Some Axiomatic and Algorithmic Perspectives on the Social Ranking Problem*

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Abstract. Several real-life complex systems, like human societies or economic networks, are formed by interacting units characterized by patterns of relationships that may generate a group-based social hierarchy. In this paper, we address the problem of how to rank the individuals with respect to their ability to "influence" the relative strength of groups in a society. We also analyse the effect of basic properties in the computation of a social ranking within specific classes of (ordinal) coalitional situations. We show that the pairwise combination of these natural properties yields either to impossibility (i.e., no social ranking exists), or to flattening (i.e., all the individuals are equally ranked), or to dictatorship (i.e., the social ranking is imposed by the relative comparison of coalitions of a given size). Then, we turn our attention to an algorithmic approach aimed at evaluating the frequency of "essential" individuals, which is a notion related to the (ordinal) marginal contribution of individuals over all possible groups.

Keywords: social ranking, coalitional power, ordinal power, axioms.

1 Introduction

Ranking is a fundamental ingredient of many real-life situations, like the ranking of candidates applying to a job, the rating of universities around the world, the distribution of power in political institutions, the centrality of different actors in social networks, the accessibility of information on the web, etc. Often, the criterion used to rank the items (e.g., agents, institutions, products, services, etc.) of a set N also depends on the interaction among the items within the subsets of N (for instance, with respect to the users' preferences over bundles of products or services). In this paper we address the following question: given a finite set N of items and a ranking over its subsets, can we derive a "social" ranking over N according to the "overall importance" of its single elements?

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For instance, consider a company with three employees $1, 2$ and 3 working in the same department. According to the opinion of the manager of the company, the job performance of the different teams $S \subseteq N = \{1, 2, 3\}$ is as follows: $\{1, 2, 3\} \succcurlyeq \{3\} \succcurlyeq \{1, 3\} \succcurlyeq \{2, 3\} \succcurlyeq \{2\} \succcurlyeq \{1, 2\} \succcurlyeq \{1\} \succcurlyeq \emptyset$ ($S \succcurlyeq T$, for each $S, T \subseteq N$, means that the performance of S is at least as good as the performance of T). Based on this information, the manager asks us to make a ranking over his three employees showing their attitude to work with others as a team or autonomously. Intuitively, 3 seems to be more influential than 1 and 2, as employee 3 belongs to the most successful teams in the above ranking. Can we state more precisely the reasons driving us to this conclusion? And what can we say if we have to decide who between 1 and 2 is more productive and deserves a promotion? In this paper we analyse different properties of ordinal social rankings in order to get some answers to such questions.

The problem studied in this paper can be seen as an ordinal counterpart of the one about how to measure the power of players in *simple games*, which are coalitional games where coalitions may be winning or not [1, 4]. However, our framework is different for at least two reasons: first, we face coalitional situations where only a qualitative (ordinal) comparison of the strength of coalitions is given; second, we look for a ranking over the single objects in N, and we do not require a quantitative assessment of the "power" of the players. As far as we know, the only attempt in the literature to generalize the notions of coalitional game and power index within an ordinal framework has been provided in Moretti [10], where, given a total preorder representing the relative strength of coalitions, a social ranking over the player set is provided according to a notion of *ordinal influence* and using the Banzhaf index [1] of a "canonical" coalitional game.

In the literature of simple games, related questions deal with the ordinal equivalence of power indices (see, for instance, [3, 6, 9]) and the analysis of the differences between rankings generated by alternative power indices on special classes of simple games (e.g, the papers [14, 8]). Similarly to our work, in Taylor and Zwicker [15] the authors investigated alternative notions of ordinal power on different classes of simple games. All the aforementioned papers focus on the notion of simple game, that is a numerical representation of a dichotomous power relation (i.e., winning or losing coalitions), a much more restricted domain than the one considered in this work, where a power relation can be any total preorder over the coalitions. In a still different context, a model of coalition formation has been introduced in Piccione and Razin [12], where the relative strength of disjoint coalitions is represented by an exogenous binary relation and the players try to maximize their position in a social ranking. We also notice a connection with some kind of "inverse problems", precisely, how to derive a ranking over the set of all subsets of N in a way that is "compatible" with a primitive ranking over the single elements of N (see, for instance, [2]; see [11] for an approach using coalitional games).

In this paper, a *social ranking* is defined as a map associating to each *power relation* (i.e., a total preorder over the set of all subsets of N) a total preorder over the elements of N. The properties for social rankings that we analyse in this paper have classical interpretations, such as *symmetry*, basically saying that the relative social ranking of "symmetric"[1] pairs of elements i, j and p, q should coincide (i.e., i is in the social ranking relation with j if and only if p is in the social ranking relation with q); or the *dominance*, saying that an element $i \in N$ should be ranked higher than an element $j \in N$ whenever i dominates j, i.e. a coalition $S \cup \{i\}$ is stronger than $S \cup \{j\}$ for each $S \subset N$ containing neither i nor j. Another property we study in this paper is the *independence of irrelevant coalitions*, saying that the social ranking between two elements i and j should only depend on their respective contributions when added to coalitions containing neither i nor j (in other words, the information needed to rank i and j is provided by the relative comparison of coalitions $U, W \subset N$ such that $U \setminus \{i\} = W \setminus \{j\}$).

We use these properties to axiomatically analyse social rankings on particular classes of power relations. We first notice that two natural properties, precisely, dominance and symmetry, are not compatible over the class of all power relations (see Theorem 1 in Section 4), despite the fact that, in some related axiomatic frameworks (see, for instance, [2]), similar axioms have been successfully used in combination. On the other hand, the properties of independence of irrelevant coalitions and symmetry, when applied in combination to a a large class of power relations, determine a flattening of the social ranking, where all the items are equivalent (see Proposition 2 in Section 4). Moreover, we prove that the property of independence of irrelevant coalitions and dominance property determine a kind of 'dictatorship of the cardinality' when a relation of strong dominance among coalitions of the same size holds: in this case, the only social ranking satisfying those two properties is the one imposed by the relation of dominance of a given cardinality $s \in \{1, \dots, |N|\}$ (see Theorem 2 in Section 5). Finally, we focus on an alternative algorithmic approach aimed at representing the influence of an item i as the number of coailtions S for which item i results to be essential [13], i.e., $S \cup \{i\}$ is strictly stronger than S.

The structure of the paper is the following. In the next section, we present some related approaches from the literature and our main contributions. Basic notions and definitions are presented in Section 2. In Section 3 we introduce and discuss some properties for social rankings. In Section 4 we study the compatibility of certain axioms and their effect on some elementary notions of social ranking. In Section 5 we focus on the analysis of social rankings that satisfy both the dominance property and the property of independence of irrelevant coalitions, and that, on particular power relations, are specified by the ordering of coalitions of the same size. In Section 6 we introduce a procedure to define a social ranking based on the cardinality of particular *essential set* and we finally provide some future research directions.

[1] Roughly speaking, two pairs of single elements i, j and p, q are said to be symmetric if, for coalitions S with the same cardinality, the number of times that $S \cup \{i\}$ is stronger than $S \cup \{j\}$ equals the number of times that $S \cup \{p\}$ is stronger than $S \cup \{q\}$, and the number of times that $S \cup \{j\}$ is stronger than $S \cup \{i\}$ equals the number of times that $S \cup \{q\}$ is stronger than $S \cup \{p\}$ (for more details, see Definition 3).

2 Preliminaries and notations

A *binary relation* R on a finite set $N = \{1, \ldots, n\}$ is a collection of ordered pairs of elements of N, i.e. $R \subseteq N \times N$. for all $x, y \in N$, the more familiar notation xRy will be often used instead of the more formal one $(x, y) \in R$. We provide some standard properties for R. *Reflexivity*: for each $x \in N$, xRx; *transitivity*: for each $x, y, z \in N$, xRy and $yRz \Rightarrow xRz$; *totality*: for each $x, y \in N$, $x \neq y$ $\Rightarrow xRy$ or yRx; *antisymmetry*: for each $x, y \in N$, xRy and $yRx \Rightarrow x = y$. A reflexive and transitive binary relation is called *preorder*. A preorder that is also total is called *total preorder*. A total preorder that also satisfies antisymmetry is called *linear order*. The notation $\neg(xRy)$ means that xRy is not true. We denote by 2^N the power set of N and we use the notations $\mathcal{T}^N$ and $\mathcal{T}^{2^N}$ to denote the set of all total preorders on N and on 2^N, respectively. Moreover, the cardinality of a set $S \in 2^N$ is denoted by $|S|$. In the remaining of the paper, we will also refer to an element $S \in 2^N$ as a *coalition* S. Consider a total preorder $\succcurlyeq \subseteq 2^N \times 2^N$ over the subsets of N. Often we will use the notation $S \succ T$ to denote the fact that $S \succcurlyeq T$ and $\neg(T \succcurlyeq S)$ (in this case, we also say that the relation between S and T is 'strict'), and the notation $S \sim T$ to denote the fact that $S \succcurlyeq T$ and $T \succcurlyeq S$ (in this case, we say that S and T are indifferent in $\succcurlyeq$). For each $i, j \in N$, $i \neq j$, and all $k = 1, \ldots, n-2$, we denote by $\Sigma_{ij}^k = \{S \in 2^{N \setminus \{i,j\}} : |S| = k\}$ the set of all subsets of N containing neither i nor j with k elements. Moreover, for each $i, j \in N$, we define the set $D_{ij}^k(\succcurlyeq) = \{S \in \Sigma_{ij}^k : S \cup \{i\} \succcurlyeq S \cup \{j\}\}$ as the set of coalitions $S \in 2^{N \setminus \{i,j\}}$ of cardinality k such that $S \cup \{i\}$ is in relation with $S \cup \{j\}$ (and, changing the ordering of i and j, the set $D_{ji}^k(\succcurlyeq) = \{S \in \Sigma_{ij}^k : S \cup \{j\} \succcurlyeq S \cup \{i\}\}$).

3 Axioms for social rankings

In the remaining of this paper, we interpret a total preorder $\succcurlyeq$ on 2^N as a *power relation*, that is, for each $S, T \in 2^N$, $S \succcurlyeq T$ stands for 'S is considered at least as strong as T according to the power relation $\succcurlyeq$'.

Given a class $\mathcal{C}^{2^N} \subseteq \mathcal{T}^{2^N}$ of power relations, we call a map $\rho : \mathcal{C}^{2^N} \longrightarrow \mathcal{T}^N$, assigning to each power relation in $\mathcal{C}^{2^N}$ a total preorder on N, a *social ranking solution* or, simply, a *social ranking*. Then, given a power relation $\succcurlyeq$, we will interpret the total binary relation $\rho(\succcurlyeq)$ associated to $\succcurlyeq$ by the social ranking ρ, as the relative power of items (e.g., agents) in a society under relation $\succcurlyeq$. Precisely, for each $i, j \in N$, $i\rho(\succcurlyeq)j$ stands for 'i is considered at least as influential as j according to the social ranking $\rho(\succcurlyeq)$', where the influence of an item is intended as its ability to join coalitions in the strongest positions of a power relation. Note that we require that $\rho(\succcurlyeq)$ is a total preorder over the elements of N, that is we always want to express the relative comparison of two items, and such a relation must be transitive. Two elements $i, j \in N$ such that $i\rho(\succcurlyeq)j$ and $j\rho(\succcurlyeq)i$ are said to be indifferent in $\rho(\succcurlyeq)$.

Let $\succcurlyeq \in \mathcal{C}^{2^N} \subseteq \mathcal{T}^{2^N}$. A social ranking $\rho : \mathcal{C}^{2^N} \longrightarrow \mathcal{T}^N$ such that $i\rho(\succcurlyeq)j \Leftrightarrow \{i\} \succcurlyeq \{j\}$ for each $i, j \in N$ is said to be *primitive* on $\succcurlyeq$ (i.e., it neglects any information contained in $\succcurlyeq$ about the comparison of coalitions of cardinality different from 1). A social ranking $\rho : \mathcal{C}^{2^N} \longrightarrow \mathcal{T}^N$ such that $i\rho(\succcurlyeq)j$ and $j\rho(\succcurlyeq)i$ for all $i, j \in N$ is said to be *unanimous* on $\succcurlyeq$ (N is an indifference class with respect to $\rho(\succcurlyeq)$).

Now we introduce some properties for social rankings. The first axiom is the *dominance* one: if each coalition S containing item i but not j is stronger than coalition S with j in the place of i, then item i should be ranked higher than item j in the society, for any $i, j \in N$. Precisely, given a power relation $\succcurlyeq \in \mathcal{T}^{2^N}$ and $i, j \in N$ we say that i *dominates* j in $\succcurlyeq$ if $S \cup \{i\} \succcurlyeq S \cup \{j\}$ for each $S \in 2^{N\setminus\{i,j\}}$ (we also say that i *strictly dominates* j in $\succcurlyeq$ if i dominates j and in addition there exists $S \in 2^{N\setminus\{i,j\}}$ such that $S \cup \{i\} \succ S \cup \{j\}$).

Definition 1 (DOM). *A social ranking $\rho : \mathcal{C}^{2^N} \longrightarrow \mathcal{T}^N$ satisfies the dominance (DOM) property on $\mathcal{C}^{2^N} \subseteq \mathcal{T}^{2^N}$ if and only if for all $\succcurlyeq \in \mathcal{C}^{2^N}$ and $i, j \in N$, if i dominates j in $\succcurlyeq$ then $i\rho(\succcurlyeq)j$ (and $\neg(j\rho(\succcurlyeq)i)$ if i strictly dominates j in $\succcurlyeq$).*

The following axiom states that the relative strength of two items $i, j \in N$ in the social ranking should only depend on their effect when they are added to each possible coalition S containing neither i nor j, and the relative ranking of the other coalitions is irrelevant. Formally:

Definition 2 (IIC). *A social ranking $\rho : \mathcal{C}^{2^N} \longrightarrow \mathcal{T}^N$ satisfies the Independence of Irrelevant Coalitions (IIC) property on $\mathcal{C}^{2^N} \subseteq \mathcal{T}^{2^N}$ iff*

$$i\rho(\succcurlyeq)j \Leftrightarrow i\rho(\sqsupseteq)j$$

for all $i, j \in N$ and all power relations $\succcurlyeq, \sqsupseteq \in \mathcal{C}^{2^N}$ such that for each $S \in 2^{N\setminus\{i,j\}}$

$$S \cup \{i\} \succcurlyeq S \cup \{j\} \Leftrightarrow S \cup \{i\} \sqsupseteq S \cup \{j\}.$$

Let $\succcurlyeq \in \mathcal{T}^{2^N}$, and let $i, j, p, q \in N$ be such that $|D^k_{ij}(\succcurlyeq)| = |D^k_{pq}(\succcurlyeq)|$ and $|D^k_{ji}(\succcurlyeq)| = |D^k_{qp}(\succcurlyeq)|$ for each $k = 0, \ldots, n-2$. Differently stated, for coalitions S of fixed cardinality, we have that the number of times that $S \cup \{i\}$ is stronger than $S \cup \{j\}$ equals the number of times that $S \cup \{p\}$ is stronger than $S \cup \{q\}$ (and the number of times that $S \cup \{j\}$ is stronger than $S \cup \{i\}$ equals the number of times that $S \cup \{q\}$ is stronger than $S \cup \{p\}$). In this symmetric situation, the following axiom states a principle of equivalence between the pairs $\{i, j\}$ and $\{p, q\}$.

Definition 3 (SYM). *A social ranking $\rho : \mathcal{C}^{2^N} \longrightarrow \mathcal{T}^N$ satisfies the symmetry (SYM) property on $\mathcal{C}^{2^N} \subseteq \mathcal{T}^{2^N}$ iff*

$$i\rho(\succcurlyeq)j \Leftrightarrow p\rho(\succcurlyeq)q$$

for all $i, j, p, q \in N$ and $\succcurlyeq \in \mathcal{C}^{2^N}$ such that $|D^k_{ij}(\succcurlyeq)| = |D^k_{pq}(\succcurlyeq)|$ and $|D^k_{ji}(\succcurlyeq)| = |D^k_{qp}(\succcurlyeq)|$ for each $k = 0, \ldots, n-2$.

Remark 1. Note that if a social ranking ρ satisfies the SYM axiom on $\mathcal{C}^{2^N} \subseteq \mathcal{T}^{2^N}$, then for every $\succcurlyeq \in \mathcal{C}^{2^N}$ and $i, j \in N$, if $|D_{ij}^k(\succcurlyeq)| = |D_{ji}^k(\succcurlyeq)|$ for each $k = 0, \ldots, n-2$, then $i\rho(\succcurlyeq)j$ and $j\rho(\succcurlyeq)i$, that is i and j are indifferent in $\rho(\succcurlyeq)$ (to see this, simply take $p = i$ and $q = j$ in Definition 3).

Remark 2. If we want to check if a given social ranking solution satisfies DOM, IIC, or SYM only partial information on $\succcurlyeq$ is needed. In fact, conditions on the ranking $\rho(\succcurlyeq)$ between two elements i and j only depend on the comparisons of subsets $S \cup \{i\}$ and $S \cup \{j\}$, for all $S \in 2^{N \setminus \{i,j\}}$.

We conclude this section with an example showing that an apparently natural procedure (namely, the majority rule) to rank the items of N may fail to provide a transitive social ranking. We first formally introduce such a procedure.

Definition 4 (Majority rule). *The majority rule (denoted by M) is the map assigning to each power relation $\succcurlyeq \in \mathcal{T}^{2^N}$ the total binary relation $M(\succcurlyeq)$ on N such that*

$$iM(\succcurlyeq)j \Leftrightarrow d_{ij}(\succcurlyeq) \geq d_{ji}(\succcurlyeq).$$

where $d_{ij}(\succcurlyeq) = \sum_{k=0}^{n-2} |D_{ij}^k(\succcurlyeq)|$ for each $i, j \in N$.

Example 1. One can easily check that the majority rule M satisfies the property of DOM, IIC and SYM on the class $\mathcal{T}^{2^N}$. On the other hand, it is also easy to find an example of power relation $\succcurlyeq$ such that $M(\succcurlyeq)$ is not transitive. Consider for instance the power relation $\succcurlyeq \in \mathcal{T}^{2^N}$ with $N = \{1, 2, 3, 4\}$ such that: $2 \succ 1 \succ 3 \succ 23 \succ 13 \succ 12 \succ 14 \succ 34 \succ 24 \succ 134 \sim 124 \sim 234$.

We rewrite the relevant information about $\succcurlyeq$ by means of Table 1 (From now, we will sometimes omit braces and commas to separate elements, for instance, ij denotes the set $\{i, j\}$). Note that $d_{12}(\succcurlyeq) = 2$, $d_{21}(\succcurlyeq) = 3$, $d_{23}(\succcurlyeq) = 2$, $d_{32}(\succcurlyeq) = 3$, $d_{13}(\succcurlyeq) = 3$ and $d_{31}(\succcurlyeq) = 2$. So, we have that $2M(\succcurlyeq)1$, $3M(\succcurlyeq)2$ and $1M(\succcurlyeq)3$, but $\neg(3M(\succcurlyeq)1)$: $M(\succcurlyeq)$ is not a transitive relation.

Table 1. The relevant information about $\succcurlyeq$ of Example 1.

1 vs. 2	2 vs. 3	1 vs. 3
$1 \prec 2$	$2 \succ 3$	$1 \succ 3$
$13 \prec 23$	$12 \prec 13$	$12 \prec 23$
$14 \succ 24$	$24 \prec 34$	$14 \succ 34$
$134 \sim 234$	$124 \sim 134$	$124 \sim 234$

4 Primitive and unanimous social rankings

In this section we study the relations between the axioms introduced in the previous section and the social ranking solutions. In the following, we show that

DOM and SYM are not compatible in a general case, for $N > 3$ (see Theorem 1), whereas SYM and IIC determine a unanimous social ranking on particular power relations.

We start with showing some consequences of using the axioms introduced in the previous section when the cardinality of the set N is 3 or 4. The analysis for cardinality $|N| = 3$ is easy since we can enumerate all the cases. As we will present in the following, the notion of complementarity plays an important role in this case. We denote by S^* the complement of the subset S ($S^* = N \setminus S$), and we say that a social ranking ρ such that $i\rho(\succcurlyeq)j \Leftrightarrow \{j\}^* \succcurlyeq \{i\}^*$ for each $i, j \in N$ is *complement primitive* on $\succcurlyeq$ (i.e., it neglects any information contained in $\succcurlyeq$ about the comparison of coalitions of cardinality different from $n-1$).

Proposition 1. *If $|N| = 3$, the only social ranking solution satisfying the DOM and SYM axioms can be either primitive or complement primitive on $\succcurlyeq \in \mathcal{T}^{2^N}$.*

Proof. Let $N = \{1, 2, 3\}$ with $1 \succcurlyeq 2 \succcurlyeq 3$. Then six cases may occur in $\succcurlyeq$: case 1) $13 \succcurlyeq 23 \succcurlyeq 12$, case 2) $13 \succcurlyeq 12 \succcurlyeq 23$, case 3) $23 \succcurlyeq 13 \succcurlyeq 12$, case 4) $12 \succcurlyeq 13 \succcurlyeq 23$, case 5) $23 \succcurlyeq 12 \succcurlyeq 13$ and case 6) $12 \succcurlyeq 23 \succcurlyeq 13$.

DOM and SYM impose that:

- case 1) by DOM :$1\rho(\succcurlyeq)2$, by SYM ($1\rho(\succcurlyeq)3$ and $2\rho(\succcurlyeq)3$) or ($3\rho(\succcurlyeq)1$ and $3\rho(\succcurlyeq)2$). Hence we have $1\rho(\succcurlyeq)2\rho(\succcurlyeq)3$ (primitive) or $3\rho(\succcurlyeq)1\rho(\succcurlyeq)2$ (complement primitive)
- case 2) by DOM :$1\rho(\succcurlyeq)2$ and $1\rho(\succcurlyeq)3$. We can have $2\rho(\succcurlyeq)3$ or $3\rho(\succcurlyeq)2$. Hence we have $1\rho(\succcurlyeq)2\rho(\succcurlyeq)3$ (primitive) or $1\rho(\succcurlyeq)3\rho(\succcurlyeq)2$ (complement primitive)
- case 3) by SYM : ($1\rho(\succcurlyeq)2$, $1\rho(\succcurlyeq)3$ and $2\rho(\succcurlyeq)3$) or ($2\rho(\succcurlyeq)1$, $3\rho(\succcurlyeq)1$ and $3\rho(\succcurlyeq)2$).
- case 4) by DOM $1\rho(\succcurlyeq)2\rho(\succcurlyeq)3$
- case 5) by DOM :$2\rho(\succcurlyeq)3$, by SYM ($1\rho(\succcurlyeq)2$ and $1\rho(\succcurlyeq)3$) or ($2\rho(\succcurlyeq)1$ and $3\rho(\succcurlyeq)1$). Hence we have $1\rho(\succcurlyeq)2\rho(\succcurlyeq)3$ (primitive) or $2\rho(\succcurlyeq)3\rho(\succcurlyeq)1$ (complement primitive)
- case 6) by DOM :$1\rho(\succcurlyeq)3$ and $2\rho(\succcurlyeq)3$. We can have $1\rho(\succcurlyeq)2$ or $2\rho(\succcurlyeq)1$. Hence we have $1\rho(\succcurlyeq)2\rho(\succcurlyeq)3$ (primitive) or $2\rho(\succcurlyeq)1\rho(\succcurlyeq)3$ (complement primitive)

Corollary 1. *If $|N| = 3$ and $\succcurlyeq \in \mathcal{T}^{2^N}$ such that for all $S, Q \subseteq N$, $S \succcurlyeq Q$ implies $Q^* \succcurlyeq S^*$ (i.e., according to [5], $\succcurlyeq$ is said to be "self-reflecting"), then a social ranking satisfying the DOM property is primitive on $\succcurlyeq$.*

Proof. Let $N = \{i, j, k\}$. Self-reflecting implies that for all $i, j \in N$ $i \succcurlyeq j \Leftrightarrow j^* \succcurlyeq i^* \Leftrightarrow ik \succcurlyeq jk$. By DOM we get for all $i, j, k \in N$ $i\rho(\succcurlyeq)j \Leftrightarrow i \succcurlyeq j \Leftrightarrow j^* \succcurlyeq i^* \Leftrightarrow ik \succcurlyeq jk$.

Next theorem shows that on the class $\mathcal{T}^{2^N}$ (all possible total preorders) the properties of DOM and SYM are not compatible.

Theorem 1. *Let $|N| > 3$. There is no social ranking solution $\rho : \mathcal{T}^{2^N} \longrightarrow \mathcal{T}^N$ which satisfies DOM and SYM on $\mathcal{T}^{2^N}$.*

Proof. We first show a particular situation where DOM and SYM are not compatible. Consider a power relation $\succcurlyeq \in \mathcal{T}^{2^N}$ with $N = \{1, 2, 3, 4\}$ and such that

$$1 \sim 2 \sim 3 \succ 13 \succ 23 \succ 12 \succ 24 \sim 14 \succ 34 \succ 1234 \sim 123 \sim 124 \sim 134 \sim 234$$

We rewrite the relevant information about $\succcurlyeq$ and the elements $1, 2$ and 3 by means of the following Table 2. By Remark 1, a social ranking solution $\rho : \mathcal{T}^{2^N} \longrightarrow \mathcal{T}^N$ which satisfies SYM should be such that $2\rho(\succcurlyeq)3$, $3\rho(\succcurlyeq)2$, $1\rho(\succcurlyeq)3$, $3\rho(\succcurlyeq)1$. By the DOM property, we should have $1\rho(\succcurlyeq)2$, and $\neg(2\rho(\succcurlyeq)1)$, which yields a contradiction with the transitivity of the ranking $\rho(\succcurlyeq)$.

Table 2. The relevant information about $\succcurlyeq$ and the elements $1, 2$ and 3.

1 vs. 2	2 vs. 3	1 vs. 3
$1 \sim 2$	$2 \sim 3$	$1 \sim 3$
$13 \succ 23$	$12 \prec 13$	$12 \prec 23$
$14 \sim 24$	$24 \succ 34$	$14 \succ 34$
$134 \sim 234$	$124 \sim 134$	$124 \sim 234$

The incompatibility between DOM and SYM also holds for power relations on 2^N with $|N| > 4$. This conclusion directly follows from the fact that one can generate power relations in $\mathcal{T}^{2^N}$, with $N \supseteq \{1, 2, 3, 4\}$, that are obtained from the power relation $\succcurlyeq$ defined above and assigning all the additional subsets of N not contained in $\{1, 2, 3, 4\}$ in the same indifference class. More precisely, the arguments used to show the incompatibility of DOM and SYM on $\succcurlyeq$ also hold for a power relation $\succcurlyeq' \in \mathcal{T}^{2^N}$ with $N \supset \{1, 2, 3, 4\}$ and such that

$$U \succcurlyeq' W :\Leftrightarrow U \succcurlyeq W$$

for all the subsets $U, W \subseteq \{1, 2, 3, 4\}$ (i.e., the subsets of $\{1, 2, 3, 4\}$ are ranked in $\succcurlyeq'$ precisely as in $\succcurlyeq$) and

$$U \succcurlyeq' W \text{ and } W \succcurlyeq' U$$

for all the other subsets of N not included in $\{1, 2, 3, 4\}$ (i.e., all the sets not contained in $\{1, 2, 3, 4\}$ are indifferent with respect to the power relation $\succcurlyeq'$).

The following proposition shows that the adoption of properties IIC and SYM yields a unanimous social ranking over all those power relations $\succcurlyeq \in \mathcal{T}^N$ such that, for some $i, j \in N$ and $k \in \{0, \ldots, |N| - 2\}$, the relation between $S \cup \{j\}$ and $S \cup \{i\}$ holds strict in the two directions for some $S \in 2^{N \setminus \{i,j\}}$ with $|S| = k$ (precisely, $D^k_{ij}(\succcurlyeq) \setminus D^k_{ji}(\succcurlyeq) \neq \emptyset$ and $D^k_{ji}(\succcurlyeq) \setminus D^k_{ij}(\succcurlyeq) \neq \emptyset$), whereas for all the cardinalities $t \neq k$, we have that $S \cup \{j\}$ and $S \cup \{i\}$ are indifferent for each $S \in 2^{N \setminus \{i,j\}}$ with $|S| = t$ (precisely, $D^t_{ji}(\succcurlyeq) = D^t_{ij}(\succcurlyeq)$).

Proposition 2. *Let $\rho : \mathcal{T}^{2^N} \longrightarrow \mathcal{T}^N$ be a social ranking satisfying IIC and SYM. Let $\succcurlyeq \in \mathcal{T}^{2^N}$, $i, j \in N$ and $k \in \{0, \ldots, |N|-2\}$ be s.t. $D^k_{ij}(\succcurlyeq) \setminus D^k_{ji}(\succcurlyeq) \neq \emptyset$ and $D^k_{ji}(\succcurlyeq) \setminus D^k_{ij}(\succcurlyeq) \neq \emptyset$, and s.t. $D^t_{ji}(\succcurlyeq) = D^t_{ij}(\succcurlyeq)$, for all $t \neq k$. Then $i\rho(\succcurlyeq)j$ and $j\rho(\succcurlyeq)i$.*

Proof. Take $i, j \in N$ such that $|D^k_{ij}(\succcurlyeq)| \geq |D^k_{ji}(\succcurlyeq)|$. Define another power relation $\sqsupseteq \in \mathcal{T}^{2^N}$ such that

$$S \cup \{i\} \succcurlyeq S \cup \{j\} \Leftrightarrow S \cup \{i\} \sqsupseteq S \cup \{j\}$$

for each $S \in 2^{N\setminus\{i,j\}}$ with $|S| = k$, and $S \sqsupseteq T$ and $T \sqsupseteq S$ for all the other coalitions $S, T \in 2^N$ with $|S| = |T| \neq k+1$. We still need to define relation $\sqsupseteq$ on the remaining coalitions of size k.

Take $l \in N \setminus \{i, j\}$. Let $\mathcal{D} \subseteq D^k_{ij}(\succcurlyeq)$ be such that $|\mathcal{D}| = |D^k_{ji}(\succcurlyeq)|$. Define the remaining comparisons in $\sqsupseteq$ as follows (an illustrative example of these cases are given in Table 3):
case 1) for each $S \in D^k_{ji}(\succcurlyeq)$ with $l \in S$, let $S \cup \{i,j\} \setminus \{l\} \sqsubseteq S \cup \{j\}$ and $S \cup \{i,j\} \setminus \{l\} \sqsupseteq S \cup \{i\}$;
case 2) for each $S \in D^k_{ji}(\succcurlyeq)$ with $l \notin S$, let $S \cup \{i\} \sqsubseteq S \cup \{l\}$ and $S \cup \{j\} \sqsupseteq S \cup \{l\}$;
case 3) For each $S \in \mathcal{D}$ with $l \in S$, let $S \cup \{i,j\} \setminus \{l\} \sqsubseteq S \cup \{j\}$ and $S \cup \{i,j\} \setminus \{l\} \sqsubseteq S \cup \{i\}$;
case 4) for each $S \in \mathcal{D}$ with $l \notin S$, let $S \cup \{i\} \sqsubseteq S \cup \{l\}$ and $S \cup \{j\} \sqsubseteq S \cup \{l\}$;
case 5) for each $S \in D^k_{ij} \setminus \mathcal{D}$ with $l \in S$, let $S \cup \{i,j\} \setminus \{l\} \sqsupseteq S \cup \{j\}$ and $S \cup \{i,j\} \setminus \{l\} \sqsupseteq S \cup \{i\}$;
case 6) for each $S \in D^k_{ij} \setminus \mathcal{D}$ with $l \notin S$, let $S \cup \{i\} \sqsupseteq S \cup \{l\}$ and $S \cup \{j\} \sqsupseteq S \cup \{l\}$.
Notice that $|D^k_{ji}(\succcurlyeq)| = |D^k_{li}(\sqsupseteq)| = |D^k_{jl}(\sqsupseteq)|$ and $|D^k_{ij}(\succcurlyeq)| = |D^k_{il}(\sqsupseteq)| = |D^k_{lj}(\sqsupseteq)|$.

Table 3. An illustrative example of the six possible cases for a power relation $\sqsupseteq$ as the one considered in Proposition 2 with $N = \{1, 2, 3, i, j, l\}$, $k = 2$ and $\mathcal{D} = \{\{1, 2\}, \{2, l\}\}$.

	i vs j	i vs. l	j vs. l
case 1): $S = \{3, l\}$	$\{3, i, l\} \sqsubseteq \{3, j, l\}$	$\{3, i, j\} \sqsubseteq \{3, j, l\}$	$\{3, i, j\} \sqsupseteq \{3, i, l\}$
case 2): $S = \{2, 3\}$	$\{2, 3, i\} \sqsubseteq \{2, 3, j\}$	$\{2, 3, i\} \sqsubseteq \{2, 3, l\}$	$\{2, 3, j\} \sqsupseteq \{2, 3, l\}$
case 3): $S = \{2, l\}$	$\{2, i, l\} \sqsupseteq \{2, j, l\}$	$\{2, i, j\} \sqsubseteq \{2, j, l\}$	$\{2, i, j\} \sqsubseteq \{2, i, l\}$
case 4): $S = \{1, 2\}$	$\{1, 2, i\} \sqsupseteq \{1, 2, j\}$	$\{1, 2, i\} \sqsubseteq \{1, 2, l\}$	$\{1, 2, j\} \sqsubseteq \{1, 2, l\}$
case 5): $S = \{1, l\}$	$\{1, i, l\} \sqsupseteq \{1, j, l\}$	$\{1, i, j\} \sqsupseteq \{1, j, l\}$	$\{1, i, j\} \sqsupseteq \{1, i, l\}$
case 6): $S = \{1, 3\}$	$\{1, 3, i\} \sqsupseteq \{1, 3, j\}$	$\{1, 3, i\} \sqsupseteq \{1, 3, l\}$	$\{1, 3, j\} \sqsupseteq \{1, 3, l\}$
	$\|D_{ij}(\sqsupseteq)\| = 4$ $\|D_{ji}(\sqsupseteq)\| = 2$	$\|D_{il}(\sqsupseteq)\| = 2$ $D_{li}(\sqsupseteq)\| = 4$	$\|D_{jl}(\sqsupseteq)\| = 4$ $\|D_{lj}(\sqsupseteq)\| = 2$

Suppose now that $i\rho(\succcurlyeq)j$. By IIC, we have $i\rho(\sqsupseteq)j$. By SYM, $j\rho(\sqsupseteq)l$ and $l\rho(\sqsupseteq)i$. By transitivity of $\rho(\sqsupseteq)$, $j\rho(\sqsupseteq)i$. By IIC we conclude that $j\rho(\succcurlyeq)i$ too. In a similar way, if we suppose $j\rho(\succcurlyeq)i$, then we end up with the conclusion that $i\rho(\succcurlyeq)j$ too, and the proof follows.

5 Dictatorship of the coalition size

In this section, we define a class of power relations (namely, the *per size-strong dominant* relations) characterized by the fact that a relation of dominance always exists with respect to coalitions of the same size, but the dominance may change with the cardinality (for instance, an element i could dominate another element j when coalitions of size s are considered, but j could dominate i over coalitions of size $t \neq s$). We first need to introduce the notion of s-strong dominance.

Definition 5. *Let* $\succcurlyeq \in \mathcal{T}^{2^N}$*,* $i, j \in N$ *and* $s \in \{0, \ldots, n-2\}$*. We say that* i *s-strong dominates* j *in* $\succcurlyeq$*, iff*

$$S \cup \{i\} \succ S \cup \{j\} \text{ for each } S \in 2^{N \setminus \{i,j\}} \text{ with } |S| = s. \tag{1}$$

Definition 6. *We say that* $\succcurlyeq \in \mathcal{T}^{2^N}$ *is per size-strong dominant (shortly, ps-sdom) iff for each* $s \in \{0, \ldots, n-2\}$ *and all* $i, j \in N$*, we have either*

$$[i \text{ } s\text{-strong dominates } j \text{ in } \succcurlyeq] \text{ or } [j \text{ } s\text{-strong dominates } i \text{ in } \succcurlyeq].$$

The set of all ps-sdom power relations is denoted by $\mathcal{S}^{2^N} \subseteq \mathcal{T}^{2^N}$*.*

Now, we study the effect of the combination of the properties of DOM and IIC on a specific instance of ps-sdom power relations where there exist elements that are always placed at the top or at the bottom in the rankings of coalitions of equal cardinality.

Example 2. Consider a power relation $\succcurlyeq \in \mathcal{S}^{2^N}$ with $N = \{1, 2, 3, 4\}$ and such that

$$1 \succ 2 \succ 3 \succ 4 \succ 34 \succ 24 \succ 14 \succ 23 \succ 13 \succ 12 \succ 123 \succ 134 \succ 124 \succ 234.$$

We rewrite the relevant information about $\succcurlyeq$ by means of Table 4.

Table 4. The relevant information about $\succcurlyeq$ of Example 2.

1 vs. 2	2 vs. 3	1 vs. 3	1 vs. 4	2 vs. 4	3 vs. 4
$1 \succ 2$	$2 \succ 3$	$1 \succ 3$	$1 \succ 4$	$2 \succ 4$	$3 \succ 4$
$13 \prec 23$	$12 \prec 13$	$12 \prec 23$	$12 \prec 24$	$12 \prec 14$	$13 \prec 14$
$14 \prec 24$	$24 \prec 34$	$14 \prec 34$	$13 \prec 34$	$23 \prec 34$	$23 \prec 24$
$134 \succ 234$	$124 \prec 134$	$124 \succ 234$	$123 \succ 234$	$123 \succ 134$	$123 \succ 124$

Note that for all $S \subseteq N \setminus \{1\}$ and each $l \in N \setminus (S \cup \{1\})$, it holds that $S \cup \{1\} \succcurlyeq S \cup \{l\}$ if $|S| \in \{0, 2\}$ (i.e., coalition $S \cup \{1\}$ is ranked above coalition $S \cup \{l\}$, for all S containing 0 or 2 elements), whereas $S \cup \{1\} \preccurlyeq S \cup \{l\}$ if $|S| = 1$ (i.e., coalition $S \cup \{1\}$ is ranked below coalition $S \cup \{l\}$, for all S containing precisely one element). So, elements 1 (or, similar, element 4) is an "extreme" element of

N in $\succcurlyeq$, where for extreme element we mean an element $i \in N$ such that, for all coalitions S of the same size and not containing i, we have either $S \cup \{i\} \succcurlyeq S \cup \{l\}$ for all $l \in N \setminus (S \cup \{i\})$, or, $S \cup \{l\} \succcurlyeq S \cup \{i\}$ for all $l \in N \setminus (S \cup \{i\})$. In Proposition 3 we argue that on this kind of power relations, a social ranking satisfying both DOM and IIC cannot rank "extreme" elements in between two others.

Proposition 3. *Let $\rho : \mathcal{S}^{2^N} \longrightarrow \mathcal{T}^N$ be a social ranking satisfying IIC and DOM on $\mathcal{S}^{2^N}$. Let $\succcurlyeq \in \mathcal{S}^{2^N}$ and $i \in N$ be such that for each $s \in \{0, \ldots, n-2\}$ either*

$$[S \cup \{i\} \succ S \cup \{j\} \text{ for all } j \in N \setminus \{i\} \text{ and } S \in 2^{N \setminus \{i,j\}} \text{ with } |S| = s] \qquad (2)$$

or

$$[S \cup \{j\} \succ S \cup \{i\} \text{ for all } j \in N \setminus \{i\} \text{ and } S \in 2^{N \setminus \{i,j\}} \text{ with } |S| = s]. \qquad (3)$$

Then, $[i\rho(\succcurlyeq)j$ for all $j \in N]$ or $[j\rho(\succcurlyeq)i$ for all $j \in N]$.

Proof. Suppose on the contrary that there exist $j, k \in N \setminus \{i\}$, such that

$$j\rho(\succcurlyeq)i \text{ and } i\rho(\succcurlyeq)k. \qquad (4)$$

Define $\sqsupseteq \in \mathcal{S}^{2^N}$ such that

$$S \cup \{i\} \sqsupset S \cup \{j\} \Leftrightarrow S \cup \{i\} \succ S \cup \{j\} \text{ for all } S \subseteq N \setminus \{i,j\}, \qquad (5)$$

$$S \cup \{i\} \sqsupset S \cup \{k\} \Leftrightarrow S \cup \{i\} \succ S \cup \{k\} \text{ for all } S \subseteq N \setminus \{i,k\}, \qquad (6)$$

and

$$S \cup \{k\} \sqsupset S \cup \{j\} \text{ for all } S \subseteq N \setminus \{j,k\}. \qquad (7)$$

(note that each coalition $S \cup \{i\}$, with $S \subseteq N \setminus \{i\}$, by condition (2) and (3), is ranked strictly higher or lower than each other coalition $S \cup \{j\}$, $j \neq i$, so condition (7) does not violate the transitivity of $\sqsupseteq$.)

By IIC, we have that $i\rho(\succcurlyeq)j \Leftrightarrow i\rho(\sqsupseteq)j$ and $i\rho(\succcurlyeq)k \Leftrightarrow i\rho(\sqsupseteq)k$. So, by relation (4), $j\rho(\sqsupseteq)i$ and $i\rho(\sqsupseteq)k$. On the other hand, by DOM we have $k\rho(\sqsupseteq)j$ and $\neg(j\rho(\sqsupseteq)k)$, which yields a contradiction with the transitivity of $\rho(\sqsupseteq)$.

Proposition 3 shows that if there is an element $i \in N$ having "contradictory" and "radical" behavior depending on the size of coalitions, then the social ranking satisfying IIC and DOM can not give him an intermediate position. In the following, we argue that if a power relation is in $\mathcal{S}^{2^N}$ and a social ranking satisfies both DOM and IIC on the set of ps-sdom power relations $\mathcal{S}^{2^N}$, then it must exist a cardinality $t^* \in \{0, \ldots, n-2\}$ whose relation of t^*-strong dominance (dictatorially) determines the social ranking. We first introduce the next lemma.

Lemma 1. *Let $i \in N$ and $\rho : \mathcal{S}^{2^N} \longrightarrow \mathcal{T}^N$ be a social ranking satisfying IIC and DOM on $\mathcal{S}^{2^N}$. There exists $t^* \in \{0, \ldots, n-2\}$ such that*

$$j\rho(\succcurlyeq)k \Leftrightarrow j \ t^*\text{-strong dominates } k \text{ in } \succcurlyeq,$$

for all $j, k \in N \setminus \{i\}$ and $\succcurlyeq \in \mathcal{S}^{2^N}$.

Proof. Given a power relation $\succcurlyeq \in \mathcal{S}^{2^N}$, define another power relation $\succcurlyeq_0 \in \mathcal{S}^{2^N}$ such that for each $S \subseteq N \setminus \{i\}$ we have

$$S \cup \{l\} \succ_0 S \cup \{i\} \text{ for all } l \in N \setminus (S \cup \{i\}), \tag{8}$$

and $U \succcurlyeq_0 W :\Leftrightarrow U \succcurlyeq W$ for all the other possible pairs of coalitions U, W whose comparison is not already considered in (8). Roughly speaking, the only difference between $\succcurlyeq_0$ and $\succcurlyeq$ is that coalitions of size s containing i are placed at the bottom of the ranking induced by $\succcurlyeq$ over the coalitions of the same size. By DOM, it follows that $l\rho(\succcurlyeq_0)i$ for every $l \in N$.

Now, for each $t \in \{0, \dots, n-2\}$, define a power relation $\succcurlyeq_t \in \mathcal{T}^{2^N}$ such that

$$S \cup \{i\} \succ_t S \cup \{l\} \text{ for each } l \in N \text{ and } S \in 2^{N \setminus \{i,l\}} \text{ with } |S| = s, \tag{9}$$

where $s \in \{0, \dots, t\}$, and $U \succcurlyeq_t W :\Leftrightarrow U \succcurlyeq_{t-1} W$ for all the other possible pairs of coalitions U, W whose comparison is not already considered in (9). So, the only difference between $\succcurlyeq_t$ and $\succcurlyeq_{t-1}$, for each $t \in \{1, \dots, n-2\}$, is that in $\succcurlyeq_t$ coalitions of size t containing i are placed at the top of the ranking induced by $\succcurlyeq_{t-1}$ over coalitions of the same size t, and all the remaining comparisons remain the same as in $\succcurlyeq_{t-1}$.

Note that by Proposition 3, we have that either $l\rho(\succcurlyeq_t)i$ for every $l \in N$, or $i\rho(\succcurlyeq_t)l$ for every $l \in N$. Moreover, By DOM, it follows that $i\rho(\succcurlyeq_{n-2})l$ for every $j \in N$. Let t^* be the smallest number in $\{0, \dots, n-2\}$ such that $l\rho(\succcurlyeq_{t^*-1})i$ for every $l \in N$ and $i\rho(\succcurlyeq_{t^*})l$ for every $l \in N$ (for the considerations above such a t^* must exist, being, at most, $t^* = n-2$). Next, we argue that for every $j, k \in N \setminus \{i\}$, the social ranking between j and k in $\succcurlyeq$ is imposed by the relation of t^*-strong dominance in $\succcurlyeq$. W.l.o.g., suppose that $S \cup \{j\} \succcurlyeq S \cup \{k\}$ (and, as a consequence, $S \cup \{j\} \succcurlyeq_{t^*} S \cup \{k\}$) for each $S \in 2^{N \setminus \{j,k\}}$, and $|S| = t^*$. Consider another power relation $\sqsupseteq \in \mathcal{T}^{2^N}$ obtained by $\succcurlyeq_{t^*}$ and such that:

$$S \cup \{j\} \sqsupset S \cup \{i\} \text{ for each } S \in 2^{N \setminus \{i,j\}} \text{ with } |S| = t^*, \tag{10}$$

$$S \cup \{i\} \sqsupset S \cup \{k\} \text{ for each } S \in 2^{N \setminus \{i,k\}} \text{ with } |S| = t^*, \tag{11}$$

$$S \cup \{j\} \sqsupset S \cup \{k\} \text{ for each } S \in 2^{N \setminus \{j,k\}} \setminus \left(2^{N \setminus \{i,j\}} \cup 2^{N \setminus \{i,k\}}\right), \text{ and } |S| = t^*, \tag{12}$$

and, finally,

$$U \sqsupseteq V :\Leftrightarrow U \succcurlyeq_{t^*} V \tag{13}$$

for all the other relevant pairs of coalitions U, W of size $s \neq t^*+1$. By IIC $j\rho(\sqsupseteq)i$ (since in $\sqsupseteq$ the comparisons between coalitions containing i and j are precisely as in $\succcurlyeq_{t^*-1}$ and, as previously stated, $j\rho(\succcurlyeq_{t^*-1})i$) and $i\rho(\sqsupseteq)k$ (since in $\sqsupseteq$ the comparisons between coalitions containing i and k are precisely as in $\succcurlyeq_{t^*}$ and, as previously stated, $i\rho(\succcurlyeq_{t^*})k$). Then, by transitivity of $\rho(\sqsupseteq)$ we have $j\rho(\sqsupseteq)k$. Note that by IIC, $j\rho(\sqsupseteq)k \Leftrightarrow j\rho(\succcurlyeq_{t^*})k \Leftrightarrow j\rho(\succcurlyeq)k$. We have then proved that whenever j t^*-dominates k, then $j\rho(\succcurlyeq)k$.

The following theorem states the "dictatorship of the coalition's size".

Theorem 2. *Let $\rho : \mathcal{S}^{2^N} \longrightarrow \mathcal{T}^N$ be a social ranking satisfying IIC and DOM on $\mathcal{S}^{2^N}$. There exists $t^* \in \{0, \dots, n-2\}$ such that*

$$i\rho(\succcurlyeq)j \Leftrightarrow i \ t^*\text{-strong dominates } j \text{ in } \succcurlyeq,$$

for all $i, j \in N$ and $\succcurlyeq \in \mathcal{S}^{2^N}$.

Proof. Given a power relation $\succcurlyeq \in \mathcal{S}^{2^N}$, let $i \in N$ and define $\succcurlyeq_{t^*}$ starting from $\succcurlyeq$ and i precisely as in the proof of Lemma 1.

Now take $k \in N \setminus \{i\}$ and apply Lemma 1 with k in the role of i. Consequently, we have that there exists $\hat{t} \in \{0, \dots, n-2\}$ such that

$$h\rho(\succcurlyeq)l \Leftrightarrow h \ \hat{t}\text{-strong dominates } l \text{ in } \succcurlyeq,$$

for each $h, l \in N \setminus \{k\}$, and in particular

$$i\rho(\succcurlyeq)l \Leftrightarrow i \ \hat{t}\text{-strong dominates } l \text{ in } \succcurlyeq,$$

for any complete power relation $\succcurlyeq \in \mathcal{S}^{2^N}$. But in the proof of Lemma 1 we have shown that

$$i\rho(\succcurlyeq)l \Leftrightarrow i \ t^*\text{-strong dominates } l \text{ in } \succcurlyeq_{t^*}$$

(remember that t^* in the proof of Lemma 1 is the smallest number in $\{0, \dots, n-2\}$ such that $l\rho(\succcurlyeq_{t^*-1})i$ for every $l \in N$ and $i\rho(\succcurlyeq_{t^*})l$ for every $l \in N$). Then it must be $\hat{t} = t^*$, and the proof follows.

6 An algorithmic approach

In view of the results provided in the previous axiomatic analysis, each combination of two axioms yields either no social ranking or an unsatisfactory one. It is worth noting that all the axioms that we studied in this paper are based on the comparison of subsets having the same number of elements. Therefore, it would be interesting to study properties based on the comparison among subsets with different cardinalities. Following this idea, an interesting property is the notion of *essential alternative* that has been introduced in Puppe [13] as a necessary condition for a power relation representing the preferences of a decision maker over menus (in this context, the preference over menus of a decision maker should reflect her or his freedom to chose a most preferred alternative from any selected menu). Given a power relation $\succcurlyeq \in \mathcal{T}^{2^N}$ and a coalition $S \in 2^N$, an element $i \in N \setminus S$ is said to be *essential* for S if $S \cup \{i\} \succ S$. In our framework, where a power relation represents the relative strength of coalitions, an item i is essential for a coalition S not containing i if coalition $S \cup \{i\}$ is strictly stronger than S. Differently stated, an item i is essential for S (not containing i), if the marginal contribution $v(S \cup \{i\}) - v(S)$ of i to $S \cup \{i\}$ is strictly positive, for every utility function $v : 2^N \to \mathbb{R}$ associated to the power relation $\succcurlyeq$ and such that $v(T) \geq v(U) :\Leftrightarrow T \succcurlyeq U$, for each $T, U \in 2^N$. Our goal in this section is to assess the influence of items in terms of the number of coalitions in which each item i is essential under a given power relation. More precisely, for each item $i \in N$ we first need to introduce the notion of *essential set* $E_i(\succcurlyeq) := \{S \in 2^{N \setminus \{i\}} : S \cup \{i\} \succ S\}$. Then we define the social ranking solution $\rho^e : \mathcal{T}^{2^N} \longrightarrow \mathcal{T}^N$ such that

$$i \rho^e(\succcurlyeq) j :\Leftrightarrow |E_i(\succcurlyeq)| \geq |E_j(\succcurlyeq)| \tag{14}$$

for each $i, j \in N$ and $\succcurlyeq \in \mathcal{T}^{2^N}$. It is easy to check that ρ^e does not satisfy any of the axioms studied in the previous sections.

Example 3. Consider the power relation $\succcurlyeq \in \mathcal{T}^{2^N}$ with $N = \{1, 2, 3, 4\}$ such that $2 \succ 4 \succ 23 \succ 123 \succ 13 \sim 134 \sim 124 \sim 234 \sim N \sim 12 \succ 14 \succ 1 \succ 3 \succ 34 \succ 24 \succ \emptyset$. Notice that the relevant information presented in Table 1 of Example 1 is still compatible with this power relation. Moreover, the essential sets for players in N are: $E_1(\succcurlyeq) = \{\emptyset, \{3\}, \{2, 4\}, \{3, 4\}\}$, $E_2(\succcurlyeq) = \{\emptyset, \{3\}, \{1\}, \{1, 3\}, \{1, 4\}, \{3, 4\}\}$, $E_3(\succcurlyeq) = \{\emptyset, \{1, 2\}, \{1, 4\}, \{2, 4\}\}$ and $E_4(\succcurlyeq) = \{\emptyset, \{1\}\}$. Consequently, accordingly to the social ranking ρ^e, 2 is the most influential item ($|E_2(\succcurlyeq)| = 6$), followed by 1 and 3 with the same score ($|E_1(\succcurlyeq)| = E_3(\succcurlyeq)| = 4$), and finally by item 4 ($|E_4(\succcurlyeq)| = 2$).

Notice that the definition of an essential set $E_i(\succcurlyeq)$, for all $i \in N$, involves the comparison of 2^{n-1} pairs of coalitions S and $S \cup \{i\}$, with $S \subseteq N \setminus \{i\}$. On the other hand, several coalitions are compared multiple times over different essential sets. So, it is computationally useful to design a procedure aimed at computing the social ranking $\rho^e(\succcurlyeq)$ avoiding those multiple comparisons (see Algorithm 1). To this aim, we first group coalitions over classes of indifferences with respect to $\succcurlyeq$: suppose we have $S_1 \succcurlyeq S_2 \succcurlyeq S_3 \succcurlyeq \ldots \succcurlyeq S_{2^n}$ then we shall write $\Sigma_1 \succ \Sigma_2 \succ \Sigma_3 \succ \ldots \succ \Sigma_l$, to denote the power relation $\succcurlyeq$, but having grouped in Σ_1 all the coalitions indifferent to S_1 (i.e., all $T \in 2^N$ s.t. $T \succcurlyeq S_1$ and $S_1 \succcurlyeq T$), in Σ_2 all the coalitions indifferent to the first coalition strictly less strong than S_1 in the ranking $\succcurlyeq$, and so on. Then, a coalition S in Σ_k is strictly stronger than any coalition in Σ_{k+1}. Notice that at each iteration k,

Algorithm 1: A procedure to find a social ranking based on the cardinality of the essential sets.

```
Input  : A power ≽ on 2^N in the form of indifference classes Σ_1 ≻ Σ_2 ≻ ... ≻ Σ_l.
Output: A vector d ∈ ℝ^N such that d_i = |E_i(≽)| for each i ∈ N.
initialisation: d_i := 0 for each i ∈ N ; X := ∅ ;
for k = 1 to l do
    X := X ∪ Σ_k;
    for every S ∈ Σ_k do
        for every i ∈ S do
            if {S \ {i}} ∉ X then
                d_i := d_i + 1;
            end
        end
    end
end
return d.
```

$k \in \{1, \ldots, l\}$, the test to establish whether i is essential for $S \in \Sigma_k$ is done by means of the **if** condition in line 6 (if $S \setminus \{i\}$ belongs to some Σ_t, $t \leq k$, then i is not essential for S).

A possible direction for future research is the open question about which axioms could be used to characterize a social ranking based on the essential sets introduced in this section. It would also be interesting to consider social ranking based on alternative definitions of essential item. For instance, consider a set of items $N = \{1, 2, 3\}$ and a power relation such that $\{2, 3\} \succ \{1, 3\} \succ \{1\} \sim \{2\}$. Clearly items 1 and 2 are essential for $\{1, 3\}$ and $\{2, 3\}$, respectively, but 2 seems "more" essential than 1, in the sense that the contribution of 2 to the power of coalition $\{2, 3\}$ is larger than the contribution of 1 to $\{1, 3\}$ ($\{1\}$ and $\{2\}$ are indifferent, but $\{2, 3\}$ is strictly stronger than $\{1, 3\}$). This kind of consideration about the "intensity" of items' contribution requires a more complex algorithmic analysis of the structure of a power relation aimed at comparing the role of elements over sets of different size, as for the particular instance described above.

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