A characterization of the 2-additive Choquet integral through cardinal information

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Abstract

In the context of Multiple criteria decision analysis, we present the necessary and sufficient conditions to represent a cardinal preferential information by the Choquet integral w.r.t. a 2-additive capacity. These conditions are based on some complex cycles called *cyclones*.

Keywords: MCDA, Choquet integral, 2-additive capacity, MACBETH methodology.

1 Introduction

Multiple criteria decision analysis aims at representing the preferences of a decision maker over options or alternatives. The Choquet integral has been proved to be a versatile aggregation function to construct overall scores [3, 7, 8] and is based on the notion of the capacity or the fuzzy measure. Used as an aggregation function, the Choquet integral arises as a generalization of the weighted sum, taking into account the interaction between criteria. In this paper we focus on the Choquet integral [7, 11] w.r.t. a 2-additive capacity.

In many situations, it is important for the Decision-Maker (DM) to construct a preference relation over the set of all alternatives X. Because it is not an easy task (the cardinality of X may be very large), we ask him to give, using pairwise comparisons, a cardinal information (a preferential information given with preference intensity) on a particular reference subset $\mathcal{B} \subseteq X$. The set $\mathcal{B}$ we use is the set of binary alternatives also called binary actions. A binary action is a fictitious alternative which takes either the neutral value $\mathbf{0}$ for all criteria, or the neutral value $\mathbf{0}$ for all criteria except for one or two criteria for which it takes the satisfactory value $\mathbf{1}$. Under these hypotheses, we present the necessary and sufficient conditions on the cardinal information for the existence of a 2-additive capacity such that the Choquet integral w.r.t. this capacity represents the preference of the DM. These conditions concern some cycles called *cyclones* in a directed graph where multiple edges are allowed between two vertices.

The basic material on the Choquet integral, binary actions and cardinal information are given in the next section. In the last section, we study the representation of a cardinal information by a Choquet integral w.r.t. a 2-additive capacity and give our main result.

2 Preliminaries

2.1 Notations and aim

Let $N = \{1, \ldots, n\}$ be a finite set of n criteria. An action (also called alternative or option) $x = (x_1, \ldots, x_n)$ is identified to an element of the Cartesian product $X = X_1 \times \cdots \times X_n$, where $X_1, \ldots, X_n$ represent the set of points of view or attributes. For a subset A of N and actions x and y, the notation $z = (x_A, y_{N-A})$ means that z is defined by $z_i = x_i$ if $i \in A$, and $z_i = y_i$ otherwise.

We assume that, given two alternatives x and y the DM is able to judge the difference of attractiveness between x and y when he prefers strictly x to y. Like in the MACBETH methodology [6], the difference of attractiveness will be provided under the form of semantic categories d_s, $s = 1, \ldots, q$ defined so that, if $s < t$, any difference of attractiveness in the class d_s is lower than any difference of attractiveness in the class d_t. MACBETH approach uses the following six semantic categories: d_1 = very weak, d_2 = weak, d_3 = moderate, d_4 = strong, d_5 = very strong, d_6 = extreme.

Our aim is to construct a preference relation over X. In practice (see [1, 10]) one can only ask DM to do pairwise comparisons of alternatives on a finite subset X' of X, X' having a small size. Hence we get a preference relation $\succsim_{X'}$ on X'. The question is then: how to construct a preference relation $\succsim_X$ on X, so that $\succsim_X$ is an extension of $\succsim_{X'}$? To this end, people usually suppose that $\succsim_X$ is representable by an overall utility function:

$$(1) \qquad x \succsim_X y \Leftrightarrow F(U(x)) \geq F(U(y))$$

where $U(x) = (u_1(x_1), \ldots, u_n(x_n))$, $u_i : X_i \to \mathbb{R}$ is called a utility function, and $F : \mathbb{R}^n \to \mathbb{R}$ is an aggregation function. Usually, we consider a family of aggregation functions characterized by a parameter vector θ (e.g., a weight distribution over the criteria). The parameter vector θ can be deduced from the knowledge of $\succsim_{X'}$, that is, we determine the possible values of θ for which (1) is fulfilled over X'. We study the case where F is the Choquet integral w.r.t. a 2-additive capacity, and X' is the set of binary actions and the parameter vector is the 2-additive capacity. The aim of this paper is to give necessary and sufficient conditions on $\succsim_{X'}$ to be represented by a Choquet integral w.r.t. a 2-additive capacity. The model obtained in X' will be then automatically extended to X.

2.2 Choquet integral w.r.t. a 2-additive capacity

The Choquet integral [8, 7, 9] is a well-known aggregation function used in multicriteria decision aiding when interactions between criteria occur. We are interested in the Choquet integral w.r.t a 2-additive capacity [4, 8]. We define this notion below.

Definition 2.1.

1. A *capacity* on N is a set function $\mu : 2^N \to [0, 1]$ such that:
   (a) $\mu(\emptyset) = 0$
   (b) $\mu(N) = 1$
   (c) $\forall A, B \in 2^N$, $[A \subseteq B \Rightarrow \mu(A) \leq \mu(B)]$ (monotonicity).
2. The *Möbius transform* (see [2]) of a capacity μ on N is a function $m : 2^N \to \mathbb{R}$ defined by:
   (2)
   $$m(T) := \sum_{K \subseteq T} (-1)^{|T \setminus K|} \mu(K), \forall T \in 2^N.$$
   When m is given, it is possible to recover the original μ by the following expression:
   $$(3) \qquad \mu(T) := \sum_{K \subseteq T} m(K), \forall T \in 2^N.$$

Definition 2.2. A capacity μ on N is said to be *2-additive* if

- For all subsets T of N such that $|T| > 2$, $m(T) = 0$;
- There exists a subset B of N such that $|B| = 2$ and $m(B) \neq 0$.

Notations We simplify our notation for a capacity μ and its Möbius transform m by using the following shorthand: $\mu_i := \mu(\{i\})$, $\mu_{ij} := \mu(\{i, j\})$, $m_i := m(\{i\})$, $m_{ij} := m(\{i, j\})$, for all $i, j \in N$, $i \neq j$. Whenever we use i and j together, it always means that they are different.

The following important lemma shows that a 2-additive capacity is entirely determined by the value of the capacity on the singletons $\{i\}$ and pairs $\{i, j\}$ of 2^N:

Lemma 1.

1. *Let μ be a 2-additive capacity on N. We have $\forall K \subseteq N$, $|K| \geq 2$,*
   $$(4) \quad \mu(K) = \sum_{\{i,j\} \subseteq K} \mu_{ij} - (|K| - 2) \sum_{i \in K} \mu_i.$$

2. *If the coefficients μ_i and μ_{ij} are given for all $i, j \in N$, then the necessary and sufficient conditions that μ is a 2-additive capacity are:*

(5) $$\sum_{\{i,j\}\subseteq N} \mu_{ij} - (n-2)\sum_{i\in N}\mu_i = 1;$$

(6) $$\mu_i \geq 0, \ \forall i \in N;$$

$\forall A \subseteq N, |A| \geq 2, \forall k \in A,$

(7) $$\sum_{i\in A\setminus\{k\}} (\mu_{ik} - \mu_i) \geq (|A|-2)\mu_k.$$

Proof. See [7]. □

Given $x := (x_1, ..., x_n) \in \mathbb{R}^n_+$, the Choquet integral w.r.t. a 2-additive capacity μ, called for short a *2-additive Choquet integral*, is given by (see [9]):

(8) $$C_\mu(x) = \sum_{i=1}^{n} v_i x_i - \frac{1}{2}\sum_{\{i,j\}\subseteq N} I_{ij}|x_i - x_j|$$

where $v_i = \sum_{K\subseteq N\setminus i} \frac{(n-|K|-1)!|K|!}{n!}(\mu(K\cup i) - \mu(K))$ represents the importance of the criterion i and corresponds to the Shapley value of μ [14]; $I_{ij} := \mu_{ij} - \mu_i - \mu_j$ represents the interaction between the two criteria i and j [7, 12].

2.3 Binary actions and cardinal information

We assume that the DM is able to identify for each criterion i two reference levels:

1. A reference level $\mathbf{1}_i$ in X_i which he considers as good and completely satisfying if he could obtain it on criterion i, even if more attractive elements could exist. This special element corresponds to the *satisficing level* in the theory of bounded rationality of Simon [15].

2. A reference level $\mathbf{0}_i$ in X_i which he considers neutral on i. The neutral level is the absence of attractiveness and repulsiveness. The existence of this neutral level has roots in psychology [16], and is used in bipolar models [17].

We set for convenience $u_i(\mathbf{1}_i) = 1$ and $u_i(\mathbf{0}_i) = 0$. For more details about these reference levels, see [8, 9].

We call a *binary action or binary alternative*, an element of the set $\mathcal{B} = \{\mathbf{0}_N, \ (\mathbf{1}_i, \mathbf{0}_{N-i}), \ (\mathbf{1}_{ij}, \mathbf{0}_{N-ij}), \ i, j \in N, i \neq j\} \subseteq X$ where

- $\mathbf{0}_N = (\mathbf{1}_\emptyset, \mathbf{0}_N) =: a_0$ is an action considered neutral on all criteria.
- $(\mathbf{1}_i, \mathbf{0}_{N-i}) =: a_i$ is an action considered satisfactory on criterion i and neutral on the other criteria.
- $(\mathbf{1}_{ij}, \mathbf{0}_{N-ij}) =: a_{ij}$ is an action considered satisfactory on criteria i and j and neutral on the other criteria.

Using the Choquet integral, we get the following consequences:

Remark 1. 1. The Choquet integral satisfies the following property [8]: if μ is a capacity then

(9) $$C_\mu(U(\mathbf{1}_A, \mathbf{0}_{N-A})) = \mu(A), \ \forall A \subseteq N.$$

2. Let μ be a 2-additive capacity. We have
$C_\mu(U(a_0)) = 0;$
$C_\mu(U(a_i)) = \mu_i = v_i - \frac{1}{2}\sum_{k\in N, \ k\neq i} I_{ik};$
$C_\mu(U(a_{ij})) = \mu_{ij} = v_i + v_j - \frac{1}{2}\sum_{k\in N, \ k\notin\{i,j\}} (I_{ik} + I_{jk}).$

The last two equations come from general relations between the capacity μ and interaction (see [7] for details). Generally, the DM knows how to compare some alternatives using his knowledge of the problem, his experience, etc. These alternatives form a set of reference alternatives and allow to determine the parameters of a model (weights, utility functions, subjective probabilities,...) in the decision process (see [10] for more details). As shown by the previous Remark 1 and Lemma 1, it should be sufficient to get some preferential information from the DM only on binary actions. To entirely determine the 2-additive capacity, this information is expressed by the following relations:

- $P = \{(x, y) \in \mathcal{B} \times \mathcal{B}$: the DM strictly prefers x to $y\}$,
- $I = \{(x, y) \in \mathcal{B} \times \mathcal{B}$: the DM is indifferent between x and $y\}$,
- For the semantic category "d_k", $k \in \{1, ..., q\}$, $P_k = \{(x, y) \in P$ such that DM judges the difference of attractiveness between x and y as belonging in the class "d_k"$\}$

Without loss of generality, we will suppose that all the relations P_k are nonempty (we can always redefine the number q when some P_k are empty). The relation P is irreflexive and asymmetric while I is reflexive and symmetric.

Definition 2.3. The *cardinal information on* $\mathcal{B}$ is the structure $\{P,\ I,\ P_1, \ldots, P_q\}$.

The cardinal information is used also in the MACBETH methodology (see [6]). Now we will suppose P to be nonempty for any cardinal information $\{P,\ I,\ P_1, \ldots, P_q\}$ ("non triviality axiom") and $P = P_1 \cup P_2 \cup \cdots \cup P_q$.

3 The representation of the cardinal information by the Choquet integral

3.1 The representation

A cardinal information $\{P,\ I,\ P_1, \ldots, P_q\}$ is said to be *representable by a 2-additive Choquet integral* if there exists a 2-additive capacity μ such that:

1. $\forall x, y \in \mathcal{B},\ x\ P\ y \Rightarrow C_\mu(x) > C_\mu(y)$,
2. $\forall x, y \in \mathcal{B},\ x\ I\ y \Rightarrow C_\mu(x) = C_\mu(y)$,
3. $\forall x, y, z, w \in \mathcal{B}$, $\forall s, t \in \{1, \ldots, q\}$ s.t. $s < t$, $\left[\left.\begin{array}{c} (x,y) \in P_t \\ (z,w) \in P_s \end{array}\right\} \Rightarrow C_\mu(x) - C_\mu(y) > C_\mu(z) - C_\mu(w) \right]$

Given a cardinal information $\{P,\ I,\ P_1, \ldots, P_q\}$, we look for the necessary and sufficient conditions on $\mathcal{B}$ for which $\{P,\ I,\ P_1, \ldots, P_q\}$ is representable by a 2-additive Choquet integral. By using the monotonicity constraints of a 2-additive capacity and Lemma 1, this problem is equivalent to look for a function $f : \mathcal{B} \to \mathbb{R}_+$ satisfying the following equations:

$$(10) \qquad \forall x, y \in \mathcal{B},\ x\ P\ y \Rightarrow f(x) > f(y),$$

$$(11) \qquad \forall x, y \in \mathcal{B},\ x\ I\ y \Rightarrow f(x) = f(y),$$

$\forall x, y, z, w \in \mathcal{B}$, $\forall s, t \in \{1, \ldots, q\}$ s.t. $s < t$,

$$(12) \qquad \left.\begin{array}{c} (x,y) \in P_t \\ (z,w) \in P_s \end{array}\right\} \Rightarrow f(x) - f(y) > f(z) - f(w),$$

$$(13) \qquad f(a_0) = 0,$$

$$(14) \qquad \forall i \in N,\ f(a_i) \geq 0,$$

$\forall A \subseteq N, |A| \geq 2, \forall i \in A,$

$$(15) \qquad \sum_{j \in A \setminus \{i\}} (f(a_{ij}) - f(a_j)) \geq (|A| - 2)\ f(a_i).$$

3.2 The binary relations M and M_A on $\mathcal{B}$

To solve our problem, we introduce the following binary relations on $\mathcal{B}$:

1. The relation $M = \{(a_{ij}, a_i), i, j \in N, i \neq j\}$ which models the natural monotonicity constraints $\mu_{ij} \geq \mu_i$, $i, j \in N$ for a capacity μ (see equation (7)).
2. The relations $M_A = \{(a_0, a_i), i \in A\}$, $\forall A \subseteq N$ such that $|A| \geq 3$.

Let us give a simple example where these two binary relations are presented:

Example 1. *Let be* $N = \{1, 2, 3, 4\}$. *We have* $M = \{(a_{12}, a_1), (a_{12}, a_2), (a_{13}, a_1),$ $(a_{13}, a_3), (a_{14}, a_1),$ $(a_{14}, a_4), (a_{23}, a_2), (a_{23}, a_3),$ $(a_{24}, a_2), (a_{24}, a_4), (a_{34}, a_3), (a_{34}, a_4)\}$, $M_{\{1,2,3\}} =: m_1 = \{(a_0, a_1), (a_0, a_2), (a_0, a_3)\}$, $M_{\{1,2,4\}} =: m_2 = \{(a_0, a_1), (a_0, a_2), (a_0, a_4)\}$, $M_{\{1,3,4\}} =: m_3 = \{(a_0, a_1), (a_0, a_3), (a_0, a_4)\}$, $M_{\{2,3,4\}} =: m_4 = \{(a_0, a_2), (a_0, a_3), (a_0, a_4)\}$ *and* $M_{\{1,2,3,4\}} =: m_5 = \{(a_0, a_1),$

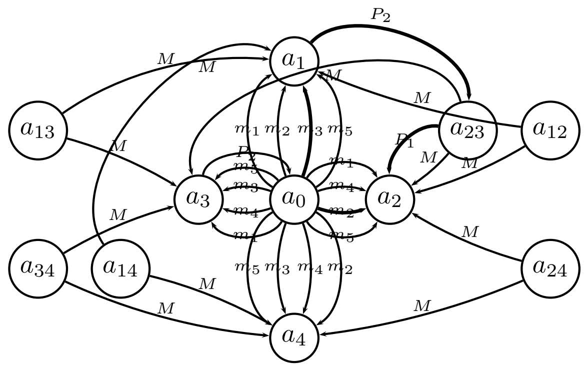

Figure 1: Graph of binary relations P_1, P_2, M and M_A on $\mathcal{B}$ with $N = \{1, 2, 3, 4\}$, $M_{\{1,2,3\}} = m_1$, $M_{\{1,2,4\}} = m_2$, $M_{\{1,3,4\}} = m_3$, $M_{\{2,3,4\}} = m_4$ and $M_{\{1,2,3,4\}} = m_5$

$(a_0, a_2), (a_0, a_3), \quad (a_0, a_4)\}$. *We assume also that* $P_1 = \{(a_{23}, a_2)\}$ *and* $P_2 = \{(a_1, a_{23}); (a_3, a_0)\}$. *All these relations are represented in the graph*[1] *of the Figure 1.*

Notations: Given a cardinal information $\{P, I, P_1, \ldots, P_q\}$, we will use the notation below:

$$\mathcal{P}' = P \cup (\bigcup_{k=1}^{q-1} P_k^{-1})$$
$$\mathcal{R} = I \cup \mathcal{P}' \cup M \cup (\bigcup_{\substack{A \subseteq N \\ |A| \geq 3}} (M_A \cup M_A^{-1}))$$

where for a general binary relation R, $R^{-1} = \{(y, x) \in \mathcal{B} \times \mathcal{B} : (x, y) \in R\}$. The relations $\mathcal{P}$, $\mathcal{P}'$ and $\mathcal{R}$ are binary relations on $\mathcal{B}$. Using all these binary relations, we define in the next section the main property for the characterization of a representation of a cardinal information by a Choquet integral w.r.t. a 2-additive capacity.

3.3 2-additive balanced cyclone and the main result

Before defining the property we call *2-additive balanced cyclone*, let us give some basic notions of graph theory we need:

[1] We assume that multiple edges exist between two vertices in the graph. This type of graph is called multigraph.

Definition 3.1. Let x, y be two elements of $\mathcal{B}$.

1. A sequence $x_1, x_2, \cdots, x_p$ of elements of $\mathcal{B}$ is a *path* of $\mathcal{R}$ from x to y if $x = x_1 \ \mathcal{R} \ x_2 \ \mathcal{R} \cdots \mathcal{R} \ x_{p-1} \ \mathcal{R} \ x_p = y$, and the sequence $x_1, x_2, \cdots, x_p$ contains at least two distinct elements.
   - A path of $\mathcal{R}$ from x to x is called a *cycle* of $\mathcal{R}$.
   - A path of $\mathcal{R}$ from x to x is called an *elementary cycle* of $\mathcal{R}$ if no element of the cycle is visited more than once (except x).
2. A path $\{x_1, x_2, ..., x_p\}$ of $\mathcal{R}$ is said to be a *strict path* from x to y if there exists i in $\{1, ..., p-1\}$ such that $x_i \ \mathcal{P}' \ x_{i+1}$. A strict path of $\mathcal{R}$ from x to x is called a *strict cycle* of $\mathcal{R}$.

Given a cycle C of $\mathcal{R}$ and a binary relation R in $\{P_k, P_k^{-1}, M, M_A, M_A^{-1}\}$, we denote by $C \cap R$ the usual intersection between the cycle C and R. For $x, y \in \mathcal{B}$, $\{(x, y)\}$ represents the set of all edges of $\mathcal{R}$ between x and y.

Remark 2. Since multiple edges are allowed between two vertices, we have in Figure 1: for the cycle C of $\mathcal{R}$ defined by $a_{23} \ P_1 \ a_2 \ {M_{\{1,2,4\}}}^{-1} \ a_0 \ M_{\{1,3,4\}} \ a_1 \ P_2 \ a_{23}$, $\big|C \cap M \cap \{(a_{23}, a_2)\}\big| = 0$ and $\big|C \cap \{(a_{23}, a_2)\}\big| = 1$.

Definition 3.2. Let $m \in \mathbb{N} \setminus \{0\}$.

1. An m-cyclone of $\mathcal{R}$ is a nonempty union of at most m cycles of $\mathcal{R}$. Thus an m-cyclone is obtained by taking pairs in $\mathcal{R}$ that altogether can be partitioned into m cycles.
2. An m-cyclone of $\mathcal{R}$ is said to be *strict* if it contains a strict cycle of $\mathcal{R}$.
3. An m-cyclone $\mathcal{C}$ of $\mathcal{R}$ is said to be *2-additive balanced* if it satisfies the two following conditions:
   (a) $\forall k \in \{1, 2, \ldots, q-1\}$, $|\mathcal{C} \cap P_k^{-1}| \leq |\mathcal{C} \cap P_{k+1}|$;
   (b) $\forall i, j \in N, i \neq j$,

$$\sum_{\substack{\{i,j\} \subseteq A \subseteq N \\ |A| \geq 3}} \frac{1}{|A|-2} \Big| \mathcal{C} \cap M_A \cap \{(a_0, a_i)\} \Big| \leq \Big| \mathcal{C} \cap M \cap \{(a_{ij}, a_j)\} \Big|$$

The notion of balanced m-cyclone was first used by Doignon in [5] for the characterization of the representation of multiple semiorders by thresholds. The same notion is also presented in [13]. We give below the main result of the paper, which is a theorem of characterization of consistent cardinal information $\{P, I, P_1, \ldots, P_q\}$ representable by a 2-additive Choquet integral:

Theorem 1. *A cardinal information $\{P, I, P_1, \ldots, P_q\}$ is representable by a 2-additive Choquet integral on $\mathcal{B}$ if and only if no strict $(q + n(n-1))$-cyclone of $\mathcal{R}$ is 2-additive balanced.*

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