = \{\sigma_{G^*} \in \Sigma_{G^*} | \sigma_{G^*} \supseteq \sigma_i(Z)\}$.*

Proof: Let $i \in N$ and $Z \in At(i)$. Let $\sigma_{G^*} \in \mu(Z)$; by definition of $\mu(Z)$, there exists an $s \in S$ such that $\mu(s) = \sigma_{G^*}$. Consider the decomposition of s into atoms, thatis, $\{s\} = Z_1(s) \cap \ldots \cap Z_n(s)$ (cf. Lemma 4). By construction of μ, the projection of $\mu(s)$ on $\{x_i^1, \ldots, x_i^{p_i}\}$ corresponds to the binary representation of $r_i(Z_i(s))$. Therefore, $\mu(s) = \sigma_{G^*}$ extends $\sigma_i(Z)$.

Conversely, let σ_{G^*} such that $\sigma_{G^*} \supseteq \sigma_i(Z)$. For every $j \leq n$, let k_j be the number whose binary representation in $\{x_j^1, \ldots, x_j^{p_j}\}$ is the projection of σ_{G^*} on $\{x_j^1, \ldots, x_j^{p_j}\}$. Let σ be defined by $\{\sigma\} = Z_1(k_1) \cap \ldots \cap Z_n(k_n)$. By construction of μ, we have $\mu(\sigma) = \sigma_{G^*}$. Moreover, $Z_i(k_i) = Z$ by atomicity, that is, $\sigma \in Z$. Therefore $\sigma_{G^*} \in \mu(Z)$. ∎

We are now ready for establishing the main result of this section.

PROPOSITION 1. *An effectivity function* Eff *satisfies*

1. *strong playability,*
2. *atomicity,*
3. *decomposability and*
4. $\mathrm{Eff}(N) = 2^{\Sigma} \setminus \varnothing$

if and only if there exists a pre-Boolean game $G = (N, V, \pi, \Gamma)$ *and a bijective function* $\mu : \Sigma \to Mod(\Gamma)$ *such that for every* $C \subseteq N$*:* $\mathrm{Eff}_G(C) = \{\mu(X) | X \in \mathrm{Eff}(C)\}$.

Proof: The right-to-left direction is Lemma 2. In order to prove the opposite direction, we show that for every $C \subseteq N$ and every $X \subseteq S$, $X \in \mathrm{Eff}(C)$ holds if and only if $\mu(X) \in \mathrm{Eff}_{G^*}(C)$.

Decomposability of both Eff and Eff_{G^*} implies that it is enough to show that for every i and every $X \subseteq S$, $X \in \mathrm{Eff}(i)$ if and only if $\mu(X) \in \mathrm{Eff}_{G^*}(i)$. Because both Eff and Eff_{G^*} satisfy coalition monotonicity, it is enough to show that for every i, $Z_i \in At_{G^*}(i)$ implies $\mu(Z_i) \in \mathrm{Eff}_{G^*}(i)$ and $T_i \in At_{G^*}(i)$ implies $\mu^{-1}(T_i) \in \mathrm{Eff}(i)$.

Let $Z_i \in At(i)$. Because $r(Z_i) \leq p_i$, we have $\sigma_i(Z_i) \models \gamma_i$, therefore $\sigma_i(Z_i) \in \mathrm{Eff}_{G^*}(i)$. By Lemma 5, $\mu(Z_i) = \{\sigma_{G^*} | \sigma_{G^*} \supseteq \sigma_i(Z_i)\}$. Therefore, $\mu(Z_i) \in \mathrm{Eff}_{G^*}(i)$.

Conversely, let $T_i \in At_{G^*}(i)$. By Lemma 3, $T_i = \{\sigma | \sigma \supseteq \sigma_i\}$ for some $\sigma_i \in \Sigma_i$. Let $\sigma_i = (\varepsilon_{i,1}.x_1^1, \ldots, \varepsilon_{i,p_i}.x_i^{p_i})$ and $q(\sigma_i) = \sum_{k=1}^{p_i} 2^{p_i - k}.\varepsilon_{i,k}$. Note that $q(\sigma_i) \leq p_i$, because $\sigma_i \in \Sigma_i$ implies $\sigma_i \models \gamma_i$. Now, let $j = r_i^{-1}(q(\sigma_i))$. Let $Z_i^j \in At(i)$ such that $r_i(Z_i^j) = j$. We have $\mu(Z_i^j) = \{\sigma | \sigma \supseteq \sigma_i\} = T_i$. Now, $Z_i^j \in \mathrm{Eff}(i)$, because $Z_i^j \in At(i)$. Therefore, $\mu^{-1}(T_i) \in \mathrm{Eff}(i)$.

We have now proven that for $C \subseteq N$ and every $X \subseteq S$, $X \in \mathrm{Eff}(C)$ holds if and only if $\mu(X) \in \mathrm{Eff}_{G^*}(C)$. We can now conclude that if Eff satisfies strong playability, atomicity, decomposability, and $\mathrm{Eff}(N) = 2^{\Sigma} \setminus \varnothing$, then there exists a game $G (= G^*)$ and an bijective function $\mu : S \to Mod(\Gamma)$ such that for every $C \subseteq N$: $\mathrm{Eff}_G(C) = \{\mu(X) | X \in \mathrm{Eff}(C)\}$. ■

A natural question is, can we obtain a similar result without the need of constraints? This is obviously not the case, because for a Boolean game G without constraints, the cardinality of Σ_G is 2^m for some m. Therefore, requiring that the cardinality of S is a power of 2 is necessary. But it is not sufficient: because the cardinality of every Σ_i is also be a power of 2, this must also be the case for $At(i)$.

We say that Eff satisfies *regular atomicity* if it satisfies atomicity and for all $i \in N$, $|At(\{i\})| = 2^{m_i}$ for some positive integer m_i. We note that regular atomicity and decomposability implies that this cardinality property propagates to every coalition, *i.e.*, for all $C \subseteq N$, $|At(C)| = 2^{m_C}$ for some positive integer m_C.

Then we have the following:

PROPOSITION 2. *A coalitional* α*-effectivity function* Eff *satisfies*

1. *strong playability,*
2. *regular atomicity,*

3. *decomposability and*

4. $\mathrm{Eff}(N) = 2^S \setminus \varnothing$

if and only if there exists a constraint-free pre-Boolean game $G = (N, V, \pi, \top)$ *and an bijective function* $\mu : S \to 2^V$ *such that for every* $C \subseteq N$*:* $\mathrm{Eff}_G(C) = \{\mu(X) | X \in \mathrm{Eff}(C)\}$.

Proof: The proof is almost identical to the proof of Proposition 1. The only difference is that in the construction of G^* is unchanged except that we don't need to define Γ. Since $|At(\{i\})|$ is a power of 2, we have $p_i = \log |At(\{i\})|$, and μ is a bijection between S and $\Sigma_{G^*} = Mod(\gamma_1 \wedge \ldots \wedge \gamma_n) = Mod(\top) = 2^V$. ■

4. Efficient coalitions

We now consider Boolean games and define *efficient coalitions*. Informally, a coalition is efficient in a Boolean game if and only if it has the ability to jointly satisfy the goals of all members of the coalition. This notion of efficient coalition is not totally new, as is coincides with the notion of successful coalition in qualitative coalitional games (QCG) introduced in (Wooldridge and Dunne, 2004).

4.1. Definition and characterization

DEFINITION 8. *Let* $G = (N, V, \pi, \Gamma, \Phi)$ *be a Boolean game. A coalition* $C \subseteq N$ *is* **efficient** *if and only if there exists* $\sigma_C \in \Sigma_C$ *such that for all* σ_{-C}*, we have* $(\sigma_C, \sigma_{-C}) \models \bigwedge_{i \in C} \varphi_i$*. The set of all efficient coalitions of a game G is denoted by* $\mathrm{EC}(G)$*. C is a* **minimal efficient coalition** *if there is no efficient coalition* $B \subset C$*.*

Note that this definition still makes sense for quasi-dichotomous Boolean games; in this case, the cost c_i is irrelevant, and a coalition is efficient if it is able to jointly satisfy the goals of its members, whatever the induced costs (which, we recall, are always smaller than the utility gain resulting from goal satisfaction): formally, C is efficient for a quasi-dichotomous Boolean game G if and only if it is efficient for G^*. (However, it no longer makes sense for generalized Boolean games with arbitrary utility functions.)

Note that the empty coalition $\varnothing$ is efficient, because $\varphi_\varnothing = \bigwedge_{i \in \varnothing} \varphi_i \equiv \top$ is always satisfied.

EXAMPLE 1. *Let* $G = (N, V, \Gamma, \pi, \Phi)$ *where* $V = \{a, b, c\}$, $N = \{1, 2, 3\}$, $\gamma_i = \top$ *for every* i, $\pi_1 = \{a\}$, $\pi_2 = \{b\}$, $\pi_3 = \{c\}$, $\varphi_1 = (\neg a \wedge b)$, $\varphi_2 = (\neg a \vee \neg c)$ *and* $\varphi_3 = (\neg b \wedge \neg c)$.

First note that $\varphi_1 \wedge \varphi_3$ *is inconsistent, therefore no coalition containing* $\{1,3\}$ *can be efficient.* $\{1\}$ *is not efficient, because* φ_1 *cannot be made true only by fixing the value of a; similarly,* $\{2\}$ *and* $\{3\}$ *are not efficient either.* $\{1,2\}$ *is efficient, because the joint strategy* $\sigma_{\{1,2\}} = \bar{a}b$ *is such that* $\sigma_{\{1,2\}} \models \varphi_1 \wedge \varphi_2$. $\{2,3\}$ *is efficient, because* $\sigma_{\{2,3\}} = \bar{b}\bar{c} \models \varphi_2 \wedge \varphi_3$. *Therefore,* $\mathrm{EC}(G) = \{\varnothing, \{1,2\}, \{2,3\}\}$.

From this simple example we see already that EC is neither downward closed nor upward closed, that is, if C is efficient, then a subset or a superset of C may not be efficient. We also see that EC is not closed under union or intersection: $\{1,2\}$ and $\{2,3\}$ are efficient, but neither $\{1,2\} \cap \{2,3\}$ nor $\{1,2\} \cup \{2,3\}$ is.

EXAMPLE 2 (kidney exchange, after (Abraham et al., 2007)).

Consider n pairs of individuals, each consisting of a recipient R_i *in urgent need of a kidney transplant, and a donor* D_i *who is ready to give one of her kidneys to save* R_i. *Because the kidney of donor* D_i *is not necessarily compatible with recipient* R_i, *a strategy for saving more people consists in considering the graph* $\langle\{1, \ldots, n\}, E\rangle$ *containing a node* $i \in 1, \ldots, n$ *for each pair* (D_i, R_i) *and containing the edge* (i, j) *whenever* D_i*'s kidney is compatible with* R_j. *A solution is any set of nodes that can be partitioned into disjoint cycles in the graph: in a solution, Donor* D_i *gives a kidney if and only if* R_i *gets one. An optimal solution (saving a maximum number of lifes) is a solution with a maximum number of nodes. The problem can be seen as the following Boolean game G:*

- $N = \{1, \ldots, n\}$;
- $V = \{g_{ij} | i, j \in \{1, \ldots, n\}\}$; g_{ij} *being true means that* D_i *gives a kidney to* R_j.
- $\pi_i = \{g_{ij}; 1 \leq j \leq n\}$;
- *for every* i, $\gamma_i = \bigwedge_{j \neq k} \neg(g_{ij} \wedge g_{ik})$ *expresses that a donor cannot give more than one kidney.*
- *for every* i, $\varphi_i = \bigvee_{(j,i) \in E} g_{ji}$ *expresses that the goal of* i *is to get a kidney that is compatible with* R_i.

For example, take $n = 5$ *and* $E = \{(1,1), (1,2), (2,3), (2,4), (2,5), (3,1), (4,2), (5,4)\}$. *Then* $G = (N, V, \Gamma, \pi, \Phi)$, *with*

- $N = \{1, 2, 3, 4, 5\}$

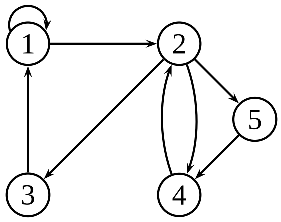
– $V = \{g_{ij} \mid 1 \le i, j \le 5\}$;

– $\forall i, \gamma_i = \bigwedge_{j \neq k} \neg(g_{ij} \wedge g_{ik})$

– $\pi_1 = \{g_{11}, g_{12}, g_{13}, g_{14}, g_{15}\}$, *and similarly for* π_2, *etc.*

– $\varphi_1 = g_{11} \vee g_{31}$; $\varphi_2 = g_{12} \vee g_{42}$; $\varphi_3 = g_{23}$; $\varphi_4 = g_{24} \vee g_{54}$; $\varphi_5 = g_{25}$.

The corresponding graph is depicted below.



Clearly enough, efficient coalitions correspond to solutions. In our example, the efficient coalitions are $\varnothing$, $\{1\}$, $\{2,4\}$, $\{1,2,4\}$, $\{1,2,3\}$, $\{2,4,5\}$ *and* $\{1,2,4,5\}$.

We have seen that the set of efficient coalitions associated with a Boolean game may not be downward closed nor upward closed, nor closed under union or non-empty intersection. However, it is possible to characterize the efficient coalitions of a Boolean game: we will show that a set of coalitions corresponds to the set of efficientcoalitions for some Boolean game if and only if (a) it contains the emptyset and (b) it is closed by union of disjoint coalitions.

We will prove this characterization result in the rest of this subsection. To do so, we will need Lemmas 6 to 9, which will we establish first.

LEMMA 6. *Let I, J be two coalitions of a Boolean game G. If I and J are efficient and $I \cap J = \varnothing$, then $I \cup J$ is efficient.*

Proof: If I is efficient, then we know that $\exists \sigma_I \in \Sigma_I$ such that $\sigma_I \models \bigwedge_{i \in I} \varphi_i$, and the same for J: $\exists \sigma_J \in \Sigma_J$ such that $\sigma_J \models \bigwedge_{j \in J} \varphi_j$. Moreover, as $I \cap J = \varnothing$, we have $(\sigma_I, \sigma_J) \models \bigwedge_{i \in I \cup J} \varphi_i$, so $I \cup J$ is an efficient coalition.

■

We now need to define the following construction. Let $\mathcal{SC}$ be a set of coalitions satisfying the following conditions:

(1) $\varnothing \in \mathcal{SC}$.

(2) for all $I, J \in \mathcal{SC}$ such that $I \cap J = \varnothing$ then $I \cup J \in \mathcal{SC}$.

Define the following Boolean game G as follows:

- $V = \{connect(i,j) | i,j \in N\}$ (all possible connections between players);
- $\forall i, \gamma_i = \top$;
- $\pi_i = \{connect(i,j) | j \in N\}$ (all connections from Player i);
- $\varphi_i = \bigvee_{I \in \mathcal{SC} | i \in I} F_I$,
  where

  $$F_I = \left(\bigwedge_{j,k \in I} connect(j,k) \right) \bigwedge \left(\bigwedge_{j \in I, k \notin I} \neg connect(j,k) \right)$$

  (Player i wants that all the players of her coalition are interconnected and that there is no connection from the coalition to the "outside" of the coalition)

We want to show that the set $\mathrm{EC}_G = \mathcal{SC}$ (where EC_G is the set of efficient coalitions for G).

Before proving that $\mathrm{EC}_G \subseteq \mathcal{SC}$, we establish the following lemmas:

LEMMA 7. *For any collection $\mathcal{SC} = \{C_i, i = 1, \ldots, q\} \subseteq 2^{2^N}$, $\bigwedge_{1 \leq i \leq q} F_{C_i}$ is satisfiable if and only if for any $i, j \in \{1, \ldots, q\}$, either $C_i = C_j$ or $C_i \cap C_j = \varnothing$.*

Proof:

1. Assume that for any $i, j \in \{1, \ldots, q\}$, either $C_i = C_j$ or $C_i \cap C_j = \varnothing$. Then $\bigwedge_{1 \leq i \leq q} F_{C_i}$ is equivalent to

   $$\left(\bigwedge_{1 \leq i \leq q} \bigwedge_{j,k \in C_i} connect(j,k) \right) \bigwedge \left(\bigwedge_{1 \leq i \leq q} \bigwedge_{j \in C_i, k \notin C_i} \neg connect(j,k) \right)$$

   $\bigwedge_{1 \leq i \leq q} F_{C_i}$ is satisfied by any interpretation assigning each $connect(j,k)$ such that j,k belong to the same C_i to true, and each $connect(j,k)$ such that $j \in C_i$ for some i and $k \notin C_i$ to false. Hence $\bigwedge_{1 \leq i \leq q} F_{C_i}$ is satisfiable.

2. Assume that for some $i, j \in \{1, \ldots, q\}$, we have $C_i \cap C_j \neq \varnothing$ and $C_i \neq C_j$. Let $k \in C_i \cap C_j$ and (without loss of generality) $l \in C_i \setminus C_j$. Then $F_{C_i} \models connect(k,l)$ and $F_{C_j} \models \neg connect(k,l)$, hence $F_{C_i} \wedge F_{C_j}$ is unsatisfiable, and a fortiori, so is $\bigwedge_{1 \leq i \leq q} F_{C_i}$.

■

We now define a *covering of a coalition I by disjoint subsets of $\mathcal{SC}$* as a tuple $\vec{C} = \langle C_i | i \in I \rangle$ of coalitions such that: **(i)** for every $k \in I$, $C_k \in \mathcal{SC}$; **(ii)** for all $C_j, C_k \in \vec{C}$, either $C_j = C_k$ or $C_j \cap C_k = \varnothing$; **(iii)** for every $i \in I$, $i \in C_i$.

Let $Cov(\mathcal{SC}, I)$ be the set of all covering of I by disjoint subsets of $\mathcal{SC}$. For instance, if $\mathcal{SC} = \{1, 24, 123, 124\}$ then $Cov(\mathcal{SC}, 12) = \{\langle 1, 24\rangle, \langle 123, 123\rangle, \langle 124, 124\rangle\}^3$, $Cov(\mathcal{SC}, 123) = \{\langle 123, 123, 123\rangle\}$, $Cov(\mathcal{SC}, 124) = \{\langle 1, 24, 24\rangle, \langle 124, 124, 124\rangle\}$ and $Cov(\mathcal{SC}, 234) = Cov(\mathcal{SC}, 1234) = \varnothing$.

LEMMA 8. *For any $I \neq \varnothing$, Φ_I is equivalent to $\bigvee_{\vec{C} \in Cov(\mathcal{SC}, I)} \bigwedge_{i \in I} F_{C_i}$.*

Proof:

$$\begin{aligned} \Phi_I &\equiv \textstyle\bigwedge_{i \in I} \varphi_i \\ &\equiv \textstyle\bigwedge_{i \in I} \bigvee_{J \in \mathcal{SC} \mid i \in J} F_J \\ &\equiv \textstyle\bigvee_{\langle C_i, i \in I\rangle \text{ such that } C_i \in \mathcal{SC} \text{ and } i \in C_i \text{ for every } i \in I} \bigwedge_{i \in I} F_{C_i} \end{aligned}$$

Now, by Lemma 7, $\bigwedge_{i \in I} F_{C_i}$ is satisfiable if and only if for all $i, j \in I$, either $C_i = C_j$ or $C_i \cap C_j = \varnothing$. Therefore, $\Phi_I \equiv \bigvee_{\vec{C} \in Cov(\mathcal{SC}, I)} \bigwedge_{i \in I} F_{C_i}$.

■

For instance, if $\mathcal{SC} = \{1, 24, 123, 124\}$ then $\Phi_{12} \equiv (F_1 \wedge F_{24}) \vee F_{123} \vee F_{124}$; $\Phi_{123} \equiv F_{123}$; $\Phi_{124} \equiv (F_1 \wedge F_{24}) \vee F_{124}$; $\Phi_{234} \equiv \bot$.

LEMMA 9. *Let $I \subseteq 2^N$. As $\forall I, J \in \mathcal{SC}$, $I \cap J = \varnothing \Rightarrow I \cup J \in \mathcal{SC}$, Φ_I is satisfiable if and only if there exists $J \in \mathcal{SC}$ such that $I \subseteq J$.*

Proof: The case $I = \varnothing$ is straightforward: $\Phi \equiv \top$ is satisfiable, and $\varnothing \in \mathcal{SC}$ by assumption, therefore there exists $J \in \mathcal{SC}$ ($J = \varnothing$) such that $I \subseteq J$.

Now, let $I \neq \varnothing$.

$\Rightarrow$ Assume Φ_I is satisfiable. By Lemma 8, Φ_I is equivalent to

$$\bigvee_{\vec{C} \in Cov(\mathcal{SC}, I)} \bigwedge_{i \in I} F_{C_i}$$

therefore there exists a $\vec{C}$ in $Cov(\mathcal{SC}, I)$ such that $\bigwedge_{i \in I} F_{C_i}$ is satisfiable, therefore $Cov(\mathcal{SC}, I)$ is not empty. Now, $\vec{C} \in Cov(\mathcal{SC}, I)$ implies that:

(i) for every $i \in I$, $C_i \in \mathcal{SC}$;

(ii) for every $i, j \in I$, either $C_i = C_j$ or $C_i \cap C_j = \varnothing$.

(iii) $I \subseteq \bigcup_{i \in I} C_i$

Now, (i), (ii) and $\forall I, J \in \mathcal{SC}, I \cap J = \varnothing \Rightarrow I \cup J \in \mathcal{SC}$ imply that $\bigcup_{i \in I} C_i \in \mathcal{SC}$, which together with (iii) proves that there exists a $J \in \mathcal{SC}$ (namely $J = \bigcup_{i \in I} C_i$) such that $I \subseteq J$.

$\Leftarrow$ Assume that there is a $J \in \mathcal{SC}$ such that $I \subseteq J$. Then $\Phi_J \models \Phi_I$, and Φ_J is consistent (consider the interpretation assigning each $connect(i, j)$ such that $i, j \in J$ to true).

■

We can now establish the characterization of the efficient coalitions of a Boolean game.

PROPOSITION 3. *Let $N = \{1, \ldots, n\}$ be a set of agents and $\mathcal{SC} \in 2^{2^N}$ a set of coalitions. There exists a Boolean game G over N such that the set of efficient coalitions for G is $\mathcal{SC}$ (i.e.* $\mathrm{EC}(G) = \mathcal{SC}$*) if and only if $\mathcal{SC}$ satisfies these two properties:*

(1) $\varnothing \in \mathcal{SC}$.

(2) *for all $I, J \in \mathcal{SC}$ such that $I \cap J = \varnothing$ then $I \cup J \in \mathcal{SC}$.*

Proof: Lemma 6 proves the ($\Rightarrow$) direction of Proposition 3. For the ($\Leftarrow$) direction, we want to show that $\mathcal{SC} \subseteq \mathrm{EC}_G$.

We first show that $\mathcal{SC} \subseteq \mathrm{EC}_G$. Let $I \in \mathcal{SC}$. If every agent $i \in I$ plays $\left(\bigwedge_{j \in I} connect(i,j)\right) \wedge \left(\bigwedge_{k \notin I} \neg connect(i,k)\right)$, then φ_i is satisfied for every $i \in I$. Hence, I is an efficient coalition for G and $\mathcal{SC}$ is included in $\mathrm{EC}(G)$.

It remains to be shown that $\mathrm{EC}_G \subseteq \mathcal{SC}$. Let I be a coalition such that $I \notin \mathcal{SC}$ (which implies $I \neq \varnothing$, because of assumption $\varnothing \in \mathcal{SC}$).

- If $I = N$ then there is no $J \in \mathcal{SC}$ such that $I \subseteq J$ (because $I \notin \mathcal{SC}$), and then Lemma 9 implies that Φ_I is unsatisfiable, therefore I cannot be efficient for G.

- Assume now that $I \neq N$ and define the following $\overline{I}$-strategy $\Sigma_{\overline{I}}$ ($\overline{I} = N \setminus I$): for every $i \in \overline{I}$, σ_i is such that for all $j \in I$, σ_i contains $\neg connect(i,j)$ (whether $connect(i,j)$ is true or false for $j \notin I$ is irrelevant). Let $\vec{C} = \langle C_i, i \in I \rangle \in Cov(\mathcal{SC}, I)$.

  We first claim that there exists $i^* \in I$ such that C_{i^*} is not contained in I. Indeed, suppose that for every $i \in I$, $C_i \subseteq I$. Then, because $i \in C_i$ holds for every i, we have $\bigcup_{i \in I} C_i = I$. Now, $C_i \in \mathcal{SC}$ for all i, and any two distinct C_i, C_j are disjoint, therefore, by Property (2) of Proposition 3, we get $I \in \mathcal{SC}$, which by assumption is false.

  Now, let $k \in C_{i^*} \setminus I$ (such a k exists because C_{i^*} is not contained in I). As i and k are in C_i, $connect(k, i^*)$ has to be true to satisfy F_{C_i}. Therefore $\sigma_k \models \neg F_{C_i}$, and a fortiori $\sigma_{\overline{I}} \models \neg F_{C_i}$, which entails $\sigma_{\overline{I}} \models \neg \bigwedge_{i \in I} F_{C_i}$.

  This being true for any $\vec{C} \in Cov(\mathcal{SC}, I)$, we have

$$\sigma_{\overline{I}} \models \bigwedge_{\vec{C} \in Cov(\mathcal{SC}, I)} \neg \bigwedge_{i \in I} F_{C_i}$$

that is, $\sigma_{\bar{I}} \models \neg \bigvee_{\vec{C} \in Cov(\mathcal{S}C, I)} \bigwedge_{i \in I} F_{C_i}$. Together with Lemma 8, this entails $\sigma_{\bar{I}} \models \neg \Phi_I$. Hence, I does not control Φ_I and I cannot be efficient for G.

■

4.2. Efficient coalitions and the core

We now relate the notion of efficient coalitions to the usual notion of core of a coalitional game. In coalitional games with ordinal preferences, the core is usually defined as follows (see e.g. (Aumann, 1967; Owen, 1982; Myerson, 1991)): a strategy profile σ is in the core of a coalitional game if and only if there exists no coalition C with a joint strategy σ_C that guarantees that all members of C are better off than with σ. Here we consider also a stronger notion of core: a strategy profile σ is in the strong core of a coalitional game if and only if there exists no coalition C with a joint strategy σ_C that guarantees that all members of C are at least as satisfied as with σ, and at least one member of C is strictly better off than with σ.

DEFINITION 9. *Let G be a Boolean game. The* **(weak) core** *of G, denoted by $WCore(G)$, is the set of strategy profiles $\sigma = (\sigma_1, \ldots, \sigma_n)$ such that there exists no $C \subseteq N$ and no $\sigma_C \in \Sigma_C$ such that for every $i \in C$ and every $\sigma_{-C} \in \Sigma_{-C}$, $(\sigma_C, \sigma_{-C}) \succ_i \sigma$.*
The **strong core** *of a Boolean game G, denoted by $SCore(G)$, is the set of strategy profiles $\sigma = (\sigma_1, \ldots, \sigma_n)$ such that there exists no $C \subseteq N$ and no $\sigma_C \in \Sigma_C$ such that for every $i \in C$ and every $\sigma_{-C} \in \Sigma_{-C}$, $(\sigma_C, \sigma_{-C}) \succeq_i \sigma$ and there is an $i \in C$ such that for every $\sigma_{-C} \in \Sigma_{-C}$, $(\sigma_C, \sigma_{-C}) \succ_i \sigma$.*

Obviously enough, this notion of weak core is equivalent to the notion of **strong Nash equilibrium** (Aumann, 1959), where coalitions form in order to correlate the strategies of their members.

The relationship between the (weak) core of a Boolean game and its set of efficient coalitions is expressed by the following simple result.

PROPOSITION 4. *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game. $\sigma \in WCore(G)$ if and only if σ satisfies at least one member of every efficient coalition, that is, for every $C \in \mathrm{EC}(G)$, $\sigma \models \bigvee_{i \in C} \varphi_i$.*

Proof: $\sigma = (\sigma_1, \ldots, \sigma_n) \notin WCore(G)$ if and only if there exist a coalition $C \subseteq N$ and a tuple $\sigma_C \in \Sigma_C$ such that (1) for every $i \in C$ and $\sigma_{-C} \in \Sigma_{-C}$, we have $(\sigma_C, \sigma_{-C}) \succ_i \sigma$. Using the specific form that utility functions have in Boolean games, (1) is equivalent to (2a) $\forall \sigma_{-C}$, $(\sigma_C, \sigma_{-C}) \models \bigwedge_{i \in C} \varphi_i$ and (2b) $\sigma \models \bigwedge_{i \in C} \neg \varphi_i$. As $\{\pi_1, \ldots, \pi_n\}$ forms a partition of V, (2a) can be written as

$\sigma_C \models \bigwedge_{i\in C} \varphi_i$. Therefore, $\sigma \in WCore(G)$ if and only if (3) for every $C \subseteq N$, either $\sigma \models \bigvee_{i\in C} \varphi_i$ or for every $\sigma_C \in \Sigma_C$, $\sigma_C \models \bigvee_{i\in C} \neg\varphi_i$. (3) can be rewritten into (4): for every $C \subseteq N$, if there exists σ_C such that $\sigma_C \models \bigwedge_{i\in C} \varphi_i$ then $\sigma \models \bigvee_{i\in C} \varphi_i$. Now, the existence of $\sigma_C \in \Sigma_C$ such that $\sigma_C \models \bigwedge_{i\in C} \varphi_i$ means that Coalition C is efficient. Therefore, $\sigma \in WCore(G)$ if and only if for every $C \subset N$, if $C \in \mathrm{EC}(G)$ then $\sigma \models \bigvee_{i\in C} \varphi_i$.

■

In particular, when no coalition of a Boolean game G is efficient, then all strategy profiles are in $WCore(G)$.

Moreover, the weak core of a Boolean game cannot be empty:

PROPOSITION 5. *For any Boolean game G, $WCore(G) \neq \varnothing$.*

Proof: We construct the following set of coalitions E as follows. First, initialize E to $\varnothing$. Then, while there exists a coalition C in $\mathrm{EC}(G)$ such that $C \cap C' = \varnothing$ holds for every $C' \in E$, pick such a C and add it to E. At the end of the algorithm, E is a set of disjoint efficient coalitions $\{C_I, i \in I\}$, therefore, by Proposition 3, $\cup_{i\in I} C_i$ is efficient. Therefore, there exists $\sigma_E \in \Sigma_E$ such that $\sigma_E \models \bigwedge_{i\in E} \varphi_i$, and E contains at least one element of every efficient coalition (if this were not the case, there would remain an efficient coalition C that intersects none of the C_i's, and the algorithm would have continued and incorporated C into E). Let σ extending σ_E. σ satisfies at least one member of every efficient coalition, therefore, by Proposition 4, $WCore(G) \neq \varnothing$.

■

The strong core of a Boolean game is harder to characterize in terms of efficient coalitions. We only have the following implication.

PROPOSITION 6. *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game, and σ be a strategy profile. If $\sigma \in SCore(G)$ then for every $C \in \mathrm{EC}(G)$ and every $i \in C$, $\sigma \models \varphi_i$.*

Proof: Let $C \in \mathrm{EC}(G)$ and assume there exists $i \in C$ such that (1) $\sigma \models \neg\varphi_i$. We want to show that $\sigma \notin SCore(G)$. Since $C \in \mathrm{EC}(G)$, (2) there exists $\sigma_C \in \Sigma_C$ such that $\sigma_C \models \bigwedge_{j\in C} \varphi_j$. Applying (1) and (2) to i leads to $\sigma_C \succ_i \sigma$, while applying (1) and (2) to $j \in C \setminus \{i\}$ leads to $\sigma_C \succeq_j \sigma$. Therefore, $\sigma \notin SCore(G)$.

■

Thus, a strategy in the strong core of G satisfies the goal of every member of every efficient coalition. The following counterexample shows that the converse does not hold.

EXAMPLE 3. *Let $G = (N, V, \Gamma, \pi, \Phi)$ be the following Boolean game: $V = \{a, b, c, d, e\}$, $N = \{1, 2, 3, 4, 5\}$, $\gamma_i = \top$ for every i, $\pi_1 = \{a\}$, $\pi_2 = \{b\}$, $\pi_3 = \{c\}$, $\pi_4 = \{d\}$, $\pi_5 = \{e\}$, $\varphi_1 = \neg a \wedge b$, $\varphi_2 = \neg a$, $\varphi_3 = d$, $\varphi_4 = c \wedge a$ and $\varphi_5 = c \wedge e$.*

This game has one efficient coalition: $\{1,2\}$.
Let $\sigma = \overline{a}bc\overline{d}\overline{e}$. *We have* $\sigma \models \varphi_1 \wedge \varphi_2 \wedge \neg\varphi_3 \wedge \neg\varphi_4 \wedge \neg\varphi_5$. *Therefore,* $\forall C \in \mathrm{EC}(G)$, $\forall i \in C$, $\sigma \models \varphi_i$.
However, $\sigma \notin SCore(G)$*:* $\exists C' = \{1,2,3,4\} \subset N$ *such that* $\exists \sigma_C = \overline{a}bcd \models \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \neg\varphi_4$. *So,* $\forall \sigma_{-C}$, $(\sigma_C, \sigma_{-C}) \succeq_1 \sigma$, $(\sigma_C, \sigma_{-C}) \succeq_2 \sigma$, $(\sigma_C, \sigma_{-C}) \succeq_4 \sigma$, *and* $(\sigma_C, \sigma_{-C}) \succ_3 \sigma$. $\sigma \notin SCore(G)$.

Note that the strong core of a Boolean game can be empty: in Example 1, the set of efficient coalitions is $\{\varnothing, \{1,2\}, \{2,3\}\}$, therefore there is no $\sigma \in \Sigma$ such that for all $C \in \mathrm{EC}(G)$, for all $i \in C$, $\sigma \models \varphi_i$, therefore, by Proposition 6, $SCore(G) = \varnothing$. However, we can show that the non-emptiness of the strong core is equivalent to the following simple condition on efficient coalitions.

PROPOSITION 7. *Let* $G = (N, V, \Gamma, \pi, \Phi)$ *be a Boolean game. We have the following:*

$Score(G) \neq \varnothing$ if and only if $\bigcup\{C \subseteq N | C \in \mathrm{EC}(G)\} \in \mathrm{EC}(G)$ – that is, if and only if the union of all efficient coalitions is efficient.

Proof: Let $MEC(G) = \bigcup_{C \subseteq N}\{C \in \mathrm{EC}(G)\}$.

$\Leftarrow$ $Score(G) \neq \varnothing$. Let $\sigma \in Score(G)$. From Proposition 6, we know that $\forall C \in \mathrm{EC}(G), \forall i \in C, \sigma \models \varphi_i$. So, $\forall i \in MEC(G)$, $\sigma \models \varphi_i$. So, $MEC(G) \in \mathrm{EC}(G)$.

$\Rightarrow$ $MEC(G) \in \mathrm{EC}(G)$. Let $\sigma_{MEC(G)} \in \Sigma_{MEC(G)}$ such that $\forall \sigma_{-MEC(G)}, \sigma_{MEC(G)} \models \Phi_{MEC(G)}$. We are looking for σ such that $\sigma \in Score(G)$.

Let $\sigma_{-MEC(G)} \in \Sigma_{-MEC(G)}$ such that $MAX = \{i | \sigma = (\sigma_{MEC(G)}, \sigma_{-MEC(G)}) \models \varphi_i\}$ be maximal for $\subseteq$. $\sigma_{-MEC(G)}$ exists, in worst case $\sigma \models \Phi_{MEC(G)}$. As MAX is maximal, we cannot find $C \subseteq N$ such that $\exists \sigma_C \in \Sigma_C$, such that $\forall \sigma_{-C} \in \Sigma_{-C}, \forall i \in C, (\sigma_C, \sigma_{-C}) \succeq_i \sigma$, and $\exists i \in C$, $(\sigma_C, \sigma_{-C}) \succ_i \sigma$. Indeed, if we assume that this coalition C exists, then $\forall i \in N$ such that $\sigma \models \varphi_i$, we have $\sigma_C \models \varphi_i$, and $\exists i \in N$ such that $\sigma \not\models \varphi_i$ and $\sigma_C \models \varphi_i$. In this case, MAX is not maximal for $\subseteq$.

■

5. Computational complexity of reasoning about efficient coalitions

We start by identifying the complexity of some key decision problems related to efficient coalitions. The key questions are: is a given coalition efficient? does there exist a nonempty efficient coalition? is a given agent member of some efficient coalition? of all nonempty efficient coalitions? In addition to these problems that are directly related to efficient coalitions, similar problems arise for the notions of weak and strong core.

PROPOSITION 8. *Deciding whether a given coalition is efficient for a Boolean game is Σ_2^p-complete, and is Σ_2^p-hard even if $n = 2$ and the coalition is a singleton.*

Proof: Membership is straightforward and hardness is a straightforward consequence of the facts that (1) the coalition reduced to the singleton $\{i\}$ is efficient if and only if i has a winning strategy and that (2) deciding whether an agent has a winning strategy in a Boolean game is Σ_2^p-complete (see (Bonzon et al., 2009)). ■

The next result addresses the problem whether there exists an efficient coalition in a Boolean game.

PROPOSITION 9. *Deciding whether there exists a non-empty efficient coalition in a Boolean game is Σ_2^p-complete, and is Σ_2^p-hard even if $n = 2$.*

Proof: Membership to Σ_2^p is immediate.

To show that deciding whether there is a non-empty efficient coalition in a Boolean game is Σ_2^p-hard (even with 2 agents), consider the following polynomial reduction from $\text{QBF}_{2,\exists}$. To each instance $Q = \exists a_1 \ldots a_p \forall b_1 \ldots b_q \varphi$ of $\text{QBF}_{2,\exists}$, let us consider the following Boolean game $G_Q = \langle N, V, \pi, \Gamma, \Phi \rangle$, where $N = \{1,2\}$, $\gamma_1 = \gamma_2 = \top$, $V = \{a_1 \ldots a_p, b_1 \ldots b_q, x\}$, $\pi_1 = \{a_1, \ldots, a_p, x\}$, $\pi_2 = \{b_1, \ldots, b_q\}$, $\varphi_1 = \varphi$ and $\varphi_2 = \neg\varphi \wedge x$. Neither $\{2\}$ nor $\{1,2\}$ can be efficient; therefore, the only possible nonempty efficient coalition is $\{1\}$. Now, it is easily seen that $\{1\}$ is efficient if and only if Q is a valid instance of $\text{QBF}_{2,\exists}$. ■

We now consider the problems of determining whether a given agent belongs to some efficient coalition, and whether she belongs to all nonempty efficient coalitions.

PROPOSITION 10.

– *deciding whether an agent i belongs to at least one efficient coalition of a Boolean game is Σ_2^p-complete, and is Σ_2^p-hard even if $n = 2$.*

– *deciding whether an agent i belongs at all nonempty efficient coalitions of a Boolean game is Π_2^p-complete, and is Π_2^p-hard even if $n = 2$.*

Proof: For both problems, the membership part of the proof is easy.

To show that deciding whether an agent i belongs to at least one efficient coalition for G is Σ_2^p-hard (even with 2 agents), consider the following polynomial reduction from $\text{QBF}_{2,\exists}$. To each instance $Q = \exists a_1 \ldots a_p \forall b_1 \ldots b_q \varphi$

of QBF$_{2,\exists}$, let us consider the following Boolean game $G_Q = \langle N, V, \pi, \Gamma, \Phi\rangle$, where $N = \{1,2\}, \gamma_1 = \gamma_2 = \top, V = \{a_1,\dots,a_p,b_1,\dots,b_q\}, \pi_1 = \{a_1,\dots,a_p\}$, $\pi_2 = \{b_1,\dots,b_q\}$, $\varphi_1 = \varphi$ and $\varphi_2 = \neg\varphi$. $\{1\}$ is efficient if and only if there exists a strategy σ_1 such that $\sigma_1 \models \varphi$, that is, if and only if I is valid. Now, $\{1,2\}$ cannot be efficient, because $\varphi_1 \wedge \varphi_2 = \top$. therefore, 1 belongs to an efficient coalition if and only if $\{1\}$ is efficient, that is, if and only if Q is valid.

To show that deciding whether an agent i belongs to all nonempty efficient coalitions for G is Π_2^p-hard (even with 2 agents), consider the following polynomial reduction from QBF$_{2,\forall}$. To each instance $Q = \exists a_1 \dots a_p \forall b_1 \dots b_q \varphi$ of QBF$_{2,\forall}$, let us consider the following Boolean game $G_Q = \langle N, V, \pi, \Gamma, \Phi\rangle$, where $N = \{1,2\}, \gamma_1 = \gamma_2 = \top, V = \{a_1 \dots a_p, b_1 \dots b_q, x\}, \pi_1 = \{a_1,\dots,a_p,x\}$, $\pi_2 = \{b_1,\dots,b_q\}$, $\varphi_1 = \neg\varphi$ and $\varphi_2 = \varphi \wedge x$. Neither $\{2\}$ nor $\{1,2\}$ can be efficient; therefore, 2 belongs to all nonempty efficient coalitions if and only if $\{1\}$ is not efficient, that is, if $\exists a_1 \dots a_p \forall b_1 \dots b_q \neg\varphi$ is not valid, or equivalently, if $\forall a_1 \dots a_p \exists b_1 \dots b_q \varphi$ is valid. ∎

Although we have stated them and proven them for standard Boolean games with dichotomous utilities, Propositions 8, 9 and 10 hold also for quasi-dichotomous Boolean games. This is trivially obtained from the fact that a coalition is efficient for a quasi-Boolean game G if and only if it is efficient for G^{*}[4] .

Given the strong relationships between efficient coalitions and the notions of weak and strong core of a Boolean game, these results allow us to derive complexity results regarding these. Note however that unlike the previous three propositions, the following three hold for standard Boolean games (with dichotomous preferences) only.

First, Proposition 4 leads the following result:

PROPOSITION 11. *Deciding if a strategy profile σ is in the weak core of a Boolean game G is Π_2^p-complete.*

Proof: Recall that $\sigma \notin WCore(G)$ if and only if there exists a coalition $C \subseteq N$ such that (a) $\sigma \models \bigwedge_{i \in C} \neg\varphi_i$ and (b) there exists a strategy $\sigma_C \in \Sigma_C$ such that $\sigma_C \models \varphi_C = \bigwedge_{i \in C} \varphi_i$.

Membership to Π_2^p is immediate, as the formulation above immediately shows that the problem to decide if a strategy profile σ is *not* in $WCore(G)$ is in Σ_2^p.

Hardness is obtained by proving that the complementary problem is Σ_2^p-complete, using a reduction of the problem of deciding the validity of a QBF$_{2,\exists}$.

Given $Q = \exists A, \forall B, \Phi$, where A and B are disjoint sets of variables and Φ is a formula of $L_{A \cup B}$, we define the following Boolean game G_Q by

– $V = A \cup B \cup \{c\}$, where c is a fresh variable ($c \notin A \cup B$);

– $N = \{p_v \mid v \in V\}$;

– for every $v \in V$, $\pi(p_v) = \{v\}$;

– for every $a_i \in A$, $\varphi_{a_i} = \varphi \wedge c$;

– for every $b_j \in B$, $\varphi_{b_j} = \top$;

– $\varphi_c = \varphi \wedge c$;

– σ is any assignment satisfying $\neg c$.

Assume Q is a positive instance of $\mathtt{QBF}_{2,\exists}$, that is, there exists a $\vec{a}$ such that for every $\vec{b}$ we have $(\vec{a},\vec{b}) \models \varphi$. Let $C = A \cup \{c\}$ and $\sigma_C = (\vec{a}, c)$. We have $\sigma_C \models \bigwedge_{v \in C} \varphi_v = \varphi \wedge c$ and $\sigma \models \bigwedge_{v \in C} \neg\varphi_v$. Therefore, $\sigma \notin WCore(G_Q)$.

Conversely, assume that there exists a coalition $C \subseteq N$ and a strategy $\sigma_C \in \Sigma_C$ such that $\sigma_C \models \varphi_C = \bigwedge_{i \in C} \varphi_i$. Note that because $\sigma \models \neg c$, we have $\sigma \models \bigwedge_{i \in C} \neg\varphi_i$, thus condition (a) is satisfied. Because $\sigma \models \neg\varphi_i$ for any $i \in C$, we must have $C \subseteq A \cup \{c\}$, which implies that $\sigma_C = c \wedge \varphi$. Define $\vec{a} \in \Sigma_A$ such that $\vec{a}$ and σ_C agree all $a_i \in C$. Because $\sigma_C \models \varphi$ we have $\vec{a} \models \varphi$, that is, for all $\vec{b}$ we have $(\vec{a},\vec{b}) \models \varphi$.

We have shown that $\sigma \in WCore(G_Q)$ if and only if Q is a positive instance of $\mathtt{QBF}_{2,\exists}$, hence the result. ■

PROPOSITION 12. *Deciding if a strategy profile σ is in the strong core of a Boolean game G is Π_2^p-complete.*

Proof:
Recall that we have $\sigma \notin SCore(G)$ if and only if there exists a coalition $C \subseteq N$ and a strategy $\sigma_C \in \Sigma_C$ such that

(c) for all $i \in C$ and all $\sigma_{-C} \in \Sigma_{-C}$ we have $(\sigma_C, \sigma_{-C}) \succeq_i \sigma$;

(d) there exist $i \in C$ and $\sigma_{-C} \in \Sigma_{-C}$ such that $(\sigma_C, \sigma_{-C}) \succ_i \sigma$;

We take the same reduction as in Proposition 11.

If Q is satisfiable, then $\sigma \notin WCore(G_Q)$, and *a fortiori* $\sigma \notin SCore(G_Q)$.

If $\sigma \notin SCore(G_Q)$ then there exist a coalition C and $\sigma_C \in \Sigma_C$ such that (c) and (d) hold. But (d) implies that $\sigma \models \neg(\varphi \wedge c)$ and $(\sigma_C, \sigma_{-C}) \models \varphi \wedge c$, therefore, $\sigma \notin WCore(G_Q)$, which implies that Q is a positive instance of $\mathtt{QBF}_{2,\exists}$.

■

PROPOSITION 13. *Deciding whether* $SCore(G)$ *is nonempty is in* Δ_2^p.

Proof: We recall that $SCore(G) \neq \emptyset$ if and only if the union of all efficient coalitions of G is efficient.

We start by noticing that C is the union of all efficient coalitions of G if and only if the following two conditions hold:

(A) for every $i \in C$, there exists an efficient coalition C_i containing i;

(B) for all $i \notin C$, no efficient coalition contains i.

Therefore, the following nondeterministic algorithm shows that $SCore(G) \neq \emptyset$:

1. $C := \emptyset$
2. for every $i \in N$ do
    3. if there exists an efficient coalition of G containing i
       then add i to C
       (else nothing)
4. check that C is efficient.

Consider the problem of Step 3, namely: given $i \in N$, check that there exists an efficient coalition of G containing i. The problem can be solved by the following nondeterministic algorithm: guess a coalition C', a strategy profile $\sigma_{C'} \in \Sigma_{C'}$, and check that $\sigma_{C'} \models \varphi_{C'}$. Thus, checking that there exists an efficient coalition of G containing i is in NP and Step 2 amounts to a linear number of NP-oracles, whereas Step 4 amounts to one more NP-oracle. Therefore, the algorithm is a deterministic algorithm using a polynomial number of NP-oracles, which shows that the problem of checking that $SCore(G) \neq \emptyset$ is in Δ_2^p.

■

So far we do not have a Δ_2^p-hardness result.

6. Efficient coalitions and dependencies between agents

We now study how the computation of efficient coalitions can be made easier by taking benefit from specific restrictions on the agents' preferences. First, the syntactical nature of goals may help us identifying efficient coalitions easily. Second, exploiting the dependencies between agents (where a dependency between i and j means that the goal φ_i of i involves a variable controlled by

j) can allow us, in some cases, to decompose the computation of efficient coalitions into independent subproblems.

We first note that when φ_i does not involve any variable controlled by j, the satisfaction of i does not depend directly on j. This is only a sufficient condition: it may be the case that the syntactical expression of φ_i mentions a variable controlled by j, but that this variable plays no role whatsoever in the satisfaction of φ_i, as variable y in $\varphi_i = x \wedge (y \vee \neg y)$. We therefore use a stronger notion of formula-variable independence (Lang et al., 2003).

DEFINITION 10. *A propositional formula* φ *is* **independent from** *a propositional variable x if there exists a formula* ψ *logically equivalent to* φ *and in which x does not appear.*

DEFINITION 11. *Let* $G = (N, V, \Gamma, \pi, \Phi)$ *be a Boolean game. The set of* **relevant variables** *for a player i, denoted by* $RV_G(i)$*, is the set of all variables* $v \in V$ *such that* φ_i *is not independent from v.*

For the sake of notation, the set of relevant variables for a player i in a given Boolean game G will be denoted by RV_i instead of $RV_G(i)$. We now easily define the *relevant players* for a given player i as the set of players controlling at least one variable of RV_i.

DEFINITION 12. *Let* $G = (N, V, \Gamma, \pi, \Phi)$ *be a Boolean game. The set of* **relevant players** *for a player i, denoted by* RP_i*,*[5] *is the set of agents* $j \in N$ *such that j controls at least one relevant variable of i:* $RP_i = \bigcup_{v \in RV_i} \pi^{-1}(v)$.

EXAMPLE 4. *Three friends (*1, 2 *and* 3*) are invited at a party.* 1 *wants to attend the party.* 2 *wants to attend if and only if* 1 *does.* 3 *wants to attend, and would like* 2 *to attend as well and* 1 *not to. This situation can be modelled by the following Boolean game* $G = (N, V, \Gamma, \pi, \Phi)$*, defined by*

- $V = \{a, b, c\}$*, with a (resp. b, c) means “*1 *(resp.* 2*,* 3*) attends the party”,*
- $N = \{1, 2, 3\}$, $\forall i$, $\gamma_i = \top$,
- $\pi_1 = \{a\}$, $\pi_2 = \{b\}$, $\pi_3 = \{c\}$,
- $\varphi_1 = a$, $\varphi_2 = a \leftrightarrow b$ *and* $\varphi_3 = \neg a \wedge b \wedge c$.

We can see that 1*’s satisfaction depends only on herself, 2’s depends on 1 and herself, whereas 3’s depends on 1, 2 and herself. Therefore, we have:* $RV_1 = \{a\}$, $RV_2 = \{a, b\}$, $RV_3 = \{a, b, c\}$, $RP_1 = \{1\}$, $RP_2 = \{1, 2\}$, $RP_3 = \{1, 2, 3\}$.

This relation between players can be seen as a directed graph containing a vertex for each player, and an edge from i to j whenever $j \in RP_i$, *i.e.* if j is a relevant player of i.

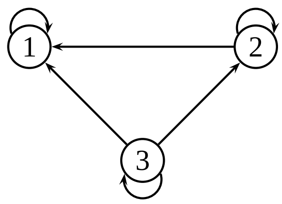
DEFINITION 13. *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game. The* **dependency graph of a Boolean game** *G is the directed graph $\mathcal{P} = \langle N, R \rangle$, with $\forall i, j \in N$, $(i, j) \in R$ (denoted by $R(i, j)$) if $j \in RP_i$.*

Thus, $R(i)$ is the set of players from which i may need some action in order to be satisfied: $j \in R(i)$ if and only if $j \in RP_i$. Remark however that $j \in R(i)$ does not imply that i *needs* some action by j to see her goal satisfied. For instance, if $\pi_1 = \{a\}$, $\pi_2 = \{b\}$ and $\varphi_1 = a \vee b$, then $2 \in R(1)$; however, 1 has a strategy for satisfying her goal (setting a to true) and therefore does not need an action by 2. Note that the dependency graph may have cycles.

We denote by R^* the transitive closure of R. $R^*(i)$ is the set of all players who have a direct or indirect influence on i. For $I \subseteq N$, we let $R(I) = \bigcup_{i \in I} R(i)$.

Example 4, continued:

The dependence graph $\mathcal{P}$ induced by G is depicted as follows:



We already know that if two disjoint coalitions I and J are efficient then their union is efficient. The converse does not hold in the general case, that is, there may exist two disjoint sets I and J such that $I \cup J$ is efficient and neither I nor J is. However, the converse holds in the following specific case:

PROPOSITION 14. *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game. Let I and J be two coalitions such that $I \cap J = \varnothing$, $I \cup J$ is efficient, $R(I) \cap J = \varnothing$ and $R(J) \cap I = \varnothing$. Then I and J are both efficient.*

Proof: The efficiency of $I \cup J$ implies that there exists $\sigma_{I \cup J} \in \Sigma_{I \cup J}$ such that $\sigma_{I \cup J} \models (\bigwedge_{i \in I \cup J} \varphi_i)$. Since $I \cap J = \varnothing$, we have this following chain of equivalences:

$$
\begin{aligned}
& \exists \sigma_I \in \Sigma_I, \exists \sigma_J \in \Sigma_J : \ (\sigma_I, \sigma_J) \models (\bigwedge_{i \in I \cup J} \varphi_i) \\
\Leftrightarrow \ & \exists \sigma_I \in \Sigma_I, \exists \sigma_J \in \Sigma_J : \ (\sigma_I, \sigma_J) \models (\bigwedge_{i \in I} \varphi_i \wedge \bigwedge_{j \in J} \varphi_j) \\
\Leftrightarrow \ & \exists \sigma_I \in \Sigma_I, \exists \sigma_J \in \Sigma_J : \ ((\sigma_I, \sigma_J) \models (\bigwedge_{i \in I} \varphi_i)) \wedge ((\sigma_I, \sigma_J) \models (\bigwedge_{j \in J} \varphi_j))
\end{aligned}
$$

Moreover, we know that $\forall i \in I, j \in J$, $j \notin RP_i$ (resp. $i \notin RP\,j$). So, $\forall i \in I, j \in J$, $\forall v \in Var(PI(\varphi_i))$, $v \notin \pi_j$ (resp. $\forall w \in Var(PI(\varphi_j))$, $w \notin \pi_i$). We know that no

player in J controls a variable of a goal of a player in I (and vice versa).
As we have $\exists \sigma_I \in \Sigma_I$, $((\sigma_I, \sigma_J) \models (\bigwedge_{i \in I} \varphi_i))$ and $\forall i \in I, j \in J, \forall v \in Var(PI(\varphi_i))$, $v \notin \pi_j$, we have: $\sigma_I \models (\bigwedge_{i \in I} \varphi_i)$ (resp. $\sigma_J \models (\bigwedge_{j \in J} \varphi_j)$.
Therefore, both I and J are efficient. ■

We now introduce the notion of *stable set* for a Boolean game[6]. A subset of agents B is stable for G if none of the agents in B has a relevant player outside B.

DEFINITION 14. *Let $G = (N, V, \pi, \Gamma, \Phi)$ be a Boolean game. $B \subseteq N$ is* **stable for** *G if and only if $R(B) \subseteq B$.*

The following proposition is straightforward but useful:

PROPOSITION 15. *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game. If $B \subseteq N$ is stable for R, then B is an efficient coalition of G ($B \subseteq \mathrm{EC}(G)$) if and only if $\varphi_B = \bigwedge_{i \in B} \varphi_i$ is consistent.*

Proof: Let B a stable set for R. Then, we have:

$$\begin{array}{ll} & \forall i \in B, \forall j \text{ such that } j \in R(i), j \in B \\ \Leftrightarrow & \forall i \in B, RP_i \subseteq B \\ \Rightarrow & \forall i \in B, \exists \sigma_B \in \Sigma_B \text{ such that } \sigma_C \models \varphi_i \\ \Leftrightarrow & \exists \sigma_B \in \Sigma_B \text{ such that } \sigma_B \models \bigwedge_{i \in B} \varphi_i \text{ if and only if } \bigwedge_{i \in B} \varphi_i \not\models \bot \end{array}$$

■

The converse is not necessarily true, as we can see on the following example:

EXAMPLE 5.
Let $G = (N, V, \Gamma, \pi, \Phi)$ be the Boolean game defined by $V = \{a, b\}$, $N = \{1, 2\}$, $\pi_1 = \{a\}$, $\pi_2 = \{b\}$, $\varphi_1 = a \vee b$ and $\varphi_2 = \top$.
The coalition $\{1\}$ is efficient, but is not stable for R: $R(\{1\}) = \{1, 2\} \not\subseteq \{1\}$.

However, the converse can be true under the very restrictive condition that the satisfaction of the goal of a player depends only on the actions of *one* player, that is, if RP_i is a singleton for every $i \in B$.

PROPOSITION 16. *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game. If $B \subseteq N$ is an efficient coalition of G ($B \subseteq \mathrm{EC}(G)$) such that $\forall i \in B$, $|RP_i| = 1$, then B is stable for R. In this case, a coalition B such that $\varphi_B = \bigwedge_{i \in B} \varphi_i \not\models \bot$ is efficient if and only if B is stable for R.*

Proof: B is an efficient coalition, so $\bigwedge_{i \in B} \varphi_i$ is consistent, and $\exists \sigma_B \in \Sigma_B$ such that $\sigma_B \models \bigwedge_{i \in B} \varphi_i$.

We know that $\forall i \in B$, $|RP_i| = 1$. So, $\exists j \in N$ such that $RP_i = \{j\}$, *i.e.* $\forall v \in Var(PI(\varphi_i))$, $v \in \pi_j$. As we have $\sigma_B \models \varphi_i$, with $\sigma_B \in \Sigma_B$, we know that B controls at least one variable in φ_i. So, $j \in B$, and thus B is stable for R. ■

In this specific case where RP_i is a singleton for every $i \in B$, we have furthermore this intuitive graph-theoretic characterization of efficient coalitions:

PROPOSITION 17. *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game such that $\forall i \in N$, $|RP_i| = 1$. For any coalition $C \subseteq N$, C is minimal efficient if and only if C forms a cycle in the dependence graph.*

Proof:

$\Rightarrow$ As $\forall i$, $|RP_i| = 1$, only one edge can go out for each player. So, if there is a cycle between p players, and if we rename these players with respect to the topological order, we have $RP_1 = \{2\}$, $RP_2 = \{3\}$, ..., $RP_{p-1} = \{p\}$, $RP_p = \{1\}$. Let $C = \{1, \ldots, p\}$. As we obviously have $R(C) = C$ ($\forall i \in C, RP_i = \{(i+1) mod\, p\} \in C$), C is stable for R.
Moreover, we know that $\forall i, j \in C$, $RP_i \neq RP_j$, so $\forall i, j \in C$, $\varphi_i \wedge \varphi_j \not\models \bot$. So, from Proposition 15, C is efficient.
Assume that $\exists I \subset C$ efficient. So, $\exists \sigma_I$ such that $\sigma_I \models \bigwedge_{i \in I} \varphi_i$. So, $\forall i \in I$, $RP_i \in I$. As $|RP_i| = 1$, and as $\forall i, j \in C$, and thus $\forall i, j \in I$, $RP_i \neq i$ and $RP_i \neq RP_j$, agents in I form a cycle. So, $I = C$. C is minimal efficient.

$\Leftarrow$ If C is stable for R, then $\forall i \in C$, $\exists j \in C$ such that $RP_i = \{j\}$. So, if $C = \{1, \ldots, p\}$, we can rename these players in order to have $RP_1 = \{2\}$, $RP_2 = \{3\}$, ..., $RP_{p-1} = \{p\}$, $RP_p = \{1\}$, and C forms a cycle in the dependance graph.

■

Another interesting issue is the study of efficient coalitions in Boolean games where goals have a specific syntactical structure. The characterization of efficient coalitions when goals are literals is a straightforward consequence of Propositions 16 and 17:

COROLLARY 1. *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game. If for every $i \in N$, φ_i is a literal, then we have the following results:*

1. *A coalition B is efficient if and only if B is stable for R.*
2. *For any coalition $B \subseteq N$, B is minimal efficient if and only if B forms a cycle in the dependence graph.*

We also have the following intuitive characterization of efficient coalitions when goals are either clauses or terms:

PROPOSITION 18. *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game.*

1. *If for every $i \in N$, φ_i is a term then for any $B \subseteq N$ such that $\bigwedge_{i \in B} \varphi_i$ is consistent, B is efficient if and only if B is stable for R.*

2. *If for every $i \in N$, φ_i is a clause such that $\bigwedge_{i \in N} \varphi_i$ is consistent, then for any $B \subseteq N$, B is efficient if and only if there exists a set of cycles in the dependence graph of G such that the nodes of the union of these cycles are exactly the members of B.*

Proof:

1. $\Rightarrow$ Let B be a stable set for R. As $\bigwedge_{i \in B} \varphi_i$ is consistent, we know from Proposition 15 that B is an efficient coalition.

   $\Leftarrow$ Let B be an efficient coalition. So $\exists \sigma_B \in \Sigma_B$ such that $\sigma_B \models \bigwedge_{i \in B} \varphi_i$. As $\forall i$, $\varphi_i = \bigwedge_{v \in Lit(\varphi_i)} v$, $\sigma_B \models \bigwedge_{v \in \bigcup_{i \in B} Lit(\varphi_i)} v$. So, $\forall v \in \bigcup_{i \in B} Lit(\varphi_i)$, $Var(v) \in \pi_B$, and then $\forall i \in B$, $RP_i \subseteq B$. B is stable for R.

2. $\Rightarrow$ Let B be an efficient coalition. So $\exists \sigma_B \in \Sigma_B$ such that $\sigma_B \models \bigwedge_{i \in B} \varphi_i$. Let decompose B in p minimal efficient coalitions[7]. We have $\forall k \in \{1, \ldots, p\}$, B_k is minimal efficient, $B_1 \cup \ldots \cup B_p = B$.
   Let $B_k \in \{B_1, \ldots, B_p\}$. As $\forall i \in N$, $\varphi_i = \bigvee_{v \in Lit(\varphi_i)} v$, we know that $\forall i \in B_k$, $\exists j \in B_k$, $\exists \sigma_j \in \Sigma_j$ such that $\sigma_j \models \varphi_i$. As B_k is minimal efficient, we know that we cannot find $C \subset B_k$ such that $\forall i \in C$, $\exists j \in C$, $\exists \sigma_j \in \Sigma_j$ such that $\sigma_j \models \varphi_i$. So, if $B_k = \{1, \ldots, m\}$ we can rename these players in the following way: let us take a player and call her 1. Then, call 2 the player in B_k such that $\sigma_2 \models \varphi_1$ (as B_k is minimal efficient, we know that 1 and 2 are two different players). Then, call 3 the player in B_k such that $\sigma_3 \models \varphi_3$ (as previously, as B_k is minimal efficient, we know that 1, 2 and 3 are three different players). We can rename all players in B_k in the same way, until $\sigma_m \models \varphi_{m-1}$. As B_k is minimal efficient, we know that $\sigma_1 \models \varphi_m$. Thus B_k forms a cycle in the dependency graph.
   As each $B_k \in \{B_1, \ldots, B_p\}$ forms a cycle in the dependency graph, B is the union of these cycles in the dependency graph.

   $\Leftarrow$ Let $B = \{1, \ldots, m\}$ be a cycle between m players. So, $\forall i \in B$, $\exists j \in B$ such that $j \in RP_i$. As $\forall i \in N$, $\varphi_i = \bigvee_{v \in Lit(\varphi_i)} v$, we know that $\forall i \in B$, $\exists j \in B$, $\exists \sigma_j \in \Sigma_j$ such that $\sigma_j \models \varphi_i$. Moreover, we know that $\bigwedge_{i \in N} \varphi_i \not\models \bot$. Then, $\exists \sigma_B \in \Sigma_B$ such that $\sigma_B \models \bigwedge_{i \in B} \varphi_i$. So, B is an efficient coalition.
   Let assume now that we have p cycles $\{B_1, \ldots B_p\}$ in the dependency graph. As seen previously, each B_i is an efficient coalition. As $\bigwedge_{i \in N} \varphi_i \not\models \bot$, $B_1 \cup \ldots \cup Bp \models \bigwedge_{i \in \{1, \ldots p\}} \bigwedge_{j \in B_i} \varphi_j$. So, $B_1 \cup \ldots \cup B_p$ is an efficient coalition.

■

Again, due to the fact that a coalition in a quasi-dichotomous Boolean game G is efficient if and only if it is efficient in the associated standard Boolean game G^*, Propositions 14, 15, 16, 17 and 18 also hold for quasi-dichotomous Boolean games.

7. Related work

Introduced in (Dunne et al., 2008), cooperative Boolean games (CBG) are a specific class of quasi-dichotomous Boolean games. In a cooperative Boolean game, as in a classical Boolean game, each agent has a goal represented by a propositional logic formula, and each agent has control over a set of Boolean variables. In addition to this, every propositional variable x_i is associated with a positive number $c(x_i)$ representing the cost, incurring for the agent who controls x_i, of making x_i true. Costs are negligible with respect to the utility of having a goal satisfied (an agent always prefers a state satisfying her goal to a state that does not), therefore cooperative Boolean games are quasi-dichotomous. Standard Boolean games are recovered by letting $c(x_i) = 0$ for all x_i.

Now, (Dunne et al., 2008) focuses on two stability concepts, one of which is highly related to our Section 4.2. This concept is also called the *core* of a (cooperative) Boolean game, and is defined as follows :

DEFINITION 15. *Let G be a Boolean game*[8]*. A strategy profile σ is* **blocked by a coalition $C \subseteq N$ through a strategy profile** σ' *if*

1. *σ and σ' coincide on all variables that are not controlled by any member of C;*
2. *coalition C strictly prefers σ' over σ: for all $i \in C$, $\sigma' \succ_i \sigma$.*

The **DHKW-core** *of G, denoted by $Core_{DHKW}(G)$, is the set of strategy profiles that are not blocked by any coalition.*

When restricted on standard (zero-cost) Boolean games, this definition differs from both our weak and strong core notions, for the following reason: σ is in the DHKW-core of a Boolean game if no coalition C has an interest to deviate from σ, *the actions of the other players being fixed*, whereas σ is in the weak or strong core if no coalition has a joint strategy which makes it better off (where the meaning of “better off” differs whether we talk about the weak or the strong core) than σ, *whatever the actions of the other players*. As remarked in (Sauro et al., 2009; Sauro and Villata, 2011), this definition corresponds to the *strong Nash equilibrium* in noncooperative game theory. The following is immediate from the definition of the DHKW-core:

PROPOSITION 19. *For any Boolean game G, the DHKW-core of G is contained in the weak core of G*[9].

The converse inclusion does not hold, as witnessed by the following example.

EXAMPLE 6. *Let* $n = 3$, $V = \{x_1, x_2, x_3\}$, $\pi(i) = \{x_i\}$, *and* $\gamma_1 = x_1 \leftrightarrow (x_2 \wedge x_3)$, $\gamma_2 = \gamma_3 = \neg\gamma_1$. *The DHKW-core of G is empty: for every* σ *satisfying* γ_2, *1 can switch* x_1 *to make* γ_1 *true, and for every* σ *satisfying* γ_1, *2 and 3 can adjust the values of* x_2 *and* x_3 *to make* γ_2 *true. However, no coalition is efficient in G, therefore, no coalition can find another way to act (than in* σ*) that ensures it to be better off, whatever the action of the other player(s), and any strategy profile is in the weak core of G.*

Note finally that deciding membership to the DHKW-core of a Boolean game is coNP-complete (whereas deciding membership to the weak and to the strong core is, in both cases, Π_2^p-complete), and that deciding the nonemptiness of the DHKW-core is Σ_2^p-complete (whereas the weak core is always nonempty, and deciding the nonemptiness of the strong core is in Δ_2^p).

Cooperative games have been further investigated in (Endriss et al., 2011), who study the design of taxation functions so as to modify a cooperative Boolean game in order to ensure that some Nash equilibrium (or all Nash equilibria) satisfies some desirable property.

Sauro, van der Torre and Villata (Sauro et al., 2009; Sauro and Villata, 2011) address the actual computation of the DHKW-core of a cooperative Boolean game, using the dependencies between players and variables and/or players, as we do for efficient coalitions in Section 6.

Another related line of work is the study of coalition formation among goal-directed agents by Boella, Sauro and van der Torre (Boella et al., 2005; Boella et al., 2006; Sauro, 2006). One of the main differences between their framework and ours is in the expression of the problem input. While we specify agents' abilities and goals separately (abilities by a control assignment function and goals by propositional formulae), Sauro (2006) defines a *power structure* consisting of a set of abstract goals $Goal(i)$ for each agent i, a power relation *pow* expressing, for every subset $Goal'$ of $Goal = \cup_i Goal(i)$, which coalitions can achieve $Goal'$, and a compatibility relation *comp* expressing which goals are jointly feasible, *i.e.*, non-conflictual. An agent is satisfied as soon as one of its goals is satisfied. Now, a pair $\langle C, E \rangle$, where $C \subseteq N$ and $E \subseteq Goal = \cup_i Goal(i)$ is *do-ut-des* if every agent i in C (a) has one of her goals satisfied: $Goal(i) \cap E \neq \emptyset$, and (b) contributed to the achievement of some of the others' goals: there exists $g_j \in E \cap Goal(j)$, $j \neq i$, such that $C \setminus \{i\}$ cannot achieve g_j.

A Boolean game can be translated into a power structure in the following way: every propositional goal φ_i is expressed as an abstract goal g_i with

$Goal(i) = \{g_i\}$, while *pow* and *comp* express respectively which coalitions can achieve which sets of goals and which sets of goals are jointly feasible. The converse translation (from power structures to Boolean games) is simple only in the special case where each agent i has a single goal g_i (the details of the translation do not present any particular interest and we omit them).

Now, in the specific case where each agent i has a single goal g_i, $\langle C, E\rangle$ is *do-ut-des* if (a') $E \supseteq Goal_C = \{g_i, i \in C\}$ and (b') for every $i \in C$ there exists $j \in C$, $j \neq i$, such that $C \setminus \{i\}$ cannot achieve g_j. This leads to the following characterization of *do-ut-des* coalitions:

OBSERVATION 1. *$\langle C, Goal_C\rangle$ is* do-ut-des *if and only if C is efficient and for every $i \in C$, $C \setminus \{i\}$ is not efficient.*

Although, clearly, any minimally efficient coalition is *do ut des*, the converse is not true: consider the Boolean game with $n = 4$, i controls x_i, $\gamma_1 = x_2$, $\gamma_2 = x_1$, $\gamma_3 = x_4$ and $\gamma_4 = x_3$; then C is *do-ut-des*, but not minimally efficient. Now, Boella *et al.* define a further refined notion : $\langle C, E\rangle$ is a *i-dud coalition* if it is *do-ut-des* and is not decomposable into two smaller *do-ut-des* coalitions. In other terms, C is *i-dud* if and only if C is efficient and cannot be decomposed into disjoint efficient subcoalitions. This stronger notion is still not equivalent to being minimally efficient: in the Boolean game where $n = 4$, i controls x_i, $\gamma_1 = x_2$, $\gamma_2 = x_1$, $\gamma_3 = x_1 \wedge x_2 \wedge x_4$ and $\gamma_4 = x_1 \wedge x_2 \wedge x_3$, the only efficient coalitions are $\{1,2\}$ and $\{1,2,3,4\}$, therefore $\{1,2,3,4\}$ is *i-dud* but not minimally efficient.

8. Conclusion

The results we have obtained are twofold – this paper can actually be seen as two independent parts.

The first part (Section 3) gives a characterization of effectivity functions induced by (pre-)Boolean games, thus allowing us to understand better the structural assumptions hidden behind the control assignment functions that define the power of agents in Boolean games. The results of Section 3 apply to pre-Boolean games, where preferences do not play any role.

The second part (Sections 4, 5 and 6) shows that Boolean games can be used as a compact representation setting for coalitional games where players have quasi-dichotomous preferences. This specificity has lead us to define an interesting notion of efficient coalitions. We have given an exact characterization of sets of coalitions that correspond to the set of efficient coalitions for a Boolean game, and several results concerning the computation of efficient coalitions. The results of Sections 4, 5 and 6 can be partitioned into two classes: those who apply to Boolean games with quasi-dichotomous preferences, and those who apply only to standard Boolean games, with dichotomous preferences.

There are many practical situations where preferences are naturally quasi-dichotomous, or even dichotomous (cf. Example 2). However, it is natural to ask whether our results of Sections 4, 5 and 6 extend to generalized Boolean games, with arbitrary preferences represented in some compact representation language (Bonzon et al., 2006; Bonzon et al., 2009). This is a challenging issue for further research. Unfortunately, this does not appear to be easy, because the notion of efficient coalition, which is dichotomous in essence, only makes sense when each agent has a primary goal whose satisfaction overweighs all possible action costs, or, in other terms, when preferences are quasi-dichotomous.

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Appendix

We show here that decomposability is equivalent to (2) whereas in the presence of decomposability and $\mathrm{Eff}(N) = 2^S \setminus \{\emptyset\}$, atomicity is equivalent to the conjunction of (3) and (4), with

$$
\begin{array}{ll}
(2) & \text{for any } C \neq \emptyset, \mathrm{Eff}^{nc}(C) = \{\cap_{i\in C} X_i : X_i \in \mathrm{Eff}^{nc}(i)\} \\
(3) & X, Y \in \mathrm{Eff}^{nc}(i) \text{ and } X \neq Y \text{ implies } X \cap Y = \emptyset \\
(4) & X \in \mathrm{Eff}^{nc}(j) \text{ and } x \in X \text{ implies } \exists Y \in \mathrm{Eff}^{nc}(i), x \in Y
\end{array}
$$

where $\mathrm{Eff}^{nc}(C)$ denotes the set of all inclusion-minimal sets in $\mathrm{Eff}(C)$.

1. **Decomposability is equivalent to (2).**
   First, we show that decomposability is equivalent to

   (D'): for every $C \subseteq N$, $X \in \mathrm{Eff}(C)$ *iff* for all $i \in C$ there exists X_i such that $X = \cap_{i\in C} X_i$

   This is easily shown by induction on the size of C: the base case is obvious, and for any $i \in C$, $X \in \mathrm{Eff}(C)$ if and only if there exists X_i and X_{-i} such that $X = X_i \cap X_{-i}$.
   Now, because $\mathrm{Eff}(C)$ is upward closed ($X \subseteq Y$ and $X \in \mathrm{Eff}(C)$ implies $Y \in \mathrm{Eff}(C)$), we have $X \in \mathrm{Eff}(C)$ *iff* $Y \in \mathrm{Eff}^{nc}(C)$ for some $Y \subseteq X$,

therefore (2) is equivalent to: $X \in \mathrm{Eff}(C)$ *iff* there exists $(Y_i)_{i\in C}$ such that for all i, $Y_i \in \mathrm{Eff}^{nc}(\{i\})$ and $X \supseteq \cap_{i\in C} Y_i$, which, again because $\mathrm{Eff}(C)$ is upward closed, is equivalent to: $X \in \mathrm{Eff}(C)$ *iff* there exists $(Y_i)_{i\in C}$ such that for all i, $Y_i \in \mathrm{Eff}(\{i\})$ and $X = \cap_{i\in C} Y_i$, which is (D').

2. **In the presence of decomposability and** $\mathrm{Eff}(N) = 2^S \setminus \{\emptyset\}$**, atomicity is equivalent to (3) and (4).**
Assume atomicity holds. Recall that $At(i)$ and $Eff^{nc}(i)$ coincide, therefore the fact that any two elements of $At(i)$ are disjoint implies (3). Now, let $X \in At(j)$ and $x \in X$. Then $x \in S$; since $At(i)$ is a partition of S, there exists $Y \in At(j)$ such that $x \in Y$, which shows (4).
Conversely, assume (3) and (4) hold, as well as decomposability and $\mathrm{Eff}(N) = 2^S \setminus \{\emptyset\}$. Let $s \in S$; then $\{s\} \in \mathrm{Eff}(N)$. By decomposability applied to $(N \setminus \{i\}, \{i\})$, there is $Y \in \mathrm{Eff}(i)$ and $Z \in \mathrm{Eff}(N \setminus \{i\}$ such that $Y \cap Z = \{s\}$. Therefore, $s \in \cup\{X | X \in At(i)\}$. Lastly, (3) implies that any two elements of $At(i)$ are disjoint, which shows that $At(i)$ is a partition of S.

Notes

[1] This result was recently shown by (Goranko et al., 2011) to be wrong for infinite game models. As the games we consider are finite, this has no impact on the rest of the paper.

[2] The strong links between our properties and the properties in (Agotnes and Alechina, 2011) that characterize injective games should not be seen as a surprise: because $\sigma = \sigma'$ if and only if $\sigma_i = \sigma'_i$ for every $i \in N$, Boolean games are injective. Our Lemmas 3 and 4 can actually be seen as a proof that our properties, which imply all of the properties (Agotnes and Alechina, 2011) (note that their property (1) is trivially satisfied for finite games) imply injectivity.

[3] There are 2 players in $I = \{1,2\}$, therefore each $\vec{C}$ in $Cov(\mathcal{SC}, 12)$ contains 2 coalitions, one for each player, satisfying (i), (ii) and (iii).

[4] We must be careful however. if the cost functions in a quasi-dichotomous Boolean game were represented explicitly, then all problems considered would be trivially polynomial. Our statement holds only if the representation is compact enough so that the representation of some cost function (such as, typically, the null cost function) has a polynomial size.

[5] Again, the set of relevant players for a Boolean game G should be denoted by $RP_G(i)$: for the ease of notation we simply write RP_i.

[6] Note that this notion has nothing to do with the classical notion of stable set in graph theory.

[7] If B is minimal efficient, then $p = 1$.

[8] Since we aim at comparing both notions in standard Boolean games, we give the definition for standard Boolean games only. The definition would be exactly the same in the general case with non-zero costs.

[9] There is no similar relationship between the DHKW-core and the strong core. Whereas both the DHKW-core and the weak core consider that a coalition is strictly better off if all its members are strictly better off, the strong core uses a weaker definition that makes the concept incomparable with the DHKW-core.

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