Cost allocation problems arising from connection situations in an interactive cooperative setting

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As in a picture, I want to fix in this page the lively moment of the conclusion of this monograph and seize the opportunity to thank those persons whose contribution was essential for its realization.

This thesis would not have existed without the excellent and enthusiastic supervision of Henk Norde, Stef Tijs and Rodica Brânzei. I met Henk Norde and Stef Tijs in 2000, during my first visit in Tilburg, and Rodica Branzei one year later, during my second visit in Tilburg. At that time, I had not much experience in doing research. Friendly and professionally at the same time, they started immediately to teach me how to do research. This thesis is a successful result of the joint work which has been done since then. To Henk, Stef and Rodica goes all my sincere gratitude and appreciation.

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During a warm Sunday afternoon of summer 2008, I was sitting at my desk, with the intention to write down an original way to express my gratitude to all

these persons. At a certain moment, my daughter Giovanna, almost four years at that time, entered in my room and, looking at me stared at a blank page, asked what I was thinking about. “A good way to thank people who helped me to write a book”, I answered, “Do you have any idea?”. She thought for few seconds, and then replied with the following sentence: “*Vi voglio bene, grazie per avermi aiutato*”. Even if I may have not succeeded in my intention, I think she did.

Stefano Moretti
July 2008
Genoa, Italy

Contents

Chapter 1

Introduction and overview

1.1 Game theory and connection situations

In this monograph Game Theory is central for studying the interaction among decision makers (which are called *players*) in connection situations, where players need to be connected directly or via other players to a source, and where connections between players and between players and the source are costly. Since the seminal book "Theory of Games and Economic Behavior" by John von Neumann and Oskar Morgenstern (1944), it is usual to divide Game Theory into two main groups of interaction situations (which are called *games*), *non-cooperative* and *cooperative* games. Non-cooperative games deal with conflict situations where players cannot make binding agreements. In cooperative games all kinds of agreement among the players are possible.

In non-cooperative games, each player will choose to act in his own interest keeping into account that the outcome of the game depends on the actions of all the players involved. Actions can be made simultaneously by players, as in the 'stone, paper, scissors' game or in 'matching pennies', or sequentially at several time moments, as in chess.

Cooperative games deal with situations where groups of players (which are called *coalitions*) coordinate their actions with the objective to end up in joint payoffs which often exceed the sum of individual payoffs. A classical application of cooperative games is in cost allocation problems (see, for instance, Young (1994)). Using cooperative games in this context, it is possible to describe a situation where the players are willing to join bigger coalitions in order to have extra monetary savings as effect of cooperation. A very simple example is a situation with two nearby towns that are considering whether to implement a joint waste collection system. Town 1 could implement a system for itself at a cost of 7 million euros, whereas town 2 could implement its waste collection system at a cost of 4 million euros. However, if they cooperate, thanks to a more efficient use of common facilities, they can implement a waste collection system at a cost of 10 million euros. This situation can be formulated as a *cooperative cost game* (or simply *cost game*) $(\{1,2\},c)$, where towns 1 and 2 are the players and the characteristic cost function c assigns to each coalition the corresponding cost of implementing a waste collection system, *i.e.* (in million euros) $c(\{1\}) = 7$, $c(\{2\}) = 4$, $c(\{1,2\}) = 10$ and $c(\emptyset) = 0$. Clearly, it makes sense to cooperate, since the two players can jointly save 1 million. Cooperation will only occur, however, if they agree on how to share the total cost of 10 million euros. Trying to solve this problem, a cost allocation that can be accepted by both towns 1 and 2 must be efficient (the total cost must be entirely shared), equitable and must provide incentives to cooperation. For instance, one could propose to share equally the cost of 10 million euros, 5 million euros for each town. The argument for equal division is that each town has an equal power to enter in a contract, so each town should support an equal burden. On the other hand, it could be the case that town 1 produces four times the waste of town 2. Then, it seems fair to propose a method based on the proportion of waste produced by the two towns. Such an allocation method would charge town 1 of 8 million euros and town 2 of 2 million euros. Surely, neither of these two proposals will be adopted. In fact, town 2 is not likely to agree to equal division, because 5 million euros exceed the cost of implementing its own collection system. On the other hand, town 1 is not likely to agree to the allocation method proportional to waste production, since 8 million exceed the cost of implementing its own system. One possible solution for cost game $(\{1,2\},c)$ is to equally divide the amount of money that 1 and 2 save by cooperation. Using this method, town 1 would pay $7 - 0.5 = 6.5$ million, and town 2 would pay $4 - 0.5 = 3.5$ million.

This allocation gives to players an incentive to cooperate, because each realizes positive savings. But, it is not the only allocation with these characteristics. Any allocation in which 1 pays at most 7 million and 2 pays at most 4 million creates no disincentives to cooperation: using game theory terminology, such an allocation is *stable*. The set of all stable allocations is the core of the cost game, a concept that will be more generally defined in Chapter 2.

Clearly, the example above is just one of the many situations in which game theory can be used to analyze a cost allocation problem. In particular, this dissertation is focused on the application of cooperative games to the analysis of cost allocation problems arising from connection situations. A connection situation takes place in the presence of a group of agents, each of which needs to be connected directly or via other agents to a source. If connections among agents are costly, then each agent will evaluate the opportunity of cooperating with other agents in order to reduce costs. In fact, if a group of agents decides to cooperate, a configuration of links which minimizes the total cost of connection is provided by a minimum cost spanning tree (mcst). A connection situation may arise facing the problem of building a network of computers that connects every computer with some server: agents are the computer users, the source is the server and the costs of links are the connection costs of each pair of computers or of a computer and the server. Another example could be the problem of building a drainage system that connects every house in a city with a water purifier. The problem of finding an mcst may be easily solved thanks to different algorithms proposed in literature (Boruvka (1926a,b), whose translations may be found in Nešetřil et al (2001), Kruskal (1956), Prim (1957), Dijkstra (1959). A historic overview of mcst problems can be found in Graham and Hell (1985).

However, finding an mcst does not guarantee that it is going to be really implemented: agents must still support the cost of the mcst and then a cost allocation problem must be addressed. This cost allocation problem was introduced by Claus and Kleitman in 1973 and has been studied with the aid of cooperative game theory since the basic paper of Bird (1976). Given a connection situation with a group of agents, Bird (1976) introduced an associated cooperative cost game (known as *mcst game*), where the players are the agents and the worth of a coalition is the minimal cost of connecting this coalition to the source via links between members of the coalition; in addition, Bird (1976) proposed an allocation method for connection situations (in this dissertation referred as the *Bird rule*) that associates with each mcst a cost allocation. After the paper of Bird, much attention has been paid to study the properties of core allocations for mcst games. Granot and Huberman (1981) proved that allocations provided by the Bird rule for connection situations are extreme points of the core of the associated mcst game. Granot and Huberman (1984) also proposed other methods which provide allocations in the core of an mcst game, with particular attention to ease computational difficulty in computing the nucleolus of an mcst game. In a similar direction, Feltkamp *et al.* (1994a,b) introduced and characterized the Proportional rule and the Equal Remaining Obligation rule for connection situations. Aarts (1994) found other extreme points of the core when the connection situation has an mcst which is a *chain*, *i.e.* a tree with only two *leaves* (a leaf of a tree is a node with only one incident edge). Kuipers (1993) introduced core elements of mcst games associated to connection situations where the cost of each link is either zero or one. The Shapley value (Shapley (1953)) of an mcst game, which is not necessarily in the core of an mcst game, was also studied and axiomatically characterized by Kar (2002).

Many cost allocation methods have been proposed, and different properties have been considered as well to make them suitable for application in a "dynamic" framework. In many applications the cardinality of the set of agents can vary in time, and also increasing or decreasing of connection costs may occur. Consider, for instance, a wireless telecommunication network where agents are operators of transmitters for traffic exchange and the source is the central hub station. Agents can decide to communicate directly with the main exchange hub, by means of powerful and very expensive transmitters, or, alternatively, can decide to cooperate and construct a wireless network of less powerful, and consequently, cheaper transmitters. Since transmissions are costly, such a situation can be handled as an mcst problem and the related cost allocation problem can be studied as an mcst game. Moreover, in such a situation, it may happen that at a given moment either new owners of transmitters can be willing to enter the network, or the cost of connection can increase (*e.g.* as a consequence of an improvement in quality and quantity of services supplied) or decrease (*e.g.* by improving telecommunication technologies). Of course, in all the connection situations that may change in time, cost allocations which are stable only in the original situation cannot guarantee cooperation among agents also under the new conditions.

Another realistic example where changes in the original connection situation may occur is in supply networks. Connection situations may be useful to answer questions regarding the implementation of clauses in supply contracts concerning transportation networks and the related cost allocation problem (Voß and Schneidereit (2002), Sharkey (1995)). In this case, agents are customer nodes of a supply chain, who all want to be connected with a central service (*i.e.* the source), directly or via other agents, and where connections are costly (*e.g.* costs due to transportation or to lead times). Stability is an important characteristic for cost allocation protocols applied to supply transportation networks, since it is a necessary condition for any subset of customers not to secede and build their own competing transportation sub-network. But, increasing of transportation costs may occur, and, consequently, other incentives to cooperation are demanded. For instance, supply contracts must take into consideration clauses for having various transport possibilities enabling, e.g., expedited delivery in cases of necessary adjustments in the lead times (Voß and Schneidereit (2002)) with corresponding increasing of transportation costs.

It should be evident that all those cost allocation problems arising from connection situations which may undergo one or more changes, require sustainable allocation methods. Therefore, the goal of this monograph is to analyze allocation methods which can keep, in the most general setting, incentives for cooperation also under modifications in the population of agents and in the structure of connection costs. For example, the question of the existence of population monotonic allocation schemes (pmas) (Sprumont (1990)) is central. A pmas provides a cost allocation vector for every coalition in a monotonic way, *i.e.* the cost allocated to some player does not increase if the coalition to which he belongs becomes larger. Another example regards cost monotonic allocation rules, that will also be studied in this monograph, where cost monotonicity means that if some connection costs go down (up), then no agents will pay more (less). To achieve this goal, the Kruskal algorithm (Kruskal (1956)) plays a key role. Roughly speaking, this algorithm works in the following way: in the first step an edge between two nodes in $N \cup \{0\}$ of minimal cost is formed. In every subsequent step, a new edge of minimal cost is formed, under the constraint that no cycles are formed. In summary, a sequence of edges is produced and after n steps an mcst appears. Since some edges may have the same cost, different mcsts may be selected by the Kruskal algorithm, depending on the ordering of the edges with respect to their increasing costs which has been considered in the Kruskal algorithm.

In this monograph, a set of cost allocation protocols is provided which *charge* the agents with "fractions" of the cost of each edge constructed in each step of the Kruskal algorithm with the possibility to control the cost allocation problem during the construction procedure (Moretti *et al.* (2005), Norde *et al.* (2004)). These protocols can be easily implemented in practical network situations (for instance, in supply transportation networks), are flexible to changes in the network situation, and meet the requirement of continuous monitoring by the agents involved. It turns out that a subclass of these cost allocation protocols coincides with the class of Obligations rules (Tijs *et al.* (2006a)). It is shown that Obligation rules are cost monotonic and induce a pmas. Interesting rules among Obligation rules are the P-value (Branzei *et al.* (2004), Feltkamp *et al.* (1994b)) and the P^τ-values, for each ordering τ of the players (Norde *et al.* (2004)). Other characteristics of the Obligation rules are that different feasible orderings of the edges lead to the same cost allocations and that all these allocations are elements of the Bird core (Bird (1976), Tijs *et al.* (2006b)). Variants of connection situations are also studied (Norde *et al.* (2004), Moretti *et al.* (2002)).

Other authors have studied cost allocation problems under modifications of the elements of the connection situations. In the paper of Kent and Skorin-Kapov (1996) the question of the existence of pmas in connection situations is central. In the paper of Dutta and Kar (2004), cost monotonic allocation rules were studied, where cost monotonicity means that an agent i does not pay more if the cost of a link involving i decreases, nothing else changing in the network. Monotonicity properties for cost allocation protocols have been also studied in Bergañtinos and Vidal-Puga (2007a). Bergañtinos and Vidal-Puga (2007b) introduced the class of optimistic transferable utility games associated to mcst situations, where the worth of a coalition is the minimal cost of connecting this coalition to the source or to a player who is not a member of the coalition. Bergañtinos and Lorenzo-Freire (2008b) introduced optimistic weighted Shapley rules for connection situations and proved that they are special Obligation rules. Later, Bergañtinos and Lorenzo-Freire (2008a) characterized the optimistic weighted Shapley rules using monotonicity properties.

Other classes of cost allocation problems related to variants in connection situations are: Steiner tree games (Megiddo (1978), Skorin-Kapov (1995)), where the cost of a coalition of agents is the minimum weight of a Steiner tree[1] that spans the coalition; minimum cost spanning forest games (Kuipers (1998)), dealing with more than one source; spanning network games (Granot and Maschler (1999), van den Nouweland *et al.* (1993)), where costs are both on the edges and on the vertices of the connection situation; hub network games (Skorin-Kapov (1998)), where some of the nodes of the connection situation serve as focal points (*i.e.* hubs); mcst extension problems (Feltkamp (1994)), which are generalized connection situations in which some network can be present initially.

More recently, Fernandez *et al.* (2004) have introduced a multi-criteria version of an mcst-game as a set-valued TU-game, and provided a family of core solutions for these games. Suijs (2003) studied mcst problems in which the connection costs are represented by random variables. Granot *et al.* (2002) introduced the class of extended tree games, where the agents want to receive a commodity flow from the root and the flow requirements of the agents can be different. Moretti (2006) introduced a class of mcst games applied to the analysis of gene expression data, where nodes in the connection situation represent genes and the cost of a link between two genes is a measure of dissimilarity between the two genes.

[1]Given a subset of nodes identified as *terminals* in a connection situation, a Steiner tree is an mcst that includes all the terminals and possibly many others. Note that for Steiner tree problems some nodes may be switching points (*i.e.* there are no users residing at them).

1.2 Overview

This dissertation mainly deals with cost games arising from mcst situations which are defined on undirected complete weighted graphs, where coalitions are not allowed to use networks which contain nodes outside the coalitions. Only Chapter 7 is devoted to variants of this kind of mcst situations.

In Chapter 2, some basic preliminaries and notations are presented. The notions of mcst situations and mcst games are formulated and illustrated on basic complete weighted graphs, that have been used throughout the monograph to illustrate also other concepts. The definitions of some basic notions in the theory of cooperative games, as the core of a game or the notion of pmas, are also introduced and illustrated with examples.

In Chapter 3, the Subtraction Algorithm is presented. This algorithm computes, for every mcst situation and each permutation on the set of players, a pmas. As a basis for this algorithm serves a decomposition theorem which guarantees that every mcst game can be written as a nonnegative combination of mcst games corresponding to $0-1$ cost functions (called *simple mcst games*). It turns out that the Subtraction Algorithm is closely related to the famous algorithm of Kruskal for the determination of mcsts. Furthermore, for each permutation τ on the set of players, the notion of P^{τ}-value is introduced, as the allocation rule for mcst situations which divides the cost of the grand coalition according to the Subtraction Algorithm initialized with τ. This chapter is based on Norde, Moretti, Tijs (2004).

In Chapter 4, the class of *Construct and Charge* (CC-) rules for mcst situations is introduced. CC-rules are defined starting from *charge systems*, and specify particular allocation protocols rooted on the Kruskal algorithm for computing an mcst. Furthermore, the chapter focuses on the class of *Obligation rules* for mcst situations. A characteristic of Obligation rules is that they assign to an mcst situation a vector of cost contributions which can be obtained as a product of a double stochastic matrix with the cost vector of edges in the optimal tree provided by the Kruskal algorithm. It is proved that special charge systems, called *conservative*, lead to a subclass of CC-rules that coincides with the class of *Obligation rules*. An interesting feature of such rules is that different feasible orderings of the edges lead to the same cost allocations. Properties of particular

Obligation rules, as the Potters value (P-value) and the P^τ-value introduced in Chapter 3, are also discussed. It turns out that the P-value equals the Equal Remaining Obligations (ERO) rule suggested by Jos Potters (which explains the name of the value) and which is studied first in Feltkamp *et al.* (1994). Furthermore, the P-value turns out to be the average of the P^τ-values. Sections 4.2-4.4 and 4.7 are based on Moretti, Tijs, Branzei, Norde (2008); section 4.5 is based on Tijs, Branzei, Moretti, Norde (2006a); section 4.6 is based on Branzei, Moretti, Norde, Tijs (2004).

In Chapter 5, it is first demonstrated that Obligation rules are cost monotonic and induce also a pmas. Then, a new way to define the irreducible core (Bird (1976)) is presented, based on a non-Archimedean semimetric. The Bird core correspondence turns out to have interesting monotonicity and additivity properties, and each stable cost monotonic allocation rule for mcst situations is a selection of the Bird core correspondence. Section 5.2 is based on Tijs, Branzei, Moretti, Norde (2006a); sections 5.3 and 5.4 are based on Tijs, Moretti, Branzei, Norde (2006b).

In Chapter 6 an axiomatic characterization of the P-value is provided, where cone-wise positive linearity of the P-value is a fundamental property and where the decomposition of an mcst situation into simple mcst situations plays a role. Using the additivity property an axiomatic characterization of the Bird core correspondence is also given. A value-theoretic interpretation of the Obligation rules using sharing values for cost games is also discussed. Section 6.2 is based on Branzei, Moretti, Norde, Tijs (2004); section 6.3 is based on Tijs, Moretti, Branzei, Norde (2006b); section 6.4 is based on Moretti, Tijs, Branzei, Norde (2005).

In Chapter 7 it is shown that, for variants of classical mcst games, a pmas does not necessarily exist. In particular, this chapter deals with monotonic mcst situations and directed mcst situations. Directed mcst situations of a special kind are studied, namely those which show up in considering the problem of connecting units (houses) in mountains with a purifier. For such problems an easy method is described to obtain an mcst. It turns out that the cores of the related cost allocation problems have a simple structure and each core element can be extended to a pmas and also to a bi-monotonic allocation scheme

(Branzei *et al.* (2001), Voorneveld *et al.* (2002)). Sections 7.2 and 7.3 are based on Norde, Moretti, Tijs (2004); section 7.4 is based on Moretti, Norde, Pham Do, Tijs (2002).

Chapter 2

Connection situations and games

2.1 Minimum cost spanning tree (mcst) situations

An (undirected) *graph* is a pair $< V, E >$, where V is a set of vertices or nodes and E is a set of edges e of the form $\{i, j\}$ with $i, j \in V$, $i \neq j$. The *complete graph* on a set V of vertices is the graph $< V, E_V >$, where $E_V = \{\{i, j\} | i, j \in V \text{ and } i \neq j\}$.

A *path* between i and j in a graph $< V, E >$ is a sequence of nodes $(i_0, i_1, \ldots, i_k)$, where $i = i_0$ and $j = i_k$, $k \geq 1$, such that $\{i_s, i_{s+1}\} \in E$ for each $s \in \{0, ..., k-1\}$ and such that all these edges are distinct. A *cycle* in $< V, E >$ is a path from i to i for some $i \in V$. A path $(i_0, i_1, \ldots, i_k)$ is *without cycles* if there do not exist $a, b \in \{0, 1, \ldots, k\}$, $a \neq b$, such that $i_a = i_b$. Two nodes $i, j \in V$ are connected in $< V, E >$ if $i = j$ or if there exists a path between i and j in E. A *connected component* of V in $< V, E >$ is a maximal subset of V with the property that any two nodes in this subset are connected in $< V, E >$.

This monograph deals with *minimum cost spanning tree* (mcst) situations, *i.e.* situations where a set $N = \{1, \ldots, n\}$ of agents is willing to be connected as cheap as possible to a source (*i.e.* a supplier of a service) denoted by 0, based on a given weight (or cost) system of connection. In the sequel we use also the notation $N' = N \cup \{0\}$, and w for the *weight function*, *i.e.* a map which assigns to each edge $e \in E_{N'}$ a non-negative number $w(e)$ representing the weight or cost of edge e.

We denote an *mcst situation* with set of users N, source 0, and weight function w by $< N', w >$ (or simply w). Further, we denote by $\mathcal{W}^{N'}$ the set of all mcst situations $< N', w >$ (or w) with node set N'. For each $S \subseteq N$, one can consider the mcst subsituation $< S', w_{|S'} >$, where $S' = S \cup \{0\}$ and $w_{|S'} : E_{S'} \to I\!R_+$ is the restriction of the weight function w to $E_{S'} \subseteq E_{N'}$, *i.e.* $w_{|S'}(e) = w(e)$ for each $e \in E_{S'}$. If $w(e) \in \{0, 1\}$ for every $e \in E_{N'}$, the weight function w is called a *simple weight function*, and we refer then to $< N', E_{N'}, w >$ as a *simple mcst situation*.

Let $< N', w >$ be an mcst situation. The *carrier* $Ca(w)$ of w is the set of edges with positive costs, *i.e.* $Ca(w) = \{e \in E : w(e) > 0\}$. Two nodes i and j are called (w, N')-*connected* if $i = j$ or if there exists a path $(i_0, \dots, i_k)$ from i to j, with $w(\{i_s, i_{s+1}\}) = 0$ for every $s \in \{0, \dots, k-1\}$. A (w, N')-*component* of N' is a maximal subset of N' with the property that any two nodes in this subset are (w, N')-connected. We denote by $C_i(w)$ the (w, N')-component to which i belongs and by $\mathcal{C}(w)$ the set of all the (w, N')-components of N'. Clearly, the collection of (w, N')-components forms a partition of N'.

The cost of a *network* $\Gamma \subseteq E_{N'}$ is $w(\Gamma) = \sum_{e \in \Gamma} w(e)$. A network Γ is a *spanning network* on $S' \subseteq N'$ if for every $e \in \Gamma$ we have $e \in E_{S'}$ and for every $i \in S$ there is a path in $< S', \Gamma >$ from i to the source. For any mcst situation $w \in \mathcal{W}^{N'}$ it is possible to determine at least one *spanning tree* on N', *i.e.* a spanning network without cycles on N', of minimum cost; each spanning tree of minimum cost is called an *mcst* for N' in w or, shorter, an mcst for w. Given a spanning network Γ on N' we define the set of edges of Γ with nodes in $S' \subseteq N'$ as the set $E^{\Gamma}_{S'} = \{\{i, j\} | \{i, j\} \in \Gamma \text{ and } i, j \in S'\}$.

Example 2.1.1 In this example we consider a minimum cost spanning tree situation arising from the problem of car pooling. Suppose that three employees of a firm consider the possibility of car pooling in order to reduce their daily travel cost. The cost of driving a car from one employee to another or from one employee to the firm are given in Figure 2.1. Here the employees are denoted

Figure 2.1: An mcst situation $< \{0, 1, 2, 3\}, w >$ (left side) and a related mcst (right side).

by 1, 2, and 3 and the firm by 0. To each edge $e \in E_{\{0,1,2,3\}}$ is assigned a non-negative number $w(e)$ representing the cost of edge e. A minimum cost spanning tree in this mcst situation $< \{0, 1, 2, 3\}, w >$ is the network $\Gamma = \{\{0, 1\}, \{1, 2\}, \{1, 3\}\}$ with cost $w(\Gamma) = 48$. This network Γ corresponds to the plan of car pooling in which employees 2 and 3 drive their car in solitude to employee 1 where all employees take one car in order to drive together to the firm. In the remaining of the thesis, to capture the attention of the reader on a certain mcst, we will represent the edges of the mcst by means of thicker lines, as it has been shown in Figure 2.2.

■

2.1.1 Algorithms for the determination of an mcst

Two famous algorithms for the determination of a minimum cost spanning tree are the algorithm of Prim (Prim (1957)) and the algorithm of Kruskal (Kruskal (1956)). Let $< N', w >$ be an mcst situation. A minimum cost spanning tree in $< N', w >$ can be obtained in the following two ways.

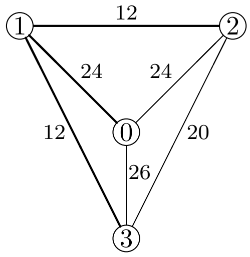


Figure 2.2: An mcst situation $< \{0,1,2,3\}, w >$ and an mcst in $< \{0,1,2,3\}, w >$ with edges denoted by thicker lines.

> **Prim's Algorithm:** In the first step construct an edge of minimal cost between a node in N and the source 0. In every subsequent step construct an edge of minimal cost between a node in N which is not connected yet with the source, directly or indirectly, and the source or with a node in N which is already connected with the source, directly or indirectly. In every step of the algorithm there is precisely one node in S which gets a connection with the source, so the algorithm stops after precisely $|N|$ steps.

> **Kruskal's Algorithm:** In the first step construct an edge between nodes in $N \cup \{0\}$ of minimal cost. In every subsequent step construct an edge between nodes in $N \cup \{0\}$ of minimal cost which does not form a cycle with the edges which have already been constructed. The algorithm also stops after precisely $|N|$ steps.

Example 2.1.2 Consider the mcst situation $< N', w >$ of Example 2.1.1, with $N' = \{0,1,2,3\}$ and w as depicted in Figure 2.1.

Prim's Algorithm may first form edge $\{0,1\}$, then $\{1,2\}$ (alternatively, $\{1,3\}$), and finally $\{1,3\}$ (alternatively, $\{1,2\}$). Having selected the edge $\{0,1\}$ in the first step, Prim's algorithm on this mcst situation determines the mcst $\{\{0,1\}, \{1,2\}, \{1,3\}\}$. On the same mcst situation, Prim's Algorithm may first form edge $\{0,2\}$, then $\{1,2\}$, and finally $\{1,3\}$, bringing to the mcst $\{\{0,2\}, \{1,2\}, \{1,3\}\}$.

Kruskal's Algorithm first forms the cheapest edge $\{1,2\}$ (alternatively, $\{1,3\}$), then $\{1,3\}$ (alternatively, $\{1,2\}$). After the first two steps of the Kruskal's Algorithm, the cheapest edges $\{1,2\}$ and $\{1,3\}$ have been formed. Since edge $\{2,3\}$ forms a cycle with the edges $\{1,2\}$ and $\{1,3\}$, it cannot be constructed. Finally, at the third step of the algorithm, one of the edges $\{0,1\}$ and $\{0,2\}$ may be formed. Depending on whether $\{0,1\}$ or $\{0,2\}$ is formed, the Kruskal's Algorithm determines the mcst $\{\{0,1\},\{1,2\},\{1,3\}\}$ or $\{\{0,2\},\{1,2\},\{1,3\}\}$, respectively.

■

2.1.2 Kruskal cones

The basic idea behind Kruskal's algorithm is to consider edges one by one according to non-decreasing cost. This idea leads to the classification of mcst situations on the basis of the orders of the edges considered in Kruskal's algorithm.

We define the set $\Sigma_{E_{N'}}$ of *linear orders* on $E_{N'}$ as the set of all bijections $\sigma : \{1,\ldots,|E_{N'}|\} \to E_{N'}$, where $|E_{N'}|$ is the cardinality of the set $E_{N'}$. For each mcst situation $< N', w >$ there exists at least one linear order $\sigma \in \Sigma_{E_{N'}}$ such that $w(\sigma(1)) \leq w(\sigma(2)) \leq \ldots \leq w(\sigma(|E_{N'}|))$. We denote by w^σ the column vector $\big(w(\sigma(1)), w(\sigma(2)), \ldots, w(\sigma(|E_{N'}|))\big)^t$.

For any $\sigma \in \Sigma_{E_{N'}}$ we define the set

$$K^\sigma = \{w \in I\!R_+^{E_{N'}} \mid w(\sigma(1)) \leq w(\sigma(2)) \leq \ldots \leq w(\sigma(|E_{N'}|))\}.$$

The set K^σ is a cone in $I\!R_+^{E_{N'}}$, which we call the *Kruskal cone with respect to* σ. One can easily see that $\bigcup_{\sigma\in\Sigma_{E_{N'}}} K^\sigma = I\!R_+^{E_{N'}}$. For each $\sigma \in \Sigma_{E_{N'}}$ the cone K^σ is a simplicial cone with generators $e^{\sigma,k} \in K^\sigma$, $k \in \{1,2,\ldots,|E_{N'}|\}$, where $e^{\sigma,1}(\sigma(j)) = 1$ for all $j \in \{1,2,\ldots,|E_{N'}|\}$, and for each $k \in \{2,\ldots,|E_{N'}|\}$

$$\begin{gathered} e^{\sigma,k}(\sigma(1)) = e^{\sigma,k}(\sigma(2)) = \ldots = e^{\sigma,k}(\sigma(k-1)) = 0 \\ \text{and} \\ e^{\sigma,k}(\sigma(k)) = e^{\sigma,k}(\sigma(k+1)) = \ldots = e^{\sigma,k}(\sigma(|E_{N'}|)) = 1. \end{gathered} \tag{2.1}$$

This implies that each $w \in K^\sigma$ can be written in a unique way as a non-negative linear combination of these generators. To be more concrete, for $w \in K^\sigma$ we have

$$w = w(\sigma(1))e^{\sigma,1} + \sum_{k=2}^{|E_{N'}|} \big(w(\sigma(k)) - w(\sigma(k-1))\big)\, e^{\sigma,k}. \tag{2.2}$$

Clearly, we can also write $\mathcal{W}^{N'} = \bigcup_{\sigma \in \Sigma_{E_{N'}}} K^\sigma$, if we identify an mcst situation $< N', w >$ with w.

Let $w \in \mathcal{W}^{N'}$ and let $\sigma \in \Sigma_{E_{N'}}$ be such that $w \in K^\sigma$. We can consider a sequence of precisely $|E_{N'}| + 1$ graphs $< N', F^{\sigma,0} >, < N', F^{\sigma,1} >, \ldots,$ $< N', F^{\sigma,|E_{N'}|} >$ such that $F^{\sigma,0} = \emptyset$, $F^{\sigma,k} = F^{\sigma,k-1} \cup \{\sigma(k)\}$ for each $k \in \{1, \ldots, |E_{N'}|\}$. For each graph $< N', F^{\sigma,k} >$, with $k \in \{0, 1, \ldots, |E_{N'}|\}$, let $\pi^{\sigma,k}$ be the partition of N' consisting of the connected components of N' in $< N', F^{\sigma,k} >$.

Remark 2.1.1 *For each $k \in \{1, \ldots, |E_{N'}|\}$, $\pi^{\sigma,k}$ is either equal to $\pi^{\sigma,k-1}$ or is obtained from $\pi^{\sigma,k-1}$ by forming the union of two elements of $\pi^{\sigma,k-1}$.*

Now, we define recursively the function $\rho^\sigma : \{0, 1, \ldots, |N|\} \to \{0, 1, \ldots, |E_{N'}|\}$ by

- $\rho^\sigma(0) = 0$

- $\rho^\sigma(j) = \min\{k \in \{\rho^\sigma(j-1) + 1, \ldots, |E_{N'}|\} | \pi^{\sigma,k} \neq \pi^{\sigma,\rho^\sigma(j-1)}\}$

for each $j \in \{1, \ldots, |N|\}$.

Note that $\pi^{\sigma,\rho^\sigma(i)} \neq \pi^{\sigma,\rho^\sigma(j)}$ for each $i, j \in \{0, 1, \ldots, |N|\}$ with $i \neq j$, and $\sigma(\rho^\sigma(1)), \ldots, \sigma(\rho^\sigma(|N|))$ correspond to the $|N|$ formed edges in the Kruskal's algorithm when the order σ of the edges is considered.

Example 2.1.3 Consider the mcst situation $< N', w >$ with $N' = \{0, 1, 2, 3\}$ and w as depicted in Figure 2.1. Note that $w \in K^\sigma$, with $\sigma(1) = \{1, 3\}$, $\sigma(2) = \{1, 2\}$, $\sigma(3) = \{2, 3\}$, $\sigma(4) = \{0, 1\}$, $\sigma(5) = \{0, 2\}$, $\sigma(6) = \{0, 3\}$. The sequence of seven graphs $< N', F^{\sigma,k} >$ and the corresponding sequence of partitions $\pi^{\sigma,k}$ are shown in the following table

k	$F^{\sigma,k}$	$\pi^{\sigma,k}$
0	$\{\emptyset\}$	$\{\{0\},\{1\},\{2\},\{3\}\}$
1	$\{\{1,3\}\}$	$\{\{0\},\{1,3\},\{2\}\}$
2	$\{\{1,3\},\{1,2\}\}$	$\{\{0\},\{1,2,3\}\}$
3	$\{\{1,3\},\{1,2\},\{2,3\}\}$	$\{\{0\},\{1,2,3\}\}$
4	$\{\{1,3\},\{1,2\},\{2,3\},\{0,1\}\}$	$\{N'\}$
5	$\{\{1,3\},\{1,2\},\{2,3\},\{0,1\},\{0,2\}\}$	$\{N'\}$
6	$\{\{1,3\},\{1,2\},\{2,3\},\{0,1\},\{0,2\},\{0,3\}\}$	$\{N'\}$

Then, $\rho^\sigma(0)=0$, $\rho^\sigma(1)=1$, $\rho^\sigma(2)=2$, $\rho^\sigma(3)=4$.

■

2.2 Cooperative game theory and mcst games

Next, we recall some basic game theoretical notions. A *cooperative cost game* or cost game is a pair (N,c), where N denotes the finite set of *players* and $c : 2^N \to I\!R$ is the *characteristic function*, with $c(\emptyset)=0$ (here 2^N denotes the power set of player set N). Often we identify a cost game (N,c) with the corresponding characteristic function c. A group of players $T \subseteq N$ is called a *coalition* and $c(T)$ is called the *cost* of this coalition. The class of all cost games with N as set of players is denoted by $\mathcal{G}^N$. Let $\mathcal{H}^N \subseteq \mathcal{G}^N$. We call a map $\psi : \mathcal{H}^N \to I\!R^N$ a *value* if it assigns to every cost game $(N,c) \in \mathcal{H}^N$ one *payoff vector* (or *cost allocation*) in $I\!R^N$. A payoff vector $x \in I\!R^N$ is *efficient* for a cost game $(N,c) \in \mathcal{G}^N$ if we have $\sum_{i\in N} x_i = c(N)$. A value ψ is *efficient* if we have that $\psi(c)$ is an efficient payoff vector for each $c \in \mathcal{H}^N$. A value $\psi : \mathcal{H}^N \to I\!R^N$ is called *linear* if $\psi(\beta v + \gamma u) = \beta\psi(v) + \gamma\psi(u)$ for all games $v,u \in \mathcal{H}^N$ and real numbers $\beta,\gamma \in I\!R$ such that $\beta v + \gamma u \in \mathcal{H}^N$.

A particular set, possibly empty, of payoff vectors of a cost game (N,c) is the *core*, which is defined as the set of efficient payoff vectors for which no coalition has an incentive to leave the grand coalition N. In formula

$$core(c) = \{x \in I\!R^N | \sum_{i\in S} x_i \le c(S)\ \forall S \in 2^N \setminus \{\emptyset\}; \sum_{i\in N} x_i = c(N)\}.$$

A game (N, c) is called a *concave game* if the marginal contribution of any player to any coalition is not less than his marginal contribution to a larger coalition, *i.e.* if it holds that

$$c(S \cup \{i\}) - c(S) \geq c(T \cup \{i\}) - c(T) \tag{2.3}$$

for all $i \in N$ and all $S \subseteq T \subseteq N \setminus \{i\}$.

We define the set Σ_N of possible orders on the set N as the set of all bijections $\tau : \{1, \ldots, |N|\} \to N$, where $|N|$ is the cardinality of the set N and where $\tau(i) = j$ means that with respect to τ, player j is in the i-th position.

Let (N, c) be a cooperative cost game. For $\tau \in \Sigma_N$, the *marginal vector* $m^\tau(c)$ is defined by

$$m_i^\tau(c) = c([i, \tau]) - c((i, \tau)) \text{ for all } i \in N,$$

where $[i, \tau] = \{j \in N : \tau^{-1}(j) \leq \tau^{-1}(i)\}$ is the set of predecessors of i with respect to τ including i, and $(i, \tau) = \{j \in N : \tau^{-1}(j) < \tau^{-1}(i)\}$ is the set of predecessors of i with respect to τ excluding i. In a coherent way with respect to previous notations, we will indicate the set $[i, \tau] \cup \{0\}$ and $(i, \tau) \cup \{0\}$ as $[i, \tau]'$ and $(i, \tau)'$, respectively. For instance, for each $k \in \{1, \ldots, |N|\}$ and for each $l \in \{2, \ldots, |N|\}$, the set $[\tau(k), \tau]' = \{0, \tau(1), \ldots, \tau(k)\}$ and $(\tau(l), \tau)' = \{0, \tau(1), \ldots, \tau(l-1)\}$, which will be denoted shorter as $[\tau(k)]'$ and $(\tau(l))'$, respectively.

The most well-known value in the theory of cost games is the *Shapley value* (Shapley (1953)). The Shapley value $\phi(c)$ of a cost game (N, c) is defined as the average of marginal vectors over all $|N|!$ possible orders in Σ_N. In formula

$$\phi_i(c) = \sum_{\tau \in \Sigma_N} \frac{m_i^\tau(c)}{|N|!} \text{ for all } i \in N. \tag{2.4}$$

A *population monotonic allocation scheme* or pmas (Sprumont (1990)) of the game (N, c) is a scheme $x = \{x_{S,i}\}_{S \in 2^N \setminus \{\emptyset\}, i \in S}$ with the properties

i) $\sum_{i \in S} x_{S,i} = c(S)$ for all $S \in 2^N \setminus \{\emptyset\}$;

ii) $x_{S,i} \geq x_{T,i}$ for all $S, T \in 2^N \setminus \{\emptyset\}$ and $i \in N$ with $i \in S \subset T$.

A pmas provides a cost allocation vector for every coalition in a monotonic way, *i.e.* the cost allocated to some player decreases if the coalition to which he belongs becomes larger.

Let $< N', w >$ be an mcst situation. The *minimum cost spanning tree game* (N, c_w) (or simply c_w), corresponding to $< N', w >$, is defined by

$$c_w(S) = \min\{w(\Gamma) | \Gamma \text{ is a spanning network on } S'\}$$

for every $S \in 2^N \backslash \{\emptyset\}$, with the convention that $c_w(\emptyset) = 0$.

We denote by $\mathcal{MCST}^N$ the class of all mcst games corresponding to mcst situations in $\mathcal{W}^{N'}$. For each $\sigma \in \Sigma_{E_{N'}}$, we denote by $\mathcal{G}^\sigma$ the set $\{c_w \mid w \in K^\sigma\}$ which is a cone. We can express $\mathcal{MCST}^N$ as the union of all cones $\mathcal{G}^\sigma$, *i.e.* $\mathcal{MCST}^N = \bigcup_{\sigma \in \Sigma_{E_{N'}}} \mathcal{G}^\sigma$, and we would like to point out that $\mathcal{MCST}^N$ itself is not a cone if $|N| \geq 2$.

Example 2.2.1 Consider the mcst situation $< N', w >$ with $N' = \{0, 1, 2, 3\}$ and w as depicted in Figure 2.1.

If $S = \{1, 2\}$ then a minimum cost spanning network for S is $\Gamma = \{\{1, 2\}, \{0, 1\}\}$ with cost 36, whereas the minimum cost spanning network for $S = \{3\}$ is $\Gamma = \{\{0, 3\}\}$ with cost 26. Proceeding in this way we find that the mcst game (N, c_w), corresponding to $< N', w >$, is given by

$$\begin{array}{lll} c_w(123) = 48, & & \\ c_w(12) = 36, & c_w(13) = 36, & c_w(23) = 44, \\ c_w(1) = 24, & c_w(2) = 24, & c_w(3) = 26. \end{array}$$

■

We call a map $F : \mathcal{W}^{N'} \to \mathbb{R}^N$ assigning to every mcst situation w a unique cost allocation in $\mathbb{R}^N$ a *solution*. A solution F is *efficient* if we have $\sum_{i \in N} F_i(w) = w(\Gamma)$ for each $w \in \mathcal{W}^{N'}$, where Γ is a spanning network on N' of minimal cost. A solution F has the *carrier* property if $F_i(w) = 0$ for each $w \in \mathcal{W}^{N'}$ and for each $i \in N$ such that i is (w, N')-connected to 0.

The core $\mathcal{C}(c_w)$ of an mcst game $c_w \in \mathcal{MCST}^N$ is nonempty (Granot and Huberman (1981), Bird (1976)) and, given an mcst Γ (with no cycles) for N' in the mcst situation w, one can easily find an element in the core looking at the algorithm of Prim described in Section 2.1.1. If one assigns the cost of an edge, which is formed in some step of the algorithm, to the player who just gets a connection with the source, directly or indirectly, then one obtains a core element of the corresponding mcst game (see Bird (1976) for more details). In the following example we will illustrate that such a procedure does not necessarily generate a pmas of the corresponding mcst game.

Example 2.2.2 Consider the complete weighted graph $< N', \tilde{w} >$ with $N' = \{0,1,2,3\}$ and cost function $\tilde{w}$ as depicted in Figure 2.3. Application of Prim's

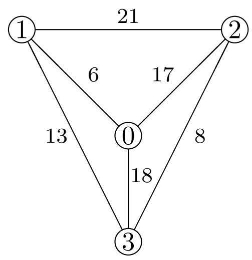


Figure 2.3: The cost function $\tilde{w}$ on $E_{\{0,1,2,3\}}$.

algorithm for the mcst situation $< \{0,1,2,3\}, \tilde{w} >$ yields the formation of edge $\{0,1\}$ first, followed by the formation of edge $\{1,3\}$ and edge $\{2,3\}$. The cost of edge $\{0,1\}$ is assigned to player 1, the cost of edge $\{1,3\}$ to player 3 and the cost of edge $\{2,3\}$ to player 2. Following the same procedure for all other coalitions we get the following table

S	1	2	3
$\{1,2,3\}$	6	8	13
$\{1,2\}$	6	17	$*$
$\{1,3\}$	6	$*$	13
$\{2,3\}$	$*$	17	8
$\{1\}$	6	$*$	$*$
$\{2\}$	$*$	17	$*$
$\{3\}$	$*$	$*$	18

This table does not provide a pmas of the corresponding mcst game $(\{1,2,3\}, c_{\tilde{w}})$: in coalition $S = \{2,3\}$ player 3 has to pay 8 which is strictly less than the amount 13 which he has to pay in the larger coalition $N = \{1,2,3\}$.

■

Chapter 3

Mcst games and population monotonic allocation schemes

3.1 Introduction

In Example 2.2.2 it has been provided an mcst situation where the allocation method introduced by Bird (1976) does not generate a pmas of the corresponding mcst game. But it is not clear, up to this point of the story, whether it is possible to find a solution for mcst situations which is always able to generate a pmas. Solving this problem is particularly valuable in applications where the cardinality of the set of agents can vary in time.

Consider for instance the mcst situation introduced in Example 2.1.1. Prim's Algorithm may first form edge $\{0, 1\}$ at the first step, then $\{1, 2\}$, and finally $\{1, 3\}$ and the Bird rule yields the core allocation $x = (24, 12, 12)$.

Suppose now that a fourth employee is asking whether he can join the car-poolers $1, 2$, and 3. The cost of driving from employee 4 to the other employees and to the firm are given in Figure 3.1, as well as a minimum spanning tree for the new situation. Application of the Bird rule to this new situation yields the

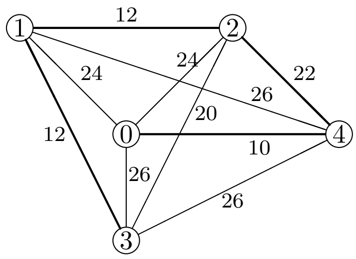


Figure 3.1: The cost of driving and a minimum spanning tree in the new situation.

allocation $x = (12, 22, 12, 10)$. In the new situation employee 2 has to pay 22, whereas in the old situation he only paid 12. Therefore, if the employees use the Bird rule in order to divide joint costs, employee 2 will veto the entrance of employee 4. Note that if Prim's Algorithm applied to the mcst situation in Example 2.1.1 would have formed edge $\{0, 2\}$ at the first step, then $\{1, 2\}$, and finally $\{1, 3\}$, bringing to the mcst $\{\{0, 2\}, \{1, 2\}, \{1, 3\}\}$, the Bird rule would have provided the allocation $(12, 24, 12)$. With respect to this allocation in the mcst situation of Figure 2.1, according to the Bird rule in the new mcst situation of Figure 3.1, no employees would have put a veto on the entrance of employee 4.

The central question in this chapter is whether every minimum cost spanning tree game has a *population monotonic allocation scheme (pmas)* (Sprumont (1990)), which is an allocation scheme that provides a core element for the game and all its subgames and which, moreover, satisfies a monotonicity condition in the sense that players have to pay less in larger coalitions. We will answer this question in the affirmative and we will provide the Subtraction Algorithm, that computes for every minimum cost spanning tree game a pmas. We will show that this algorithm is closely related to Kruskal's algorithm for finding a minimum spanning tree (Kruskal (1956)). The Subtraction Algorithm is based upon a decomposition theorem, which shows that every minimum cost spanning tree game can be written as a non-negative combination of minimum cost spanning tree games corresponding to simple mcst situations.

This chapter is organized as follows. In Section 3.2 the decomposition theorem is provided and minimum cost spanning tree games corresponding to simple mcst situations are studied. The Subtraction Algorithm is presented in Section 3.3. This chapter is based on Norde, Moretti, Tijs (2004).

3.2 Simple mcst games and the decomposition theorem

If the cost of all edges in $Ca(w)$ are lowered by the cost of an edge in $Ca(w)$ with minimal cost we are left with a cost function with smaller carrier. The following lemma establishes a relation between the corresponding mcst games.

Lemma 3.2.1 *Let $w \in \mathcal{W}^{N'}$ be a cost function with $Ca(w) \neq \emptyset$ and let $\alpha := \min\{w(l) : l \in Ca(w)\}$. Let w' be the simple cost function defined by $w'(l) := 1$ if $l \in Ca(w)$ and $w'(l) := 0$ otherwise. Let w'' be the cost function defined by $w''(l) := w(l) - \alpha w'(l)$ for every $l \in E_{N'}$. Finally, let c, c' and c'' be the mcst games corresponding to w, w' and w'' respectively. Then, we have $w = \alpha w' + w''$ and $c = \alpha c' + c''$.*

Proof It follows by definition that $w = \alpha w' + w''$. In order to prove that $c = \alpha c' + c''$, *i.e.* $c(S) = \alpha c'(S) + c''(S)$ for every $S \in 2^N \backslash \{\emptyset\}$, let $S \in 2^N \backslash \{\emptyset\}$. Let Γ' be a minimum cost spanning network for S in w' without cycles, *i.e.* Γ' is a minimum cost spanning tree for S in w'. Write $\Gamma' = L^0 \cup L^1$ where $L^0 := \{l \in \Gamma' : w'(l) = 0\}$ and $L^1 := \{l \in \Gamma' : w'(l) = 1\}$. Clearly, $|\Gamma'| = |L^0| + |L^1|$. Since Γ' is a tree we also have $|\Gamma'| = |S|$. Hence, $c'(S) = w'(\Gamma') = |L^1| = |S| - |L^0|$.
It suffices to show that there exists a minimum cost spanning tree Γ'' for S in w'' with $L^0 \subseteq \Gamma''$. Since then Γ'' contains at most $|\Gamma'' \backslash L^0| = |S| - |L^0|$ edges in $Ca(w')$ and hence $w'(\Gamma'') \leq |S| - |L^0| = w'(\Gamma')$. Therefore, Γ'' is also a minimum cost spanning tree for S in w'. Having $w = \alpha w' + w''$ and the fact that Γ'' is a minimum cost spanning tree for S in both w' and w'' we may conclude that Γ'' is also a minimum cost spanning tree for S in w. So, $c(S) = w(\Gamma'') = \alpha w'(\Gamma'') + w''(\Gamma'') = \alpha c'(S) + c''(S)$.
In order to show that there is a minimum cost spanning tree Γ'' for S in w'' with $L^0 \subseteq \Gamma''$ take an arbitrary minimum cost spanning tree Γ for S in w''. If $L^0 \subseteq \Gamma$ the proof is finished. If $L^0 \not\subseteq \Gamma$ choose an $l \in L^0 \backslash \Gamma$. Since $\Gamma \cup \{l\}$

contains a cycle C, whereas Γ', and hence L^0, do not contain cycles, we can find an edge $l' \in C$ with $l' \notin L^0$. Define $\tilde{\Gamma} := (\Gamma \cup \{l\}) \backslash \{l'\}$. Since $w''(l) = 0$ and $w''(l') \geq 0$ we find that also $\tilde{\Gamma}$ is a minimum cost spanning tree for S in w''. Moreover $|\tilde{\Gamma} \cap L^0| = |\Gamma \cap L^0| + 1$. Repeating this argument results in the tree Γ'' with the desired properties. ■

The following decomposition theorem shows that every minimum cost spanning tree game can be written as a non-negative combination of minimum cost spanning tree games corresponding to simple mcst situations.

Theorem 3.2.2 *Let $w \in \mathcal{W}^{N'}$ be a cost function with $Ca(w) \neq \emptyset$ and let c be the corresponding mcst game. Then, there exists a sequence of simple cost functions $w_1, \ldots, w_k$, with $Ca(w) = Ca(w_1) \supset Ca(w_2) \supset \cdots \supset Ca(w_k)$, and positive numbers $\alpha_1, \ldots, \alpha_k$ such that*

$$w = \sum_{j=1}^{k} \alpha_j w_j. \tag{3.1}$$

Moreover, if $c_1, \ldots, c_k$ are the mcst games corresponding to $w_1, \ldots, w_k$ respectively, we have

$$c = \sum_{j=1}^{k} \alpha_j c_j. \tag{3.2}$$

Proof The proof is by induction to $|Ca(w)|$.
If $|Ca(w)| = 1$ then $Ca(w)$ has a unique element, say l^*. Defining $\alpha := w(l^*)$ and the simple cost function w_1 by $w_1(l^*) := 1$ and $w_1(l) := 0$ if $l \neq l^*$ we clearly have $w = \alpha_1 w_1$. Moreover, if c_1 is the mcst game corresponding to w_1 one easily verifies that $c = \alpha_1 c_1$.
Now, let $m \in I\!N, m \geq 2$ and suppose that the assertion has been proved for every cost function w with $|Ca(w)| \leq m - 1$. Consider a cost function w with $|Ca(w)| = m$. According to Lemma 3.2.1 there is a simple cost function w_1, namely the simple cost function with the same carrier as w, a positive number α_1 and a cost function w'' with $Ca(w'') \subset Ca(w)$ such that $w = \alpha_1 w_1 + w''$. Moreover, if c_1 and c'' are the mcst games corresponding to w_1 and w'' respectively, we have $c = \alpha_1 c_1 + c''$. Application of the induction hypothesis to w'' finishes the proof. ■

Remark 3.2.1 In order to prove relation (3.1), one may directly observe that $\mathcal{W}^{N'} = \bigcup_{\sigma \in \Sigma_{E_{N'}}} K^\sigma$, where K^σ is the Kruskal cone with respect to σ, $\sigma \in \Sigma_{E_{N'}}$, introduced in Section 2.1.2. As we already remarked in relation (2.2), $w \in K^\sigma$ can be written in a unique way as a non-negative linear combination of the generators of the simplicial cone K^σ.

Example 3.2.1 Consider the cost function w of Example 2.1.1. Note that $w \in K^\sigma$, with $\sigma(1) = \{1,3\}$, $\sigma(2) = \{1,2\}$, $\sigma(3) = \{2,3\}$, $\sigma(4) = \{0,1\}$, $\sigma(5) = \{0,2\}$, $\sigma(6) = \{0,3\}$. Hence, by relation (2.2), we have

$$\begin{aligned}
& w = w(\sigma(1))e^{\sigma,1} + \textstyle\sum_{k=2}^{|E_{N'}|} \big(w(\sigma(k)) - w(\sigma(k-1))\big)\, e^{\sigma,k} \\
& = 12e^{\sigma,1} + (12-12)e^{\sigma,2} + (20-12)e^{\sigma,3} \\
& +(24-20)e^{\sigma,4} + (24-24)e^{\sigma,5} + (26-24)e^{\sigma,6} = \\
& = 12e^{\sigma,1} + 8e^{\sigma,3} + 4e^{\sigma,4} + 2e^{\sigma,6}.
\end{aligned}$$

In terms of Theorem 3.2.2 we may write

$$w = \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 + \alpha_4 w_4$$

where $\alpha_1 = 12, \alpha_2 = 8, \alpha_3 = 4$ and $\alpha_4 = 2$, and the simple cost functions $w_1, \ldots, w_4$ are specified by

edge l	$\{1,3\}$	$\{1,2\}$	$\{2,3\}$	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$
$w_1(l) = e^{\sigma,1}(l)$	1	1	1	1	1	1
$w_2(l) = e^{\sigma,3}(l)$	0	0	1	1	1	1
$w_3(l) = e^{\sigma,4}(l)$	0	0	0	1	1	1
$w_4(l) = e^{\sigma,6}(l)$	0	0	0	0	0	1

Computing the mcst games $c_1, \ldots, c_4$ corresponding to $w_1, \ldots, w_4$ respectively, we get

coalition S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$c_1(S)$	1	1	1	2	2	2	3
$c_2(S)$	1	1	1	1	1	2	1
$c_3(S)$	1	1	1	1	1	1	1
$c_4(S)$	0	0	1	0	0	0	0

One easily verifies that $\sum_{i=1}^{4} \alpha_i c_i$ coincides with the mcst game c, as computed in Example 2.2.1.

■

3.3 The Subtraction Algorithm for population monotonic allocation scheme (pmas)'s generation

In this section we will focus on simple cost functions. We show that an mcst game corresponding to a simple cost function has a population monotonic allocation scheme. Using Theorem 3.2.2 we obtain as a corollary that every mcst game has a population monotonic allocation scheme.

Let w be a simple cost function and let $S \in 2^N \backslash \{\emptyset\}$ be a coalition. Two nodes i and j in $S \cup \{0\}$ are (w, S')-*connected* if there exists a sequence of nodes $i = i_0, \ldots, i_k = j$ in $S \cup \{0\}$ with $w(\{i_s, i_{s+1}\}) = 0$ for every $s \in \{0, \ldots, k-1\}$. A (w, S')-*component* of $S \cup \{0\}$ is a maximal subset of $S \cup \{0\}$ with the property that any two nodes in this subset are (w, S')-connected. The number of (w, S')-components is denoted by $n(w, S')$. Clearly, the collection of (w, S')-components forms a partition of $S \cup \{0\}$.

Lemma 3.3.1 *Let w be a simple cost function and let c be the corresponding mcst game. Then, we have*

$$c(S) = n(w, S') - 1$$

for every $S \in 2^N \backslash \{\emptyset\}$.

Proof Let $S \in 2^N \backslash \{\emptyset\}$. If $n(w, S') = 1$ then $S \cup \{0\}$ is the unique (w, S')-component. Therefore, $\Gamma = \{l \in E_{N'} : l \subseteq S \cup \{0\}, w(l) = 0\}$ is a spanning network for S with $w(\Gamma) = 0$. Hence, $c(S) = 0 = n(w, S') - 1$.
Now, suppose $n(w, S') \geq 2$. Let $C_0, C_1, \ldots, C_k$ $(k \geq 1)$ be all (w, S')-components. Clearly, $S \cup \{0\} = \cup_{i=0}^{k} C_i$ and $n(w, S') = k+1$. Without loss of generality

we may assume that $0 \in C_0$. For every $i \in \{1, \ldots, k\}$ select some node $n_i \in C_i$. Consider the network

$$\Gamma = \{l \in E_{N'} : l \subseteq S \cup \{0\}, w(l) = 0\} \cup \{\{n_i, 0\} : i \in \{1, \ldots, k\}\}.$$

The network Γ is a spanning network for S: nodes in C_0 are connected with source 0 via edges in Γ of zero cost, and nodes in C_i with $i \in \{1, \ldots, k\}$ are connected with the source via node n_i. Moreover $w(\Gamma) = k$. It suffices to show that for any spanning tree Γ' for S we have $w(\Gamma') \geq k$, since then Γ is a minimum cost spanning network for S in w, and hence we have $c(S) = w(\Gamma) = k = n(w, S') - 1$. So, let Γ' be a spanning tree for S. Define, for every $i \in \{0, \ldots, k\}$, $\Gamma_i := \Gamma' \cap \{l \in E_{N'} : l \subseteq C_i, w(l) = 0\}$. Since Γ', and hence Γ_i, does not contain cycles we have $|\Gamma_i| \leq |C_i| - 1$ for every $i \in \{0, \ldots, k\}$. Write $\Gamma' = L^0 \cup L^1$ where $L^0 := \{l \in \Gamma' : w(l) = 0\}$ and $L^1 := \{l \in \Gamma' : w(l) = 1\}$. Since $L^0 \subseteq \cup_{i=0}^{k} \Gamma_i$ we have

$$|L^0| \leq \sum_{i=0}^{k} |\Gamma_i| \leq \sum_{i=0}^{k} |C_i| - (k+1) = |S| + 1 - (k+1) = |S| - k.$$

Therefore,

$$w(\Gamma') = |L^1| = |\Gamma'| - |L^0| = |S| - |L^0| \geq k.$$

■

Example 3.3.1 Consider the complete weighted graph $< N', w >$ with $N' = \{0, \ldots, 8\}$ and simple cost function w specified by $\{l \in E_{N'} : w(l) = 0\} = \{\{0,1\}, \{2,3\}, \{2,4\}, \{3,4\}, \{4,5\}, \{6,7\}\}$. Let c be the corresponding mcst game. The edges with zero cost are depicted in Figure 3.2. Clearly, $\{0,1\}$, $\{2,3,4,5\}$, $\{6,7\}$ and $\{8\}$ are all (w, N)-components. Consequently, $c(N) = n(w, N) - 1 = 4 - 1 = 3$. If we consider for example coalition $S = \{2,3,5,6\}$ we get that $\{0\}$, $\{2,3\}$, $\{5\}$ and $\{6\}$ are all (w, S')-components. Consequently, we also have $c(S) = n(w, S') - 1 = 4 - 1 = 3$.

■

In order to show that an mcst game corresponding to a simple cost function has a pmas we need some more notation. In the following, if $w \in \mathcal{W}^{N'}$ is a simple cost function, $S \in 2^N \backslash \{\emptyset\}$ and $i \in S$ then the (w, S')-component to which i belongs is denoted by $C_i(w, S)$.

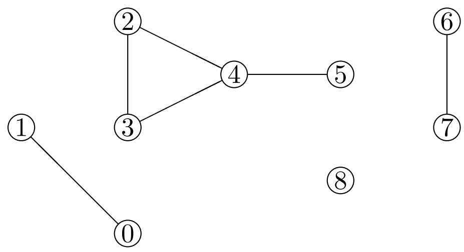


Figure 3.2: The cost function of Example 3.3.1.

Definition 3.3.1 Let $w \in \mathcal{W}^{N'}$ be a simple cost function and let $\tau \in \Sigma_N$. The scheme $x^{\tau,w} = (x^{\tau,w}_{S,i})_{S \in 2^N \setminus \{\emptyset\}, i \in S}$ is defined in the following way:

$$x^{\tau,w}_{S,i} = \begin{cases} 0 & \text{if } 0 \in C_i(w,S), \\ 0 & \text{if } 0 \notin C_i(w,S) \text{ and } \tau^{-1}(i) \neq \min\limits_{j \in C_i(w,S)} \tau^{-1}(j), \\ 1 & \text{if } 0 \notin C_i(w,S) \text{ and } \tau^{-1}(i) = \min\limits_{j \in C_i(w,S)} \tau^{-1}(j), \end{cases}$$

for every $S \in 2^N \setminus \{\emptyset\}$ and for every $i \in S$.

The scheme $x^{\tau,w}$ provides for every coalition $S \in 2^N \setminus \{\emptyset\}$ a division of the cost $c(S)$ in the following way: all members of the (w,S')-component containing the source 0 do not have to pay anything whereas the (unit) cost of all other (w,S')-components is allocated to the member in the component with the lowest index according to τ.

Example 3.3.2 Consider the simple cost function w of Example 3.3.1 and let $\tau \in \Sigma_N$ be given by $\tau^{-1}(1) = 2$, $\tau^{-1}(2) = 7$, $\tau^{-1}(3) = 5$, $\tau^{-1}(4) = 3$, $\tau^{-1}(5) = 6$, $\tau^{-1}(6) = 8$, $\tau^{-1}(7) = 1$ and $\tau^{-1}(8) = 4$. Then, $x^{\tau,w}_{N,1} = x^{\tau,w}_{N,2} = x^{\tau,w}_{N,3} = x^{\tau,w}_{N,5} = x^{\tau,w}_{N,6} = 0$ and $x^{\tau,w}_{N,4} = x^{\tau,w}_{N,7} = x^{\tau,w}_{N,8} = 1$. Moreover, for $S = \{2,3,5,6\}$ we get $x^{\tau,w}_{S,2} = 0$ and $x^{\tau,w}_{S,3} = x^{\tau,w}_{S,5} = x^{\tau,w}_{S,6} = 1$.

■

In the following lemma we prove that the scheme $x^{\tau,w}$ is a pmas for the mcst game corresponding to simple cost function w.

Lemma 3.3.2 *Let w be a simple cost function, c_w the corresponding mcst game, and $\tau \in \Sigma_N$. Then, $x^{\tau,w}$ is a pmas for c_w.*

Proof Let $S \in 2^N \backslash \{\emptyset\}$. Every (w, S')-component which does not contain the source 0 contains precisely one player $i \in S$ with $x^{\tau,w}_{S,i} = 1$. Therefore,

$$\sum_{i \in S} x^{\tau,w}_{S,i} = n(w, S') - 1 = c(S).$$

Now, let $i \in N$ and $S, T \in 2^N \backslash \{\emptyset\}$ be such that $i \in S \subset T$. In order to show that $x^{\tau,w}_{S,i} \geq x^{\tau,w}_{T,i}$ it suffices to show that $x^{\tau,w}_{T,i} = 1$ implies $x^{\tau,w}_{S,i} = 1$. So, assume $x^{\tau,w}_{T,i} = 1$, *i.e.* $0 \notin C_i(w, T)$ and

$$\tau^{-1}(i) = \min_{j \in C_i(w,T)} \tau^{-1}(j).$$

Obviously, we have $C_i(w, S) \subseteq C_i(w, T)$, which implies $0 \notin C_i(w, S)$ and

$$\tau^{-1}(i) = \min_{j \in C_i(w,S)} \tau^{-1}(j).$$

Therefore, $x^{\tau,w}_{S,i} = 1$. ∎

As a corollary we get the main theorem of this section.

Theorem 3.3.3 *Every mcst game has a pmas.*

Proof The theorem follows directly from Theorem 3.2.2, Lemma 3.3.2 and the observation that if $x^1 = (x^1_{S,i})_{S \in 2^N \backslash \{\emptyset\}, i \in S}$ is a pmas for game c^1 and $x^2 = (x^2_{S,i})_{S \in 2^N \backslash \{\emptyset\}, i \in S}$ is a pmas for game c^2, then we have that $\alpha x^1 + \beta x^2 := (\alpha x^1_{S,i} + \beta x^2_{S,i})_{S \in 2^N \backslash \{\emptyset\}, i \in S}$ is a pmas for $\alpha c^1 + \beta c^2$ for every $\alpha \geq 0$ and every $\beta \geq 0$. ∎

A basis for an algorithm that finds a pmas in any mcst game is provided by Theorem 3.2.2 and Lemma 3.3.2. Let $w \in \mathcal{W}^{N'}$ with $Ca(w) \neq \emptyset$ and let $\sigma \in \Sigma_{E_{N'}}$ be such that $w \in K^\sigma$. As we already remarked in relation (2.2), $w \in K^\sigma$ can be written in a unique way as a non-negative linear combination of the generators of the simplicial cone K^σ, in formula

$$w = w(\sigma(1))e^{\sigma,1} + \sum_{k=2}^{|E_{N'}|} \big(w(\sigma(k)) - w(\sigma(k-1))\big)\, e^{\sigma,k}.$$

Note that $Ca(e^{\sigma,1}) \supset \cdots \supset Ca(e^{\sigma,k})$. Theorem 3.2.2 tells us that the same decomposition is true for the mcst games c_w and $c_1, \ldots, c_{|E_{N'}|}$, corresponding to w and $e^{\sigma,1},\ldots,e^{\sigma,|E_{N'}|}$, respectively:

$$c_w = w(\sigma(1))c_1 + \sum_{k=2}^{|E_{N'}|} \big(w(\sigma(k)) - w(\sigma(k-1))\big)c_k.$$

Subsequently, fix some permutation $\tau \in \Sigma_N$. Compute, for every $k \in \{1,\ldots,|E_{N'}|\}$, the scheme $x^{\tau,e^{\sigma,k}}$. According to Lemma 3.3.2 the scheme $x^{\tau,e^{\sigma,k}}$ is a pmas for c_k for every $k \in \{1,\ldots,|E_{N'}|\}$. Define the scheme $P^{\tau,w}$ in the following way:

$$P^{\tau,w} := w(\sigma(1))x^{\tau,e^{\sigma,1}} + \textstyle\sum_{k=2}^{|E_{N'}|} \big(w(\sigma(k)) - w(\sigma(k-1))\big)x^{\tau,e^{\sigma,k}} \tag{3.3}$$

or, alternatively,

$$P^{\tau,w} := \Big(\textstyle\sum_{k=1}^{|E_{N'}|-1} \big(x^{\tau,e^{\sigma,k}} - x^{\tau,e^{\sigma,k+1}}\big)w(\sigma(k))\Big) + w(\sigma(|E_{N'}|))x^{\tau,e^{\sigma,|E_{N'}|}}. \tag{3.4}$$

For the same arguments used to prove Theorem 3.3.3, we have that $P^{\tau,w}$ is a pmas for the mcst game c_w. For the sake of completeness note that for $w = 0$ the vectors $P^{\tau,w}_{S,i} := 0$, for every $S \in 2^N\backslash\{\emptyset\}$ and each $i \in S$, determine a pmas for the corresponding minimum cost spanning tree game $c_w = 0$.

Example 3.3.3 Consider the cost function w of Example 2.1.1 and the corresponding mcst game c introduced in Example 2.2.1. As we already noted in Example 2.1.3 and Example 3.2.1, we have that $w \in K^\sigma$, with $\sigma(1) = \{1,3\}$, $\sigma(2) = \{1,2\}$, $\sigma(3) = \{2,3\}$, $\sigma(4) = \{0,1\}$, $\sigma(5) = \{0,2\}$, $\sigma(6) = \{0,3\}$ and, by relation (2.2),

$$\begin{aligned}
&w = w(\sigma(1))e^{\sigma,1} + \textstyle\sum_{k=2}^{|E_{N'}|} \big(w(\sigma(k)) - w(\sigma(k-1))\big)\ e^{\sigma,k}\\
&= 12e^{\sigma,1} + (12-12)e^{\sigma,2} + (20-12)e^{\sigma,3}\\
&+(24-20)e^{\sigma,4} + (24-24)e^{\sigma,5} + (26-24)e^{\sigma,6} =\\
&= 12e^{\sigma,1} + 8e^{\sigma,3} + 4e^{\sigma,4} + 2e^{\sigma,6}.
\end{aligned}$$

Let $\tau \in \Sigma_N$ be given by $\tau^{-1}(i) = i$ for every $i \in N$. By relation (3.3) we have that $P^{\tau,w}$ is given by the following sum

$$P^{\tau,w} = 12x^{\tau,e^{\sigma,1}} + 8x^{\tau,e^{\sigma,3}} + 4x^{\tau,e^{\sigma,4}} + 2x^{\tau,e^{\sigma,6}}$$

=

S	1	2	3
$\{1,2,3\}$	12	12	12
$\{1,2\}$	12	12	$*$
$\{1,3\}$	12	$*$	12
$\{2,3\}$	$*$	12	12
$\{1\}$	12	$*$	$*$
$\{2\}$	$*$	12	$*$
$\{3\}$	$*$	$*$	12

+

S	1	2	3
$\{1,2,3\}$	8	0	0
$\{1,2\}$	8	0	$*$
$\{1,3\}$	8	$*$	0
$\{2,3\}$	$*$	8	8
$\{1\}$	8	$*$	$*$
$\{2\}$	$*$	8	$*$
$\{3\}$	$*$	$*$	8

+

S	1	2	3
$\{1,2,3\}$	4	0	0
$\{1,2\}$	4	0	$*$
$\{1,3\}$	4	$*$	0
$\{2,3\}$	$*$	4	0
$\{1\}$	4	$*$	$*$
$\{2\}$	$*$	4	$*$
$\{3\}$	$*$	$*$	4

+

S	1	2	3
$\{1,2,3\}$	0	0	0
$\{1,2\}$	0	0	$*$
$\{1,3\}$	0	$*$	0
$\{2,3\}$	$*$	0	0
$\{1\}$	0	$*$	$*$
$\{2\}$	$*$	0	$*$
$\{3\}$	$*$	$*$	2

=

S	1	2	3
$\{1,2,3\}$	24	12	12
$\{1,2\}$	24	12	$*$
$\{1,3\}$	24	$*$	12
$\{2,3\}$	$*$	24	20
$\{1\}$	24	$*$	$*$
$\{2\}$	$*$	24	$*$
$\{3\}$	$*$	$*$	26

■

An alternative way of describing the procedure used to define scheme $P^{\tau,w}$, is the following algorithm.

Subtraction Algorithm for the computation of a pmas of an mcst game.

Initialization: Let $< N', w >$ be an mcst situation and let $\tau \in \Sigma_N$. Define $x = \{x_{S,i}\}_{S \in 2^N \setminus \{\emptyset\}, i \in S}$ by $x_{S,i} := 0$ for every $S \in 2^N \setminus \{\emptyset\}, i \in S$.

Algorithm: WHILE $w \neq 0$

DO $\alpha := \min\{w(l) : l \in E_{N'}, w(l) > 0\}$

FOR every $S \in 2^N \backslash \{\emptyset\}, i \in S$:

BEGIN

IF $0 \notin C_i(w, S)$ and

$\tau^{-1}(i) = \min_{j \in C_i(w,S)} \tau^{-1}(j)$

THEN $x_{S,i} := x_{S,i} + \alpha$

END

FOR every $l \in E_{N'}$ with $w(l) > 0$:

BEGIN

$w(l) := w(l) - \alpha$

END

END

Output: A pmas $(x_{S,i})_{S \in 2^N \backslash \{\emptyset\}, i \in S}$ for the game c_w such that $x_{S,i} = P^{\tau,w}_{S,i}$ for each $S \in 2^N \setminus \{\emptyset\}$ and $i \in S$.

In the following two examples we illustrate the Subtraction Algorithm.

Example 3.3.4 Consider the cost function w of Example 2.1.1 and the corresponding mcst game c_w introduced in Example 2.2.1.

Let $\tau \in \Sigma_N$ be given by $\tau^{-1}(i) = i$ for every $i \in N$. In every step of the Subtraction Algorithm some of the coefficients $x_{S,i}$ will be raised by some amount α. Which coefficients $x_{S,i}$ will be raised? Coefficient $x_{S,i}$ will be raised if there is no path in $S \cup \{0\}$ of zero cost from i to source 0 ($0 \notin C_i(w, S)$), and if there is no path in $S \cup \{0\}$ of zero cost from i to some node $j \in S$ with $\tau^{-1}(j) < \tau^{-1}(i)$.

In the first step of the algorithm $\alpha = 12$, the cost of edges $\{1, 2\}$ and $\{1, 3\}$. All coefficients $x_{S,i}$ will be raised by 12. Since all edges have positive cost, at the end of step 1 the cost of every edge will be lowered by 12, so $w(\{1, 2\}) = w(\{1, 3\}) = 0$.

In the second step of the algorithm $\alpha = 8$, the cost of edge $\{2, 3\}$. Since there is no path from $i \in \{1, 2, 3\}$ to source 0 of cost zero and $\{1, 2\}, \{1, 3\}$ is a path of zero cost, according to τ all coefficients $x_{S,1}$ with $1 \in S$ are raised by 8 together with $x_{\{2,3\},2}$,$x_{\{2,3\},3}$, $x_{\{2\},2}$ and $x_{\{3\},3}$. At the end of step 2 the cost of every

edge with positive cost will be lowered by 8, so $w(\{2,3\}) = 0$.
In the third step of the algorithm $\alpha = 4$, the cost of edges $\{0,1\}$ and $\{0,2\}$. Since there is no path from $i \in \{1,2,3\}$ to source 0 with zero cost, according to τ all coefficients $x_{S,1}$ with $1 \in S$ are raised by 4 together with $x_{\{2\},2}$, $x_{\{3\},3}$ and $x_{\{2,3\},2}$ (since 2 and 3 are connected via a zero cost path and $\tau^{-1}(2) < \tau^{-1}(3)$). At the end of step 2 the cost of every edge with positive cost will be lowered by 4, so $w(\{0,1\}) = w(\{0,2\}) = 0$.
In step 4 we have $\alpha = 2$, the cost of edge $\{0,3\}$. Now, all players are (w, S')-connected with the source, for each $S \in 2^N \setminus \{\emptyset, \{3\}\}$. Hence, only $x_{\{3\},3}$ is raised by 2. At the end of step 4, the cost of the unique edge with positive cost, edge $\{0,3\}$, is lowered by 2 and the algorithm stops. The pmas of the corresponding mcst game, created in the Subtraction Algorithm, is given by

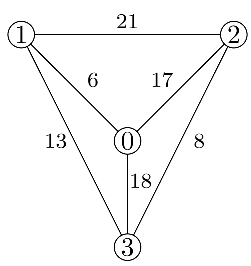
S	1	2	3
$\{1,2,3\}$	$12+8+4$	12	12
$\{1,2\}$	$12+8+4$	12	$*$
$\{1,3\}$	$12+8+4$	$*$	12
$\{2,3\}$	$*$	$12+8+4$	$12+8$
$\{1\}$	$12+8+4$	$*$	$*$
$\{2\}$	$*$	$12+8+4$	$*$
$\{3\}$	$*$	$*$	$12+8+4+2$

$=$

S	1	2	3
$\{1,2,3\}$	24	12	12
$\{1,2\}$	24	12	$*$
$\{1,3\}$	24	$*$	12
$\{2,3\}$	$*$	24	20
$\{1\}$	24	$*$	$*$
$\{2\}$	$*$	24	$*$
$\{3\}$	$*$	$*$	26

■

Example 3.3.5 Consider the complete weighted graph $< N', w >$ with $N' = \{0,1,2,3\}$ and cost function w as depicted in the following figure. Let $\tau \in \Sigma_N$



be given by $\tau^{-1}(i) = i$ for every $i \in N$. In the first step of the algorithm $\alpha = 6$,

the cost of edge $\{0,1\}$. Since all edges have positive cost all coefficients $x_{S,i}$ will be raised by 6. At the end of step 1 the cost of every edge will be lowered by 6; so, $w(\{0,1\}) = 0$.
In the second step of the algorithm $\alpha = 2$, the cost of edge $\{2,3\}$. Since edge $\{0,1\}$ is a path from 1 to source 0 of cost zero no coefficient $x_{S,1}$ with $1 \in S$ will be raised in this step (and in subsequent) steps, whereas all other coefficients are raised by 2. At the end of step 2 the cost of every edge with positive cost will be lowered by 2; so, $w(\{0,1\}) = w(\{2,3\}) = 0$.
In the third step of the algorithm $\alpha = 5$, the cost of edge $\{1,3\}$. Since edge $\{2,3\}$ is a path of zero cost which connects player 3 with player 2, which has a lower index according to τ ($\tau^{-1}(2) < \tau^{-1}(3)$), the coefficients $x_{\{1,2,3\},3}$ and $x_{\{2,3\},3}$ will not be raised in this and further steps. All coefficients, which remain to be raised, are increased by 5 and at the end of step 3 the cost of every edge with positive cost will be lowered by 5; so, $w(\{0,1\}) = w(\{1,3\}) = w(\{2,3\}) = 0$.
In step 4 we have $\alpha = 4$, the cost of edge $\{0,2\}$. Since player 2 is $(w, \{0,1,2,3\})$-connected with the source, via path $\{0,1\}$, $\{1,3\}$, $\{2,3\}$, and player 3 is $(w, \{0,1,3\})$-connected with the source, via path $\{0,1\}$, $\{1,3\}$, the corresponding coefficients will not be raised anymore, whereas all coefficients, which remain to be raised, are increased by 4. At the end of step 4 the cost of every edge with positive cost will be lowered by 4; so, $w(\{0,1\}) = w(\{0,2\}) = w(\{1,3\}) = w(\{2,3\}) = 0$.
In step 5 we have $\alpha = 1$, the cost of edge $\{0,3\}$. Since edge $\{0,2\}$ is a path from 2 to source 0 of zero cost the coefficients $x_{\{1,2\},2}$, $x_{\{2,3\},2}$, and $x_{\{2\},2}$ will not be raised any further. The only coefficient which remains to be raised, $x_{\{3\},3}$, is increased by 1. At the end of step 5 every edge with positive cost is lowered by 1; so, $w(\{0,2\}) = 0$.
In step 6 we have $\alpha = 3$, the cost of edge $\{1,2\}$. Since edge $\{0,3\}$ is a path from 3 to source 0 of zero cost coefficient $x_{\{3\},3}$ will not be raised anymore. The cost of the only edge with positive cost, edge $\{1,2\}$, is lowered by 3 and the algorithm stops. The pmas of the corresponding mcst game, created in the

Subtraction Algorithm, is given by

S	1	2	3
$\{1,2,3\}$	6	$6+2+5$	$6+2$
$\{1,2\}$	6	$6+2+5+4$	$*$
$\{1,3\}$	6	$*$	$6+2+5$
$\{2,3\}$	$*$	$6+2+5+4$	$6+2$
$\{1\}$	6	$*$	$*$
$\{2\}$	$*$	$6+2+5+4$	$*$
$\{3\}$	$*$	$*$	$6+2+5+4+1$

$=$

S	1	2	3
$\{1,2,3\}$	6	13	8
$\{1,2\}$	6	17	$*$
$\{1,3\}$	6	$*$	13
$\{2,3\}$	$*$	17	8
$\{1\}$	6	$*$	$*$
$\{2\}$	$*$	17	$*$
$\{3\}$	$*$	$*$	18

Moreover, note that player 2 in coalition $\{1,2,3\}$ has to pay the cost of edge $\{1,3\}$, although he does not belong to this edge. In every step of the Subtraction Algorithm a multiple of a simple cost function is subtracted from cost function w; the same multiple of the pmas of the mcst game corresponding to this simple cost function is added to $\{x_{S,i}\}_{S\in 2^N\setminus\{\emptyset\},i\in S}$.

■

Remark 3.3.1 *Consider an mcst situation $< N', w >$. Let $\tau \in \Sigma_N$ and let $P^{\tau,w}$ be the pmas generated by the Subtraction Algorithm or via relation (3.3). Let $P^\tau(w) := P_N^{\tau,w}$ be the corresponding core allocation in the mcst game (N, c_w). In the next chapters, the properties of the allocation vectors $P^\tau(w)$ (to which we will refer as P^τ-values), for each $w \in \mathcal{W}^{N'}$ and $\tau \in \Sigma_N$, will be studied.*

Remark 3.3.2 *Consider a complete weighted graph $< N', w >$ where, in order to simplify arguments, all edges have different positive cost. Let $\tau \in \Sigma_N$. Moreover, let $\Gamma = \{l_1, \ldots, l_n\}$ be the unique minimum cost spanning tree for N with $w(l_1) < w(l_2) < \cdots < w(l_n)$. So, according to Kruskal's algorithm, edge l_1 forms in the first step, edge l_2 in the second step, etc. Let $i_1 \in N$ be the unique player which is connected via network $\Gamma_1 = \{l_1\}$ with the source 0 or with some node $j \in N$ with $\tau^{-1}(j) < \tau^{-1}(i_1)$, let i_2 be the unique player in $N\setminus\{i_1\}$ which is connected via network $\Gamma_2 = \{l_1, l_2\}$ with the source 0 or with some node $j \in N$ with $\tau^{-1}(j) < \tau^{-1}(i_2)$, etc. Note that in Example 3.3.5 we have $l_1 = \{0,1\}$,*

$l_2 = \{2,3\}$, $l_3 = \{1,3\}$, and $i_1 = 1$, $i_2 = 3$ and $i_3 = 2$. One easily verifies that $P^{\tau}_{i_k}(w) = w(l_k)$ for every $k \in \{1,\ldots,n\}$. Stated differently, the Subtraction Algorithm allocates the cost of an edge which forms in some step of Kruskal's algorithm to the player which gets a connection with the source or with a player with a lower index according to τ. This procedure to allocate the cost of an edge which forms in some step of Kruskal's algorithm will be formalized in Example 4.2.4 to define the charge system $\mathcal{C}^{\tau}$.

Chapter 4

Construct and Charge rules

4.1 Introduction

As we already said in Chapter 2, to *construct* an mcst two methods are mainly used: Prim's algorithm (Prim (1957)) and Kruskal's algorithm (Kruskal (1956)). Both algorithms determine an mcst where exactly one edge is constructed in every step of the algorithm. The total number of steps equals n. To divide the cost of an mcst among the agents, both algorithms are suitable to define cost allocation protocols which *charge* the agents with "fractions" of the cost of each edge constructed in each step of the procedure.

In Feltkamp *et al.* (1994a,b), Norde *et al.* (2004), Branzei *et al.* (2004) and Tijs *et al.* (2006a) particular allocation protocols based on Kruskal's algorithm are studied. Recently, (Moretti *et al.* (2005)), we have discovered that we can embed all these allocation protocols on mcst situations in a larger class of *Construct and Charge* rules, formally introduced in Section 4.4. Construct and Charge rules have been studied already in Feltkamp et al. (1994b) for *minimum cost spanning extension* (mcse) situations. These mcse situations are generalized mcst situations in which some network can be present initially, which has to be extended to a network connecting every player to the source. Feltkamp et al. (1994b) proved that the allocations provided by Construct and Charge rules are in the core of the game corresponding to an mcse situation (in case no network is present initially, an mcse situation is an mcst situation, and the game is the corresponding mcst game).

Some Construct and Charge rules are independent of the selected mcst, but others are not. For example, the Proportional rule (Feltkamp *et al.* (1994b)) is a Construct and Charge rule and may provide different cost allocations on the same mcst situation, depending on the feasible orderings of the edges with respect to increasing costs. The ERO-rule introduced in Feltkamp *et al.* (1994a,b) and rebaptized as the P-value (Branzei *et al.* (2004)), the P^{τ}-values (Norde *et al.* (2004), Branzei *et al.* (2004)) and the Obligation rules (Tijs *et al.* (2006a)) are Construct and Charge rules which do not depend on the mcst selected, providing a unique cost allocation on each mcst situation.

The aim of this chapter is to introduce and characterize a class of rules for mcst situations, which we call 'conservative Construct and Charge' rules. An interesting feature of such rules is that different feasible orderings of the edges lead to the same cost allocations. It turns out that the subclass of conservative Construct and Charge rules coincides with the class of Obligations rules (Tijs *et al.* (2006a)), that will be introduced in Section 4.5.

In Section 4.2 the definition of a charge system is introduced, specific examples are given and some basic properties are studied. In Section 4.3 conservative charge systems are introduced and a related concept of potential is discussed. Based on charge systems and orderings of the edges with respect to increasing costs, the definition of a Construct and Charge rule for mcst situations is given in Section 4.4, together with some examples and properties for such rules. In section 4.5 Obligation rules are introduced starting from the general notion of obligation map, and some basic properties are studied. In Section 4.7 the connection of Construct and Charge rules with Obligation rules is studied.

Sections 4.2-4.4 and 4.7 are based on Moretti, Tijs, Branzei, Norde (2008); section 4.5 is based on Tijs, Branzei, Moretti, Norde (2006a); section 4.6 is based on Branzei, Moretti, Norde, Tijs (2004).

4.2 Charge systems

To introduce charge systems we need some additional notations. Let $N = \{1, \ldots, n\}$ and $\Delta(N) = \{x \in \mathbb{R}^N_+ | \sum_{i \in N} x_i = 1\}$. We denote by $\mathcal{E}_{N'}$ the set of n-vectors of edges which form a spanning tree on N', *i.e.*

$$\mathcal{E}_{N'} = \{(a_1, \ldots, a_n) \in (E_{N'})^n | \{a_1, \ldots, a_n\} \text{ is a spanning network on } N'\}.$$

Note that the number of edges which form a spanning tree on N' is n.

Given an element $\mathbf{a} = (a_1, \ldots, a_n) \in (E_{N'})^n$, we denote by $\mathbf{a}_{|j}$ the restriction of $\mathbf{a}$ to the first j components, that is $\mathbf{a}_{|j} = (a_1, \ldots, a_j)$ for each $j \in N$. Further, for each $j \in N$, we denote by $\Pi(\mathbf{a}_{|j})$ the partition of N' such that

$$\Pi(\mathbf{a}_{|j}) = \{T \subseteq N' | T \text{ is a connected component in } < N', \{a_1, \ldots, a_j\} >\}.$$

Example 4.2.1 Consider the spanning tree depicted in Figure 4.1 on $N' = \{0, 1, 2, 3, 4\}$. Vectors $\mathbf{a} = (\{2,3\}, \{0,1\}, \{3,4\}, \{0,3\})$ and $\mathbf{b} = (\{3,4\}, \{2,3\}, \{0,1\}, \{0,3\},)$ are elements of $\mathcal{E}_{\{0,1,2,3,4\}}$. Note that $\mathbf{a}_{|3} = (\{2,3\}, \{0,1\}, \{3,4\})$ and $\mathbf{b}_{|3} = (\{3,4\}, \{2,3\}, \{0,1\})$ implying that $\Pi(\mathbf{a}_{|3}) = \Pi(\mathbf{b}_{|3}) = \{\{0,1\}, \{2,3,4\}\}$.

■

Summing up, each element $\mathbf{a} \in \mathcal{E}_{N'}$ tells the "history" of the spanning network

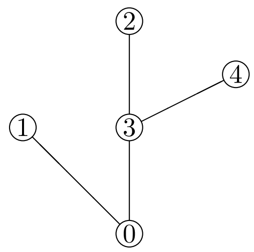


Figure 4.1: A spanning tree on $N' = \{0, 1, 2, 3, 4\}$.

formation, that is adding the edge a_j to the already formed graph $\mathbf{a}_{|j-1}$, for each $j \in N$ (note that when the first edge a_1 is formed, the already formed graph is $< N', \emptyset >$. So, $\Pi(\mathbf{a}_{|0})$ is the singleton partition of N'.).

Now, let $\theta \in \Theta(N')$, where $\Theta(N')$ is the family of partitions of N', and let $T \subseteq N'$. If T is a subset of a certain element of the partition θ, we denote this element as $S(\theta, T)$.

Definition 4.2.1 A *charge system* $\mathcal{C}$ on N is a set of functions $\mathcal{C} = \{C^1, \ldots, C^n\}$ with $C^j : \{\mathbf{a}_{|j} | \mathbf{a} \in \mathcal{E}_{N'}\} \to \Delta(N)$ for each $j \in N$ satisfying the following properties:

(Connection property):	$C_i^j(\mathbf{a}_{\vert j}) = 0$ for each $i \in S(\Pi(\mathbf{a}_{\vert j-1}), \{0\})$, each $j \in N$, and each $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{E}_{N'}$;
(Involvement property):	$C_i^j(\mathbf{a}_{\vert j}) = 0$ for each $i \in N \setminus S(\Pi(\mathbf{a}_{\vert j}), a_j)$ each $j \in N$, and each $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{E}_{N'}$;
(Total aggregation property):	$\sum_{j=1}^n C_i^j(\mathbf{a}_{\vert j}) = 1$ for each $i \in N$, and each $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{E}_{N'}$.

A charge system specifies how to charge agents during the construction of a spanning tree. Let $\mathbf{a} \in \mathcal{E}_{N'}$. First, the cost of each edge a_j, for each $j \in N$, should be totally charged among agents as soon as a_j is formed. This requirement makes a charge system promptly adaptable to modified situations, where edges are formed according to different orders (for instance, due to a change in the route of transportation).

The connection property says that agents already connected to the source in $\mathbf{a}_{|j-1}$ should not be charged anymore. This property accounts for the fact that there is no interest for agents already connected to the source in contributing to the construction of other edges in the network.

The involvement property specifies that only agents who are connected to nodes in a_j in the graph $\mathbf{a}_{|j}$ (i.e. agents involved in forming a_j) should be charged with fractions of the cost of a_j. This property is particularly valuable in supply transportation networks, because the continuous control on the charge procedure is simpler for customers which are directly involved in the construction of the edges.

The total aggregation property says that when the construction of the spanning network corresponding to $\mathbf{a}$ is completed, each agent has been charged for a total amount of fractions equal to 1. This property is a natural *a priori* requirement of fairness in a charge system, since it guarantees that all agents have duties on the same amount of total fractions of edges of a spanning tree.

The charge systems in Examples 4.2.2 and 4.2.3 will play a role in Section 4.7 to define special Construct and Charge rules. The intuition behind the charge system of Example 4.2.2 is to charge each agent in a connected component according to his 'remaining obligation (RO)'. At the start the RO is 1 for every agent. If in some step of the algorithm the connected component of an agent i is linked to some other connected component, then agent i is charged according to the following rule: if i is linked to a component containing the source, then i is charged by his RO (leaving a RO of 0 for this agent); otherwise, if i is linked to a component not containing the source, then i is charged half of his RO (leaving a RO that is half of his RO in the previous step). The charge system of Example 4.2.3 charges the agents involved in forming the edge a_j, for each $j \in N$, taking into account the cardinality of their connected components in the graphs $\mathbf{a}_{|j-1}$ and $\mathbf{a}_{|j}$. As a result of this procedure, at each stage $j \in N$, agents in the same connected component have the same RO.

Example 4.2.2 Consider the charge system $\tilde{\mathcal{C}} = \{\tilde{C}^1, \ldots, \tilde{C}^n\}$ on N such that for each $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{E}_{N'}$ and for each $i, j \in N$

$$\tilde{C}_i^j(\mathbf{a}_{|j}) = \begin{cases} \frac{1}{2} r_i^j & \text{if } i \in S(\Pi(\mathbf{a}_{|j}), a_j) \\ & \text{and } 0 \notin S(\Pi(\mathbf{a}_{|j}), a_j), \\ & \\ r_i^j & \text{if } \{0, i\} \subseteq S(\Pi(\mathbf{a}_{|j}), a_j) \\ & \text{and } 0 \notin S(\Pi(\mathbf{a}_{|j-1}), \{i\}), \\ & \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

where the *remaining obligation* r^j is defined as

$$r_i^j = 1 - \sum_{k=1}^{j-1} \tilde{C}_i^k(\mathbf{a}_{|k}) \tag{4.2}$$

for each $j \in N$, $j > 1$, and $r_i^1 = 1$ for each $i \in N$.

The involvement property and the connection property of functions $\tilde{C}^1, \ldots, \tilde{C}^n$ are a direct consequence of relation (4.1). For the total aggregation property of functions $\tilde{C}^1, \ldots, \tilde{C}^n$, first note that, by relation (4.2), for each $i, j \in N$ such that $i \in S(\Pi(\mathbf{a}_{|j}), a_j)$ and $0 \notin S(\Pi(\mathbf{a}_{|j}), a_j)$, the quantity $\sum_{k=1}^{j} \tilde{C}_i^k(\mathbf{a}_{|k}) < 1$. Then, by relation (4.1) the total aggregation property follows immediately.

In order to prove that function $\tilde{C}^j$, $j \in N$, takes values in $\Delta(N)$, we first note by relations (4.1) that $\tilde{C}_i^j \geq 0$, for all $i, j \in N$. Second, we prove by induction to j that the sum $\sum_{i \in N} \tilde{C}_i^j(\mathbf{a}_{|j}) = 1$ for each $j \in N$.

If $j = 1$ we have that $\sum_{i \in N} \tilde{C}_i^1(\mathbf{a}_{|1}) = \sum_{i \in a_1, i \neq 0} \tilde{C}_i^1(\mathbf{a}_{|1}) = 1$.

Now, let $j \in \{2, \ldots, n\}$ and suppose that $\sum_{i \in N} \tilde{C}_i^k(\mathbf{a}_{|k}) = 1$ for every $k \in \{1, \ldots, j-1\}$. Let $z \in a_j$ be one of the two nodes of edge a_j such that $0 \notin S(\Pi(\mathbf{a}_{|j-1}), \{z\})$ and let $K_z \subseteq \{1, \ldots, j-1\}$ be the set of indices k such that a_k is contained in $S(\Pi(\mathbf{a}_{|j-1}), \{z\})$, in formula $K_z = \{k \in \{1, \ldots, j-1\} | a_k \subseteq S(\Pi(\mathbf{a}_{|j-1}), \{z\})$. Note that $|K_z| = |S(\Pi(\mathbf{a}_{|j-1}), \{z\})| - 1$, since $|K_z|$ edges are needed to construct a spanning tree on $S(\Pi(\mathbf{a}_{|j-1}), \{z\})$. We have

$$\begin{aligned}
&\textstyle\sum_{i \in S(\Pi(\mathbf{a}_{|j-1}), \{z\})} \sum_{k=1}^{j-1} \tilde{C}_i^k(\mathbf{a}_{|k}) \\
&= \textstyle\sum_{k=1}^{j-1} \sum_{i \in S(\Pi(\mathbf{a}_{|j-1}), \{z\})} \tilde{C}_i^k(\mathbf{a}_{|k}) \\
&= \textstyle\sum_{k \in K_z} \sum_{i \in S(\Pi(\mathbf{a}_{|j-1}), \{z\})} \tilde{C}_i^k(\mathbf{a}_{|k}) \\
&= \textstyle\sum_{k \in K_z} \sum_{i \in N} \tilde{C}_i^k(\mathbf{a}_{|k}) = |K_z|,
\end{aligned} \tag{4.3}$$

where the second equality follows from the involvement property which specifies that $\tilde{C}_i^k(\mathbf{a}_{|k}) = 0$ for each $i \in S(\Pi(\mathbf{a}_{|j-1}), \{z\})$ and $k \in \{1, \ldots, j-1\} \setminus K_z$; the third equality follows from the involvement property which specifies that $\tilde{C}_i^k(\mathbf{a}_{|k}) = 0$ for each $i \in N \setminus S(\Pi(\mathbf{a}_{|j-1}), \{z\})$ and $k \in K_z$; finally, the last equality follows from the induction hypothesis. When edge a_j is constructed, a new partition of nodes $\Pi(\mathbf{a}_{|j})$ forms. By the connection property, only nodes which were not yet connected to 0 in $\Pi(\mathbf{a}_{|j-1})$ are charged. Then, we distinguish two cases:

case 1) $a_j = \{u, v\} \in E_{N'}$, $0 \notin S(\Pi(\mathbf{a}_{|j}), \{u\})$, $0 \notin S(\Pi(\mathbf{a}_{|j}), \{v\})$. We have

$$\begin{aligned}
&\textstyle\sum_{i\in N} \tilde{C}_i^j(\mathbf{a}_{|j})\\
&= \textstyle\sum_{i\in S(\Pi(\mathbf{a}_{|j-1}),\{u\})} \frac{1}{2} r_i^j + \sum_{i\in S(\Pi(\mathbf{a}_{|j-1}),\{v\})} \frac{1}{2} r_i^j\\
&= \textstyle\sum_{i\in S(\Pi(\mathbf{a}_{|j-1}),\{u\})} \frac{1}{2}\big(1 - \sum_{k=1}^{j-1} \tilde{C}_i^k(\mathbf{a}_{|k})\big)\\
&+ \textstyle\sum_{i\in S(\Pi(\mathbf{a}_{|j-1}),\{v\})} \frac{1}{2}\big(1 - \sum_{k=1}^{j-1} \tilde{C}_i^k(\mathbf{a}_{|k})\big)\\
&= \tfrac{1}{2}\big(|S(\Pi(\mathbf{a}_{|j-1}), \{u\})| - |K_u|\big) + \tfrac{1}{2}\big(|S(\Pi(\mathbf{a}_{|j-1}), \{v\})| - |K_v|\big)\\
&= \tfrac{1}{2}\big(|S(\Pi(\mathbf{a}_{|j-1}), \{u\})| - |S(\Pi(\mathbf{a}_{|j-1}), \{u\})| + 1\big)\\
&+ \tfrac{1}{2}\big(|S(\Pi(\mathbf{a}_{|j-1}), \{v\})| - |S(\Pi(\mathbf{a}_{|j-1}), \{v\})| + 1\big) = 1.
\end{aligned}$$

where the first equality follows by relation (4.1) and the involvement property, and the third equality from relation (4.3).

case 2) $a_j = \{u, v\} \in E_{N'}$, $0 \notin S(\Pi(\mathbf{a}_{|j}), \{u\})$, $0 \in S(\Pi(\mathbf{a}_{|j}), \{v\})$. We have

$$\begin{aligned}
&\textstyle\sum_{i\in N} \tilde{C}_i^j(\mathbf{a}_{|j})\\
&= \textstyle\sum_{i\in S(\Pi(\mathbf{a}_{|j-1}),\{u\})} r_i^j\\
&= \textstyle\sum_{i\in S(\Pi(\mathbf{a}_{|j-1}),\{u\})} \big(1 - \sum_{k=1}^{j-1} \tilde{C}_i^k(\mathbf{a}_{|k})\big)\\
&= \big(|S(\Pi(\mathbf{a}_{|j-1}), \{u\})| - |K_u|\big)\\
&= \big(|S(\Pi(\mathbf{a}_{|j-1}), \{u\})| - |S(\Pi(\mathbf{a}_{|j-1}), \{u\})| + 1\big) = 1,
\end{aligned}$$

where the first equality follows by relation (4.1) and the involvement property, and the third equality from relation (4.3).

We may conclude that $\tilde{C}^1, \ldots, \tilde{C}^n$ constitute a charge system.

■

Example 4.2.3 Consider the set of functions $\hat{\mathcal{C}} = \{\hat{C}^1, \ldots, \hat{C}^n\}$ on N such that for each $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{E}_{N'}$ and for each $j \in N$

$$\hat{C}_i^j(\mathbf{a}_{|j}) = \begin{cases} \frac{1}{|S(\Pi(\mathbf{a}_{|j-1}),\{i\})|} - \frac{1}{|S(\Pi(\mathbf{a}_{|j}),\{i\})|} & \text{if } 0 \notin S(\Pi(\mathbf{a}_{|j}), a_j) \\[2ex] \frac{1}{|S(\Pi(\mathbf{a}_{|j-1}),\{i\})|} & \text{if } \{0, i\} \subseteq S(\Pi(\mathbf{a}_{|j}), a_j) \\ & \text{and } 0 \notin S(\Pi(\mathbf{a}_{|j-1}), \{i\}), \\[2ex] 0 & \text{otherwise,} \end{cases} \tag{4.4}$$

for each $i \in N$.

In order to check that the functions $\hat{C}^1, \ldots, \hat{C}^n$ constitute a charge system, we first show that functions $\hat{C}^1, \ldots, \hat{C}^n$ take values in $\Delta(N)$. Note that for each $j \in N$ such that $a_j = \{u, v\} \in E_{N'}$ and $0 \notin S(\Pi(\mathbf{a}_{|j}), a_j)$ we have that

$$\begin{aligned}
&\textstyle\sum_{i \in N} \hat{C}_i^j(\mathbf{a}_{|j}) \\
&= \textstyle\sum_{i \in S(\Pi(\mathbf{a}_{|j-1}), \{u\})} \frac{1}{|S(\Pi(\mathbf{a}_{|j-1}), \{u\})|} - \frac{1}{|S(\Pi(\mathbf{a}_{|j}), \{u\})|} \\
&+ \textstyle\sum_{i \in S(\Pi(\mathbf{a}_{|j-1}), \{v\})} \frac{1}{|S(\Pi(\mathbf{a}_{|j-1}), \{v\})|} - \frac{1}{|S(\Pi(\mathbf{a}_{|j}), \{v\})|} \\
&= \textstyle\sum_{i \in S(\Pi(\mathbf{a}_{|j-1}), \{u\})} \frac{1}{|S(\Pi(\mathbf{a}_{|j-1}), \{u\})|} + \sum_{i \in S(\Pi(\mathbf{a}_{|j-1}), \{v\})} \frac{1}{|S(\Pi(\mathbf{a}_{|j-1}), \{v\})|} \\
&- \textstyle\sum_{i \in S(\Pi(\mathbf{a}_{|j}), a_j)} \frac{1}{|S(\Pi(\mathbf{a}_{|j}), a_j)|} = 1 + 1 - 1 = 1.
\end{aligned}$$

Differently, for each $j \in N$ such that $0 \in S(\Pi(\mathbf{a}_{|j}), a_j)$ we have that

$$\textstyle\sum_{i \in N} \hat{C}_i^j(\mathbf{a}_{|j}) = \sum_{i \in S(\Pi(\mathbf{a}_{|j-1}), \{m\})} \frac{1}{|S(\Pi(\mathbf{a}_{|j-1}), \{m\})|} = 1,$$

where $m \in S(\Pi(\mathbf{a}_{|j}), a_j)$ is such that $0 \notin S(\Pi(\mathbf{a}_{|j-1}), \{m\})$.

The connection property of functions $\hat{C}^1, \ldots, \hat{C}^n$ directly follows by relation (4.4) .

To prove that $\hat{C}^1, \ldots, \hat{C}^n$ satisfy the involvement property we note that if for $i, j \in N$ we have that $i \notin S(\Pi(\mathbf{a}_{|j}), a_j)$, then it follows that $S(\Pi(\mathbf{a}_{|j-1}), \{i\})$ $= S(\Pi(\mathbf{a}_{|j}), \{i\})$, since nothing is changed in the connected component of agent i from stage $j-1$ to stage j. Consequently, by relation (4.4), we have that $\hat{C}_i^j(\mathbf{a}_{|j}) = 0$.

Finally, to prove that functions $\hat{C}^1, \ldots, \hat{C}^n$ satisfy the total aggregation property, first note that for each $i \in N$, we have that $\sum_{j=1}^n \hat{C}_i^j(\mathbf{a}_{|j}) = \sum_{j=1}^k \hat{C}_i^j(\mathbf{a}_{|j})$, where $k \in N$ is such that $\{0, i\} \subseteq S(\Pi(\mathbf{a}_{|k}), a_k)$ and $0 \notin S(\Pi(\mathbf{a}_{|k-1}), \{i\})$. Consequently, for $k = 1$, by relation (4.4) we have that

$$\sum_{j=1}^{1} \hat{C}_i^j(\mathbf{a}_{|j}) = \frac{1}{|S(\Pi(\mathbf{a}_{|0}), \{i\})|} = 1.$$

For $k > 1$ we have that

$$\begin{aligned}
&\textstyle\sum_{j=1}^k \hat{C}_i^j(\mathbf{a}_{|j}) = \left(\sum_{j=1}^{k-1} \frac{1}{|S(\Pi(\mathbf{a}_{|j-1}), \{i\})|} - \frac{1}{|S(\Pi(\mathbf{a}_{|j}), \{i\})|} \right) + \frac{1}{|S(\Pi(\mathbf{a}_{|k-1}), \{i\})|} \\
&= \frac{1}{|S(\Pi(\mathbf{a}_{|0}), \{i\})|} - \frac{1}{|S(\Pi(\mathbf{a}_{|k-1}), \{i\})|} + \frac{1}{|S(\Pi(\mathbf{a}_{|k-1}), \{i\})|} = \frac{1}{|S(\Pi(\mathbf{a}_{|0}), \{i\})|} = 1.
\end{aligned}$$

■

Example 4.2.4 Given a bijection $\tau \in \Sigma_N$, consider the set of functions $\mathcal{C}^\tau = \{C^{\tau,1}, \dots, C^{\tau,n}\}$ on N be such that for each $\mathbf{a} = (a_1, \dots, a_n) \in \mathcal{E}_{N'}$ and for each $i \in N$

$$C_i^{\tau,1}(\mathbf{a}_{|1}) = \begin{cases} 1 & \text{if } \tau^{-1}(i) = \max\{\tau^{-1}(k) | k \in S(\Pi(\mathbf{a}_{|1}), a_1) \setminus \{0\}\}, \\ \\ 0 & \text{otherwise,} \end{cases}$$

and for each $j \in \{2, \dots, n\}$

$$C_i^{\tau,j}(\mathbf{a}_{|j}) = \begin{cases} 1 & \text{if } \tau^{-1}(i) = \max\{\tau^{-1}(k) | k \in S(\Pi(\mathbf{a}_{|j}), a_j) \setminus \{0\} \\ & \text{and } C_k^{\tau,l}(\mathbf{a}_{|l}) = 0 \text{ for all } l < j\}, \\ \\ 0 & \text{otherwise.} \end{cases}$$

One can easily verify that the functions $C^{\tau,1}, \dots, C^{\tau,n}$ take values in $\Delta(N)$ and satisfy the connection property, involvement property and total aggregation property. Hence these functions constitute a charge system.

■

Example 4.2.5 Consider the set of functions $\check{\mathcal{C}} = \{\check{C}^1, \dots, \check{C}^n\}$ on N such that for each $\mathbf{a} = (a_1, \dots, a_n) \in \mathcal{E}_{N'}$ and each $i \in N$

$$\check{C}_i^1(\mathbf{a}_{|1}) = \begin{cases} \frac{1}{2} & \text{if } 0 \notin S(\Pi(\mathbf{a}_{|1}), a_1) \text{ and } i \in S(\Pi(\mathbf{a}_{|1}), a_1) \\ \\ 1 & \text{if } \{0, i\} \subseteq S(\Pi(\mathbf{a}_{|1}), a_1), \\ \\ 0 & \text{otherwise,} \end{cases}$$

and for each $j \in \{2, \dots, n\}$

$$\check{C}_i^j(\mathbf{a}_{|j}) = \begin{cases} \min\{1 - \sum_{k=1}^{j-1} \check{C}_i^k(\mathbf{a}_{|k}), \alpha\} & \text{if } i \in S(\Pi(\mathbf{a}_{|j}), a_j), \\ \\ 0 & \text{otherwise.} \end{cases}$$

where $\alpha \in I\!R_+$ is a real number such that

$$\sum_{i \in S(\Pi(\mathbf{a}_{|j}), a_j) \setminus \{0\}} \min\{1 - \sum_{k=1}^{j-1} \check{C}_i^k(\mathbf{a}_{|k}), \alpha\} = 1. \tag{4.5}$$

From relation (4.5) it follows directly that the functions $\check{C}^1, \ldots, \check{C}^n$ take values in $\Delta(N)$. One can easily check that the functions $\check{C}^1, \ldots, \check{C}^n$ also satisfy the connection property, involvement property and total aggregation property. Hence, these functions constitute a charge system.

■

Next example shows a numerical application of the charge systems introduced in Examples 4.2.2 - 4.2.5.

Example 4.2.6 Consider the spanning tree depicted in Figure 4.1 of Example 4.2.1 and consider the charge systems $\tilde{\mathcal{C}}$, $\hat{\mathcal{C}}$, $\mathcal{C}^\tau$ with $\tau(i) = i$ for each $i \in \{1, \ldots, n\}$, and $\check{\mathcal{C}}$, respectively introduced in Examples 4.2.2 - 4.2.5. In Tables 4.1 - 4.4 we show the respective charge systems corresponding to **a** and **b** of Example 4.2.1.

j	1	2	3	4
$\tilde{C}^j(\mathbf{a}_{\vert j})$	$(0, \frac{1}{2}, \frac{1}{2}, 0)^t$	$(1, 0, 0, 0)^t$	$(0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2})^t$	$(0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2})^t$
$\tilde{C}^j(\mathbf{b}_{\vert j})$	$(0, 0, \frac{1}{2}, \frac{1}{2})^t$	$(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4})^t$	$(1, 0, 0, 0)^t$	$(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4})^t$

Table 4.1: The charge system of Example 4.2.2 for **a** and **b** of Example 4.2.1.

j	1	2	3	4
$\hat{C}^j(\mathbf{a}_{\vert j})$	$(0, \frac{1}{2}, \frac{1}{2}, 0)^t$	$(1, 0, 0, 0)^t$	$(0, \frac{1}{6}, \frac{1}{6}, \frac{2}{3})^t$	$(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3},)^t$
$\hat{C}^j(\mathbf{b}_{\vert j})$	$(0, 0, \frac{1}{2}, \frac{1}{2})^t$	$(0, \frac{2}{3}, \frac{1}{6}, \frac{1}{6})^t$	$(1, 0, 0, 0)^t$	$(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})^t$

Table 4.2: The charge system of Example 4.2.3 for **a** and **b** of Example 4.2.1.

j	1	2	3	4
$C^{\tau,j}(\mathbf{a}_{\vert j})$	$(0, 0, 1, 0)^t$	$(1, 0, 0, 0)^t$	$(0, 0, 0, 1)^t$	$(0, 1, 0, 0,)^t$
$C^{\tau,j}(\mathbf{b}_{\vert j})$	$(0, 0, 0, 1)^t$	$(0, 0, 1, 0)^t$	$(1, 0, 0, 0)^t$	$(0, 1, 0, 0,)^t$

Table 4.3: The charge system of Example 4.2.4, with $\tau(i) = i$ for each $i \in \{1, \ldots, 4\}$, for **a** and **b** of Example 4.2.1.

■

j	1	2	3	4
$\check{C}^j(\mathbf{a}_{\vert j})$	$(0,\frac{1}{2},\frac{1}{2},0)^t$	$(1,0,0,0)^t$	$(0,\frac{1}{3},\frac{1}{3},\frac{1}{3})^t$	$(0,\frac{1}{6},\frac{1}{6},\frac{2}{3})^t$
$\check{C}^j(\mathbf{b}_{\vert j})$	$(0,0,\frac{1}{2},\frac{1}{2})^t$	$(0,\frac{1}{3},\frac{1}{3},\frac{1}{3})^t$	$(1,0,0,0)^t$	$(0,\frac{2}{3},\frac{1}{6},\frac{1}{6})^t$

Table 4.4: The charge system of Example 4.2.5 for **a** and **b** of Example 4.2.1.

4.3 Conservative Charge systems

In this section, special charge systems, which we call *conservative*, will play a role. Consider a charge system $\mathcal{C} = \{C^1, \ldots, C^n\}$ on N. We define the *aggregate contribution* of the charge system $\mathcal{C}$ on $\mathbf{a}_{|j}$, for each $j \in N$ and for each $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{E}_{N'}$, as the n-vector $\mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j})$ calculated via the following formula

$$\mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j}) = \sum_{k=1}^{j} C^k(\mathbf{a}_{|k}). \tag{4.6}$$

Definition 4.3.1 Let $\mathcal{C} = \{C^1, \ldots, C^n\}$ be a charge system on N. We call $\mathcal{C}$ a *conservative charge system* if for all $j \in N$ and for each pair $\mathbf{a}, \mathbf{b} \in \mathcal{E}_{N'}$, with $\Pi(\mathbf{a}_{|j}) = \Pi(\mathbf{b}_{|j})$ we have

$$\mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j}) = \mathrm{A}^{\mathcal{C}}(\mathbf{b}_{|j}). \tag{4.7}$$

The peculiarity of conservative charge systems is that they preserve the aggregate contribution from the network construction history, *i.e.* the aggregate contribution corresponding to $\mathbf{a}_{|j}$, for $\mathbf{a} \in \mathcal{E}_{N'}$ and $j \in N$, is only dependent on the partition of N' induced by the connected components in $< N', \{a_1, \ldots, a_j\} >$.

Example 4.3.1 It is easy to check that the charge system $\tilde{\mathcal{C}}$ of Example 4.2.2 is not conservative. Consider, for instance, $\mathrm{A}^{\tilde{\mathcal{C}}}(\mathbf{a}_{|3})$ and $\mathrm{A}^{\tilde{\mathcal{C}}}(\mathbf{b}_{|3})$ in Example 4.2.1. As we noted in Example 4.2.1, $\Pi(\mathbf{a}_{|3}) = \Pi(\mathbf{b}_{|3})$ but, from Table 4.1 in Example 4.2.6, we have that $\mathrm{A}^{\tilde{\mathcal{C}}}(\mathbf{a}_{|3}) = (1, \frac{3}{4}, \frac{3}{4}, \frac{1}{2})^t \neq (1, \frac{1}{2}, \frac{3}{4}, \frac{3}{4})^t = \mathrm{A}^{\tilde{\mathcal{C}}}(\mathbf{b}_{|3})$.

Now, consider the charge system $\hat{\mathcal{C}}$ introduced in Examples 4.2.3. For each $i, j \in N$ and each $\mathbf{a} \in \mathcal{E}_{N'}$ we have

$$\mathrm{A}_i^{\hat{\mathcal{C}}}(\mathbf{a}_{|j}) = \begin{cases} 1 - \frac{1}{|S(\Pi(\mathbf{a}_{|j}),\{i\})|} & \text{if } 0 \notin S(\Pi(\mathbf{a}_{|j}), \{i\}) \\ \\ 1 & \text{otherwise.} \end{cases}$$

Note that $\mathrm{A}_i^{\hat{\mathcal{C}}}(\mathbf{a}_{|j})$ is only dependent on the partition of N' induced by the connected components in $< N', \{a_1, \ldots, a_j\} >$, for each $i, j \in N$, *i.e.* $\hat{\mathcal{C}}$ is a conservative charge system.

Now, consider the charge system $\mathcal{C}^\tau$ introduced in Examples 4.2.4. For each $j \in \{1, \ldots, n\}$, each $i \in N$ and each $\mathbf{a} \in \mathcal{E}_{N'}$ we have

$$\mathrm{A}_i^{\mathcal{C}^\tau}(\mathbf{a}_{|j}) = \begin{cases} 0 & \text{if } \tau^{-1}(i) = \min\{\tau^{-1}(k) | k \in S(\Pi(\mathbf{a}_{|j}), \{i\}) \setminus \{0\}\}, \\ \\ 1 & \text{otherwise.} \end{cases}$$

Note that $\mathrm{A}_i^{\mathcal{C}^\tau}(\mathbf{a}_{|j})$ is only dependent on the partition of N' induced by the connected components in $< N', \{a_1, \ldots, a_j\} >$, for each $j \in \{1, \ldots, n\}$ and each $i \in N$, *i.e.* $\mathcal{C}^\tau$ is a conservative charge systems.

Differently, it is easy to check that the charge system $\check{\mathcal{C}}$ of Example 4.2.5 is not conservative. Consider, for instance, $\mathrm{A}^{\check{\mathcal{C}}}(\mathbf{a}_{|3})$ and $\mathrm{A}^{\check{\mathcal{C}}}(\mathbf{b}_{|3})$ in Example 4.2.1. As we noted in Example 4.2.1, $\Pi(a_{|3}) = \Pi(b_{|3})$ but, from Table 4.4 in Example 4.2.6, we have that $\mathrm{A}^{\check{\mathcal{C}}}(\mathbf{a}_{|3}) = (1, \frac{5}{6}, \frac{5}{6}, \frac{1}{3})^t \neq (1, \frac{1}{3}, \frac{5}{6}, \frac{5}{6})^t = \mathrm{A}^{\check{\mathcal{C}}}(\mathbf{b}_{|3})$.

■

Now, let $\mathcal{C}$ be a conservative charge system on N. We introduce the notion of *potential* with respect to $\mathcal{C}$, denoted by $P^{\mathcal{C}}$, which is a function on $2^{N'} \setminus \{\emptyset\}$ with values in $\mathbb{R}^N$.

Definition 4.3.2 Let $\mathcal{C} = \{C^1, \ldots, C^n\}$ be a conservative charge system on N. For each $S \in 2^{N'} \setminus \{\emptyset\}$, consider an element $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{E}_{N'}$ such that $\Pi(\mathbf{a}_{|j}) = \{S, \{i\}_{i \in N' \setminus S}\}$, with $j \in N$.

We define the *potential* of S with respect to the conservative charge system $\mathcal{C}$ as the unique[1] aggregate contribution corresponding to the partition $\{S, \{i\}_{i \in N' \setminus S}\}$; in formula

$$P^{\mathcal{C}}(S) := \mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j}).$$

[1]Let $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathcal{E}_{N'}$, and $S \in 2^{N'} \setminus \{\emptyset\}$ be such that $\Pi(\mathbf{a}_{|j}) = \Pi(\mathbf{b}_{|j}) = \{S, \{i\}_{i \in N' \setminus S}\}$, with $j \in N$. Recall that by Definition 4.3.1, we have $\mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j}) = \mathrm{A}^{\mathcal{C}}(\mathbf{b}_{|j})$. So, the aggregate contribution corresponding to $\{S, \{i\}_{i \in N' \setminus S}\}$ is unique.

The name of potential is inspired from physics where each conservative vector field has a potential. In a connection situation, an intuitive interpretation of the potential $P^{\mathcal{C}}(S)$, $S \in 2^{N'} \setminus \{\emptyset\}$, is as the level of "connection work" done by nodes in N when $\{S, \{i\}_{i \in N' \setminus S}\}$ is the current set of connected components and the conservative charge system $\mathcal{C}$ is used. Note that at the beginning of the connection process, when no edges are formed and all the connected components are singletons, the level of connection work performed by nodes should be zero. This motivates us to use the convention that $P_i^{\mathcal{C}}(\{j\}) = P_i^{\mathcal{C}}(\{0\}) = 0$ for all $i, j \in N$.

Other elementary properties of $P^{\mathcal{C}} : 2^{N'} \setminus \{\emptyset\} \to I\!R_+^N$ are collected in the following lemma, which will play a role in Section 4.7 to prove Theorem 4.7.2.

Lemma 4.3.1 *Let $\mathcal{C} = \{C^1, \ldots, C^n\}$ be a conservative charge system on N, let $P^{\mathcal{C}}$ be the potential w.r.t. $\mathcal{C}$ and let $S \in 2^{N'} \setminus \{\emptyset\}$. Then,*

(c.1) if $0 \in S$ then $P^{\mathcal{C}}(S) = e^{S \setminus \{0\}}$;

(c.2) $\sum_{i \in S \setminus \{0\}} P_i^{\mathcal{C}}(S) = \sum_{i \in N} P_i^{\mathcal{C}}(S) = |S| - 1$;

(c.3) if $S \subseteq T \subseteq N'$, then $P^{\mathcal{C}}(S) \leq P^{\mathcal{C}}(T)$.

[Here $e^{S \setminus \{0\}} \in I\!R_+^N$ is such that $e_i^{S \setminus \{0\}} = 1$ for each $i \in S \setminus \{0\}$ and $e_i^{S \setminus \{0\}} = 0$ for each $i \in N \setminus S$.]

Proof

(c.1) Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{E}_{N'}$ and $j \in N$ be such that $\Pi(\mathbf{a}_{|j}) = \{S, \{i\}_{i \in N' \setminus S}\}$. Then, for each $i \in N \cap S$

$$P_i^{\mathcal{C}}(S) = \mathrm{A}_i^{\mathcal{C}}(\mathbf{a}_{|j}) = \sum_{k=1}^{j} C_i^k(\mathbf{a}_{|k}) = 1 - \sum_{k=j+1}^{n} C_i^k(\mathbf{a}_{|k}) = 1,$$

where the third equality follows from the total aggregation property of $\mathcal{C}$ and the fourth equality follows from the connection property of $\mathcal{C}$. From the involvement property, we have $P_i^{\mathcal{C}}(S) = 0$ for each $i \in N \setminus S$, which finally proves property (c.1).

(c.2) If $0 \in S$ then property (c.2) follows directly from property (c.1).

Now, consider the case $0 \notin S$. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{E}_{N'}$ and $j \in N$ be such that $\Pi(\mathbf{a}_{|j}) = \{S, \{i\}_{i \in N' \setminus S}\}$. First, note that since $0 \notin S$, $j = |S| - 1$. Then,

$$\begin{aligned} &\textstyle\sum_{i \in S} P_i^{\mathcal{C}}(S) = \sum_{i \in S} \mathrm{A}_i^{\mathcal{C}}(\mathbf{a}_{|j}) = \sum_{i \in S} \sum_{k=1}^{j} C_i^k(\mathbf{a}_{|k}) \\ &= \textstyle\sum_{k=1}^{j} \sum_{i \in S} C_i^k(\mathbf{a}_{|k}) = \sum_{k=1}^{j} 1 = |S| - 1, \end{aligned}$$

where the fourth equality follows from the involvement property. By the involvement property it follows too that $P_i^{\mathcal{C}}(S) = 0$ for each $i \in N \setminus S$, which finally proves property (c.2).

(c.3) Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{E}_{N'}$ and $j, l \in N$ with $l \geq j$ be such that $\Pi(\mathbf{a}_{|j}) = \{S, \{i\}_{i \in N' \setminus S}\}$ and $\Pi(\mathbf{a}_{|l}) = \{T, \{i\}_{i \in N' \setminus T}\}$. Then,

$$\begin{aligned} &P^{\mathcal{C}}(S) = \mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j}) = \textstyle\sum_{k=1}^{j} C^k(\mathbf{a}_{|k}) \\ &\leq \textstyle\sum_{k=1}^{j} C^k(\mathbf{a}_{|k}) + \sum_{k=j+1}^{l} C^k(\mathbf{a}_{|k}) \\ &= \textstyle\sum_{k=1}^{l} C^k(\mathbf{a}_{|k}) = \mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|l}) = P^{\mathcal{C}}(T), \end{aligned}$$

which concludes the proof of property (c.3).

■

Proposition 4.3.1 *Let $\mathcal{C} = \{C^1, \ldots, C^n\}$ be a conservative charge system on N. Let $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathcal{E}_{N'}$ and $j \in N$ be such that $\Pi(\boldsymbol{a}_{|j}) = \{S_1, S_2, \ldots, S_m\}$, with $S_1, S_2, \ldots, S_m \subset N'$ and $m \leq n$. Then,*

$$\mathrm{A}^{\mathcal{C}}(\boldsymbol{a}_{|j}) = \sum_{r=1}^{m} P^{\mathcal{C}}(S_r).$$

Proof Let $r \in \{1, 2, \ldots, m\}$. Determine $b^r(1), \ldots, b^r(p^r) \in \{1, \ldots, j\}$ such that $\Pi(a_{b^r(1)}, a_{b^r(2)}, \ldots, a_{b^r(p^r)}) = \{S_r, \{i\}_{i \in N' \setminus S_r}\}$, where $p^r = |S_r| - 1$.

Then, for each $i \in N \setminus S_r$, by the involvement property of $\mathcal{C}$

$$P_i^{\mathcal{C}}(S_r) = \mathrm{A}_i^{\mathcal{C}}(a_{b^r(1)}, a_{b^r(2)}, \ldots, a_{b^r(p^r)}) = 0,$$

whereas for each $i \in N \cap S_r$

$$\begin{aligned} &P_i^{\mathcal{C}}(S_r) = \mathrm{A}_i^{\mathcal{C}}(a_{b^r(1)}, a_{b^r(2)}, \ldots, a_{b^r(p^r)}) \\ &= \mathrm{A}_i^{\mathcal{C}}(a_{b^r(1)}, \ldots, a_{b^r(p^r)}, (a_s)_{s \in \{1, \ldots, j\} \setminus \{b^r(1), \ldots, b^r(p^r)\}}) \\ &= \mathrm{A}_i^{\mathcal{C}}(a_1, a_2, \ldots, a_j) = \mathrm{A}_i^{\mathcal{C}}(\mathbf{a}_{|j}), \end{aligned}$$

where the second equality follows from the involvement property in the edge sequence $(a_{b^r(1)}, a_{b^r(2)}, \ldots, a_{b^r(j)})$, and the third equality follows from the fact that $\mathcal{C}$ is conservative. Consequently, $\sum_{r=1}^{m} P^{\mathcal{C}}(S_r) = \mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j})$. ∎

4.4 Construct & Charge rules

At this point, we have all the ingredients to introduce the definition of a Construct & Charge rule.

Definition 4.4.1 Let $\mathcal{C} = \{C^1, \ldots, C^n\}$ be a charge system on N. Let $\sigma \in \Sigma_{E_{N'}}$ and let K^σ be the Kruskal cone w.r.t. σ. The *Construct and Charge (CC-)rule* w.r.t. $\mathcal{C}$ and σ is the map $\chi^{\mathcal{C},\sigma} : K^\sigma \to I\!R^N$ given by

$$\chi^{\mathcal{C},\sigma}(w) = \sum_{r=1}^{n} w(\sigma(\rho^\sigma(r)))C^r(\sigma(\rho^\sigma(1)), \ldots, \sigma(\rho^\sigma(r))) \tag{4.8}$$

for each mcst situation w in the cone K^σ.

Remark 4.4.1 *Note that the CC-rule $\chi^{\tilde{\mathcal{C}},\sigma}$, where $\tilde{\mathcal{C}}$ is the charge system of Example 4.2.2, corresponds to the Proportional rule introduced in Feltkamp et al. (1994a).*

The following example illustrates the CC-rules corresponding to the charge systems introduced in Examples 4.2.2 - 4.2.5.

Example 4.4.1 Consider the mcst situation $< N', w >$ of Example 2.1.1. Let σ be as in Example 2.1.3 and $\sigma'(1) = \{1,2\}$, $\sigma'(2) = \{1,3\}$, $\sigma'(3) = \{2,3\}$, $\sigma'(4) = \{0,1\}$, $\sigma'(5) = \{0,2\}$, $\sigma'(6) = \{0,3\}$. Now, we apply Definition 4.4.1 to the charge systems introduced in Examples 4.2.2 - 4.2.5 to calculate the allocations provided by the corresponding CC-rules on $< N', w >$.
The charge system $\tilde{\mathcal{C}}$ of Example 4.2.2 leads to

$$\begin{aligned} &\chi^{\tilde{\mathcal{C}},\sigma}(w) \\ &= 12 * (\tfrac{1}{2}, 0, \tfrac{1}{2})^t + 12 * (\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4})^t + 24 * (\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4})^t \\ &= (15, 18, 15)^t, \end{aligned}$$

and

$$\begin{aligned}
&\chi^{\tilde{\mathcal{C}},\sigma'}(w)\\
&= 12 * (\tfrac{1}{2}, \tfrac{1}{2}, 0)^t + 12 * (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2})^t + 24 * (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2})^t\\
&= (15, 15, 18)^t.
\end{aligned}$$

Note that $\chi^{\tilde{\mathcal{C}},\sigma}(w) \neq \chi^{\tilde{\mathcal{C}},\sigma'}(w)$.

The charge system $\hat{\mathcal{C}}$ of Example 4.2.3 leads to

$$\begin{aligned}
&\chi^{\hat{\mathcal{C}},\sigma}(w)\\
&= 12 * (\tfrac{1}{2}, 0, \tfrac{1}{2})^t + 12 * (\tfrac{1}{6}, \tfrac{2}{3}, \tfrac{1}{6})^t + 24 * (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3})^t\\
&= (16, 16, 16)^t,
\end{aligned}$$

and

$$\begin{aligned}
&\chi^{\hat{\mathcal{C}},\sigma'}(w)\\
&= 12 * (\tfrac{1}{2}, \tfrac{1}{2}, 0)^t + 12 * (\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{2}{3})^t + 24 * (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3})^t\\
&= (16, 16, 16)^t.
\end{aligned}$$

Note that $\chi^{\hat{\mathcal{C}},\sigma}(w) = \chi^{\hat{\mathcal{C}},\sigma'}(w)$.

The charge system $\mathcal{C}^\tau$ of Example 4.2.4, with $\tau(i) = i$ for each $i \in \{1, 2, 3\}$, leads to

$$\begin{aligned}
&\chi^{\mathcal{C}^\tau,\sigma}(w)\\
&= 12 * (0, 0, 1)^t + 12 * (0, 1, 0)^t + 24 * (1, 0, 0)^t\\
&= (12, 12, 24)^t,
\end{aligned}$$

and

$$\begin{aligned}
&\chi^{\mathcal{C}^\tau,\sigma'}(w)\\
&= 12 * (0, 1, 0)^t + 12 * (0, 0, 1)^t + 24 * (1, 0, 0)^t\\
&= (12, 12, 24)^t.
\end{aligned}$$

Note that $\chi^{\mathcal{C}^\tau,\sigma}(w) = \chi^{\mathcal{C}^\tau,\sigma'}(w)$.

The charge system $\check{\mathcal{C}}$ of Example 4.2.5 leads to

$$\begin{aligned}
&\chi^{\check{\mathcal{C}},\sigma}(w)\\
&= 12 * (\tfrac{1}{2}, 0, \tfrac{1}{2})^t + 12 * (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3})^t + 24 * (\tfrac{1}{6}, \tfrac{4}{6}, \tfrac{1}{6})^t\\
&= (14, 20, 14)^t,
\end{aligned}$$

and

$$\begin{aligned}
&\chi^{\check{\mathcal{C}},\sigma'}(w)\\
&= 12 * (\tfrac{1}{2}, \tfrac{1}{2}, 0)^t + 12 * (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3})^t + 24 * (\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{4}{6})^t\\
&= (14, 14, 20)^t.
\end{aligned}$$

Note that $\chi^{\check{\mathcal{C}},\sigma}(w) \neq \chi^{\check{\mathcal{C}},\sigma'}(w)$.

■

Definition 4.4.2 Let $\mathcal{C} = \{C^1, \dots, C^n\}$ be a charge system on N. We say that $\mathcal{C}$ has the *patch property* if for all $\sigma_1, \sigma_2 \in \Sigma_{E_{N'}}$:

$$\chi^{\mathcal{C},\sigma_1}(w) = \chi^{\mathcal{C},\sigma_2}(w)$$

for each w in the cone $K^{\sigma_1} \cap K^{\sigma_2}$.

If $\mathcal{C} = \{C^1, \dots, C^n\}$ has the patch property, then we can define $\chi^{\mathcal{C}}$ by

$$\chi^{\mathcal{C}}(w) = \chi^{\mathcal{C},\sigma}(w) \tag{4.9}$$

for all $w \in \mathcal{W}^{N'}$, where $\sigma \in \Sigma_{E_{N'}}$ is such that $w \in K^\sigma$. We will call $\chi^{\mathcal{C}}$ the CC-rule with respect to $\mathcal{C}$.

Remark 4.4.2 *Example 4.4.1 shows that the charge system $\tilde{\mathcal{C}}$ introduced in Example 4.2.2 and the charge system $\check{\mathcal{C}}$ introduced in Example 4.2.5 do not satisfy the patch property, so we cannot define $\chi^{\tilde{\mathcal{C}}}$ and $\chi^{\check{\mathcal{C}}}$ via relation (4.9).*

Theorem 4.4.1 *Let $\mathcal{C} = \{C^1, \dots, C^n\}$ be a charge system on N. If $\mathcal{C}$ has the patch property, then $\mathcal{C}$ is conservative.*

Proof Suppose that $\mathcal{C}$ has the patch property and it is not conservative. Then, we can find a $j \in N$ and a pair $\mathbf{a} = (a_1, \dots, a_n), \mathbf{b} = (b_1, \dots, b_n) \in \mathcal{E}_{N'}$, with $\Pi(\mathbf{a}_{|j}) = \Pi(\mathbf{b}_{|j})$ and $\mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j}) \neq \mathrm{A}^{\mathcal{C}}(\mathbf{b}_{|j})$.

Suppose $\Pi(\mathbf{a}_{|j}) = \{S_1, S_2, \dots, S_m\}$ and take $w \in \mathcal{W}^{N'}$ such that

$$w(\{i,j\}) = \begin{cases} 0 & \text{if there exists } r \in \{1, \dots, m\} \text{ s.t. } i, j \in S_r, \\ \\ 1 & \text{otherwise,} \end{cases}$$

for each $\{i,j\} \in E_{N'}$. Let $\sigma_1 \in \Sigma_{E_{N'}}$ be such that $\sigma_1(\rho^{\sigma_1}(k)) = a_k$ for each $k \in \{1, \dots, j\}$ and $\sigma_1(\rho^{\sigma_1}(l)) = d_l$ for each $l \in \{j+1, \dots, n\}$, where $(a_1, \dots, a_j, d_{j+1}, \dots, d_n) \in \mathcal{E}_{N'}$ and ρ^{σ_1} is defined as in Section 2.1.2.

Let $\sigma_2 \in \Sigma_{E_{N'}}$ be such that $\sigma_2(\rho^{\sigma_2}(k)) = b_k$ for each $k \in \{1, \dots, j\}$ and $\sigma_2(\rho^{\sigma_2}(l)) = d_l$ for each $l \in \{j+1, \dots, n\}$, with $(b_1, \dots, b_j, d_{j+1}, \dots, d_n) \in \mathcal{E}_{N'}$.

In addition, σ_1 and σ_2 can be chosen such that $w \in K^{\sigma_1} \cap K^{\sigma_2}$. We have

$$\begin{aligned}
&\chi^{\mathcal{C},\sigma_1}(w) \\
&= \textstyle\sum_{r=1}^{j} w(a_r)C^r(\mathbf{a}_{|r}) + \sum_{r=j+1}^{n} w(d_r)C^r(a_1,\ldots,a_j,d_{j+1},\ldots,d_r) \\
&= \textstyle\sum_{r=j+1}^{n} C^r(a_1,\ldots,a_j,d_{j+1},\ldots,d_r) \\
&= e^N - \textstyle\sum_{r=1}^{j} C^r(\mathbf{a}_{|r}) \\
&= e^N - \mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j}),
\end{aligned}$$

where the third equality follows from the total aggregation property.

Similarly,

$$\chi^{\mathcal{C},\sigma_2}(w) = e^N - \sum_{r=1}^{j} C^r(\mathbf{b}_{|r}) = e^N - \mathrm{A}^{\mathcal{C}}(\mathbf{b}_{|j}).$$

So, $\chi^{\mathcal{C},\sigma_1}(w) \neq \chi^{\mathcal{C},\sigma_2}(w)$, which yields a contradiction with the fact that $\mathcal{C}$ has the patch property. ■

4.5 Obligation rules

A different approach to define allocation protocols is rooted on the concept of obligations maps (Tijs *et al.* 2006) instead of the concept of charge systems. Surprisingly, the two approaches are strongly connected, as it will be shown in the next section.

Let $\Delta(N) = \{x \in \mathbb{R}_+^N | \sum_{i\in N} x_i = 1\}$. The sub-simplex $\Delta(S)$ of $\Delta(N)$ given by $\Delta(S) = \{x \in \Delta(N) | \sum_{i\in S} x_i = 1\}$ is called, for reasons to be clarified later, the set of *obligation vectors* of S.

An *obligation function* is a map $o : 2^N \setminus \{\emptyset\} \to \Delta(N)$ assigning to each $S \in 2^N \setminus \{\emptyset\}$ an obligation vector

$$o(S) \in \Delta(S) \tag{4.10}$$

in such a way that for each $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subset T$ and for each $i \in S$

$$o_i(S) \geq o_i(T). \tag{4.11}$$

Such an obligation function o on $2^N \setminus \{\emptyset\}$ induces an *obligation map* $\hat{o} : \Theta(N') \to \mathbb{R}^N$, where $\Theta(N')$ is the family of partitions of N', and

$$\hat{o}(\theta) = \sum_{S \in \theta, 0 \notin S} o(S) \tag{4.12}$$

for each $\theta \in \Theta(N')$.

Note that if $\theta = \{N'\}$, then the resulting empty sum is assumed, by definition, to be the n-vector of zeroes: $\hat{o}(\theta) = 0 \in \mathbb{R}^N$.

Obligation maps are basic ingredients for Obligation rules: specifically, they play a central role in defining the cost allocation protocol along the step-wise connection procedure. Three interesting types of obligation maps are provided in examples 4.5.1-4.5.3.

Example 4.5.1 Let $o^* : 2^N \setminus \{\emptyset\} \to \Delta(N)$ be defined by $o^*(S) = \frac{e^S}{|S|}$ for each $S \in 2^N \setminus \{\emptyset\}$, where e^S is the n-vector such that $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \in N \setminus S$. Then, o^* is an obligation function and the corresponding obligation map is

$$\hat{o}_i^*(\theta) = \begin{cases} |S(\theta, \{i\})|^{-1} & \text{if } 0 \notin S(\theta, \{i\}) \\ \\ 0 & \text{otherwise,} \end{cases} \tag{4.13}$$

for each $\theta \in \Theta(N')$ and each $i \in N$. Here $S(\theta, \{i\}) \in \theta$ is the partition element to which i belongs.

Note that $o^*(S)$ is the barycenter of $\Delta(S)$ and for $N = \{1, 2, 3, 4\}$, $\theta = \{\{1, 2\}, \{0, 3\}, \{4\}\}$ we have $\hat{o}^*(\theta) = (\frac{1}{2}, \frac{1}{2}, 0, 1)$.

■

Example 4.5.2 Given $\tau \in \Sigma_N$, let o^τ on $2^N \setminus \{\emptyset\}$ be the obligation function such that for each $S \in 2^N \setminus \{\emptyset\}$ and $i \in N$

$$o_i^\tau(S) = \begin{cases} 1 & \text{if } \tau^{-1}(i) = \min\{\tau^{-1}(k) | k \in S\} \\ \\ 0 & \text{otherwise.} \end{cases}$$

If $N = \{1, 2, 3, 4\}$, $\theta = \{\{1, 2\}, \{0, 3\}, \{4\}\}$ and $\tau^{-1}(i) = i$ for each $i \in N$, then $\hat{o}^\tau(\theta) = o^\tau(\{1, 2\}) + o^\tau(\{4\}) = (1, 0, 0, 1)$.

■

Example 4.5.3 Let $\nu \in \mathbb{R}^N_{++}$ be a vector of strictly positive real values. Let $o^\nu : 2^N \setminus \{\emptyset\} \to \Delta(N)$ be defined by

$$o_i^\nu(S) = \begin{cases} \frac{\nu_i}{\sum_{j\in S}\nu_j} & \text{if } i \in S \\ \\ 0 & \text{otherwise.} \end{cases}$$

Then, o^ν is an obligation function. Note that if $\nu_i = 1$ for each $i \in N$, then $o_i^\nu(S) = o_i^*(S)$ for each $S \in 2^N \setminus \{\emptyset\}$, where $o^*(S)$ is as in Example 4.5.1.

■

One can easily check that the obligation maps in Examples 4.5.1-4.5.3 satisfy condition (4.11). Next example shows a map $o : 2^N \setminus \{\emptyset\} \to \Delta(N)$ assigning to each $S \in 2^N \setminus \{\emptyset\}$ an obligation vector $o(S) \in \Delta(S)$ in such a way that condition (4.11) is not satisfied implying that it is not an obligation function.

Example 4.5.4 Let $U \subset N$ and let $o^U : 2^N \setminus \{\emptyset\} \to \Delta(N)$ be such that

$$o_i^U(S) = \begin{cases} |S|^{-1} & \text{if } i \in S \text{ and } U \nsubseteq S \\ |U|^{-1} & \text{if } i \in U \text{ and } U \subseteq S \\ 0 & \text{otherwise.} \end{cases}$$

Then, if $N = \{1,2,3,4\}$ and $U = \{2,3\}$, $o^U(\{1,2,4\}) = (\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3})$ and $o^U(\{1,2,3,4\}) = (0, \frac{1}{2}, \frac{1}{2}, 0)$. So, o^U is not an obligation function since it does not satisfy condition (4.11).

■

Remark 4.5.1 *Let $o^\bullet, o^\circ : 2^N \setminus \{\emptyset\} \to \Delta(N)$ be two distinct obligation functions. For each $\alpha \in [0,1]$ let $o^\alpha : 2^N \setminus \{\emptyset\} \to \mathbb{R}^N$ be defined by $o^\alpha(S) = \alpha o^\bullet(S) + (1-\alpha)o^\circ(S)$ for each $S \in 2^N \setminus \{\emptyset\}$. Then, $\sum_{i\in S} o_i^\alpha(S) = \sum_{i\in S}\big(\alpha o_i^\bullet(S) + (1-\alpha)o_i^\circ(S)\big) = 1$. Moreover, since condition (4.11) holds both for $o^\bullet$ and o°, condition (4.11) holds for their convex combination o^α too. Therefore, o^α is an obligation function which induces the corresponding obligation map $\hat{o}^\alpha(\theta) = \alpha\hat{o}^\bullet(\theta) + (1-\alpha)\hat{o}^\circ(\theta)$ for each $\theta \in \Theta(N')$.*

Definition 4.5.1 Let $\hat{o}$ be an obligation map on $\Theta(N')$. Let $\sigma \in \Sigma_{E_{N'}}$. The *contribution matrix* w.r.t $\hat{o}$ and σ is the matrix $D^{\sigma,\hat{o}} \in I\!R^{N \times |E_{N'}|}$ where

$$D^{\sigma,\hat{o}}_{ik} = \hat{o}_i(\pi^{\sigma,k-1}) - \hat{o}_i(\pi^{\sigma,k})$$

for each $i \in N$ and each $k \in \{1, \ldots, |E_{N'}|\}$.

Some characteristics of the contribution matrix are given in the following proposition.

Proposition 4.5.1 *Let $\hat{o}$ be an obligation map on $\Theta(N')$. Let $\sigma \in \Sigma_{E_{N'}}$. Then, $D^{\sigma,\hat{o}}$ is a non-negative matrix for which each row sum is equal to 1 and the $\rho^\sigma(j)$-th column sum is equal to 1 for each $j \in \{1, \ldots, n\}$, whereas each k-th column sum with $k \in \{1, \ldots, |E_{N'}|\} \setminus \{\rho^\sigma(j) | j \in \{1, \ldots, n\}\}$ is equal to 0.*

Proof First, note that by Remark 2.1.1 and the definition of obligation map the matrix $D^{\sigma,\hat{o}}$ is non-negative.

The sum of the elements in each row $i \in N$ is equal to 1 because

$$\sum_{k=1}^{|E_{N'}|} \big(\hat{o}_i(\pi^{\sigma,k-1}) - \hat{o}_i(\pi^{\sigma,k})\big) = \hat{o}_i(\pi^{\sigma,0}) - \hat{o}_i(\pi^{\sigma,|E_{N'}|}) = 1 - 0 = 1$$

for each $i \in N$.

The $\rho^\sigma(j)$-th column sums, for each $j \in \{1, \ldots, n\}$, are equal to 1 because

$$\begin{aligned}&\textstyle\sum_{i \in N} D^{\sigma,\hat{o}}_{i\rho^\sigma(j)} = \sum_{i \in N} \big(\hat{o}_i(\pi^{\sigma,\rho^\sigma(j)-1}) - \hat{o}_i(\pi^{\sigma,\rho^\sigma(j)})\big) = \\ &= \textstyle\sum_{i \in N} \hat{o}_i(\pi^{\sigma,\rho^\sigma(j)-1}) - \sum_{i \in N} \hat{o}_i(\pi^{\sigma,\rho^\sigma(j)}) = \\ &= \big(|\pi^{\sigma,\rho^\sigma(j)-1}| - 1\big) - \big(|\pi^{\sigma,\rho^\sigma(j)}| - 1\big) = 1\end{aligned}$$

for each $j \in \{1, \ldots, n\}$, where in the last equality we use Remark 2.1.1. The k-th column sums, for each $k \in \{1, \ldots, |E_{N'}|\} \setminus \{\rho^{\sigma(j)} | j \in \{1, \ldots, n\}\}$, are equal to 0 because $\pi^{\sigma,k-1} = \pi^{\sigma,k}$ and then

$$\textstyle\sum_{i \in N} D^{\sigma,\hat{o}}_{ik} = \sum_{i \in N} \big(\hat{o}_i(\pi^{\sigma,k-1}) - \hat{o}_i(\pi^{\sigma,k})\big) = 0.$$

$\blacksquare$

Definition 4.5.2 Let $\hat{o}$ be an obligation map on $\Theta(N')$. Let $\sigma \in \Sigma_{E_{N'}}$. We define the map $\phi^{\sigma,\hat{o}} : K^\sigma \to I\!R^N$ by

$$\phi^{\sigma,\hat{o}}(w) = D^{\sigma,\hat{o}} w^\sigma, \tag{4.14}$$

for each mcst situation w in the cone K^σ.
[Recall that w^σ is defined in Section 2.1.2 as the column vector $\big(w(\sigma(1)), w(\sigma(2)), \ldots, w(\sigma(|E_{N'}|))\big)^t$.]

Onwards, let $e^k \in I\!R^{|E_{N'}|}$ be the column vector such that $e^k_t = 1$ if $t = k$ and $e^k_t = 0$ for each $t \in \{1, \ldots, |E_{N'}|\} \setminus \{k\}$. From Proposition 4.5.1 it follows directly that the matrix $\bar{D}^{\sigma,\hat{o}} \in I\!R^{N \times n}$ defined by

$$\bar{D}^{\sigma,\hat{o}} e^j = D^{\sigma,\hat{o}} e^{\rho^\sigma(j)} \tag{4.15}$$

for each $j \in \{1, \ldots, n\}$ is a double stochastic matrix (*i.e.* all entries are non-negative and each row sum and each column sum is equal to 1), and

$$\phi^{\sigma,\hat{o}}(w) = \bar{D}^{\sigma,\hat{o}} \big(w(\sigma(\rho^\sigma(1))), \ldots, w(\sigma(\rho^\sigma(n)))\big)^t. \tag{4.16}$$

Remark 4.5.2 *Let $\sigma \in \Sigma_{E_{N'}}$. Note that for each mcst situation w in the cone K^σ and for each $j \in \{1, \ldots, n\}$ we have*

$$\bar{D}^{\sigma,\hat{o}^*} e^j = \hat{C}^j,$$

where $\hat{o}^*$ is the obligation map defined in Example 4.5.1; $\bar{D}^{\sigma,\hat{o}^*}$ is defined by relation (4.15) on the contribution matrix w.r.t. $\hat{o}^*$ and σ; $\bar{D}^{\sigma,\hat{o}^*} e^j$ is the j-th column of the double stochastic matrix $\bar{D}^{\sigma,\hat{o}^*}$; $\hat{C}^j$ is the j-th element of the charge system $\hat{\mathcal{C}}$ defined in Example 4.2.3. Consequently, for each $w \in K^\sigma$ we have

$$\phi^{\sigma,\hat{o}^*}(w) = \chi^{\hat{\mathcal{C}},\sigma}(w),$$

where $\phi^{\sigma,\hat{o}^*}$ is the map defined by relations (4.16) w.r.t. σ and $\hat{o}^*$, and $\chi^{\hat{\mathcal{C}},\sigma}$ is the Construct & Charge rule w.r.t. $\hat{\mathcal{C}}$ and σ.

Remark 4.5.3 *Let $\sigma \in \Sigma_{E_{N'}}$. Note that for each mcst situation w in the cone K^σ and for each $j \in \{1, \ldots, n\}$ we have*

$$\bar{D}^{\sigma,\hat{o}^\tau} e^j = C^{\tau,j},$$

where $\hat{o}^\tau$ is the obligation map defined in Example 4.5.2; $\bar{D}^{\sigma,\hat{o}^\tau}$ is defined by relation (4.15) on the contribution matrix w.r.t. $\hat{o}^\tau$ and σ; $\bar{D}^{\sigma,\hat{o}^\tau}e^j$ is the j-th column of the double stochastic matrix $\bar{D}^{\sigma,\hat{o}^\tau}$; $C^{\tau,j}$ is the j-th element of the charge system $\mathcal{C}^\tau$ defined in Example 4.2.4. Consequently, for each $w \in K^\sigma$ we have

$$\phi^{\sigma,\hat{o}^\tau}(w) = \chi^{\mathcal{C}^\tau,\sigma}(w),$$

where $\phi^{\sigma,\hat{o}^\tau}$ is the map defined by relations (4.16) w.r.t. σ and $\hat{o}^\tau$, and $\chi^{\mathcal{C}^\tau,\sigma}$ is the Construct & Charge rule w.r.t. $\mathcal{C}^\tau$ and σ.

In order to define Obligation rules properly on the set $\mathcal{W}^{N'}$, we need Lemma 4.5.1. In the sequel, recall that, for each $t \in \{1,\ldots,|E_{N'}|\}$, w^σ_t is the t-th coordinate of the vector w^σ.

Lemma 4.5.1 *Let $\hat{o}$ be an obligation map on $\Theta(N')$; let $\sigma \in \Sigma_{E_{N'}}, w \in K^\sigma$. Suppose that, for some $t \in \{1,\ldots,\ |E_{N'}|-1\}$, $w^\sigma_t = w^\sigma_{t+1}$. Then, for the ordering $\sigma' \in \Sigma_{E_{N'}}$ such that $\sigma'(i) = \sigma(i)$ for each $i \in \{1,\ldots,|E_{N'}|\} \setminus \{t,t+1\}$, $\sigma'(t) = \sigma(t+1)$ and $\sigma'(t+1) = \sigma(t)$, we have that $w \in K^{\sigma'}$ and $\phi^{\sigma,\hat{o}}(w) = \phi^{\sigma',\hat{o}}(w)$.*

Proof It is obvious that $w \in K^{\sigma'}$. Let $a = w^\sigma_t$. Note that $\hat{o}(\pi^{\sigma,k}) = \hat{o}(\pi^{\sigma',k})$ for all $k \in \{1,\ldots,|E_{N'}|\}$ with $k \neq t$. This implies that $w^\sigma_k\ D^{\sigma,\hat{o}}e^k = w^{\sigma'}_k\ D^{\sigma',\hat{o}}e^k$ for all $k \in \{1,\ldots,|E_{N'}|\}$ with $k \notin \{t,t+1\}$ and

$$\begin{aligned}
&w^{\sigma'}_t D^{\sigma',\hat{o}}e^t + w^{\sigma'}_{t+1}D^{\sigma',\hat{o}}e^{t+1} = \\
&= a(\hat{o}(\pi^{\sigma',t-1}) - \hat{o}(\pi^{\sigma',t})) + a(\hat{o}(\pi^{\sigma',t}) - \hat{o}(\pi^{\sigma',t+1})) = \\
&= a(\hat{o}(\pi^{\sigma',t-1}) - \hat{o}(\pi^{\sigma',t+1})) = a(\hat{o}(\pi^{\sigma,t-1}) - \hat{o}(\pi^{\sigma,t+1})) = \\
&= a(\hat{o}(\pi^{\sigma,t-1}) - \hat{o}(\pi^{\sigma,t})) + a(\hat{o}(\pi^{\sigma,t}) - \hat{o}(\pi^{\sigma,t+1})) = \\
&= w^\sigma_t D^{\sigma,\hat{o}}e^t + w^\sigma_{t+1}D^{\sigma,\hat{o}}e^{t+1}.
\end{aligned} \tag{4.17}$$

So, $D^{\sigma,\hat{o}}w^\sigma = D^{\sigma',\hat{o}}w^{\sigma'}$ or, equivalently, $\phi^{\sigma,\hat{o}}(w) = \phi^{\sigma',\hat{o}}(w)$. ∎

By repeatedly using Lemma 4.5.1 we obtain

Proposition 4.5.2 *Let $\hat{o}$ be an obligation map on $\Theta(N')$. If $w \in K^\sigma \cap K^{\sigma'}$ with $\sigma,\sigma' \in \Sigma_{E_{N'}}$, then $\phi^{\sigma,\hat{o}}(w) = \phi^{\sigma',\hat{o}}(w)$.*

This proposition makes it possible to define an Obligation rule with respect to an obligation map on $\Theta(N')$ as a map on $\mathcal{W}^{N'}$.

Definition 4.5.3 Let $\hat{o}$ be an obligation map on $\Theta(N')$. The *Obligation* (O-)*rule* w.r.t. $\hat{o}$ is the map $\phi^{\hat{o}} : \mathcal{W}^{N'} \to I\!R^N$ defined by

$$\phi^{\hat{o}}(w) = \phi^{\sigma,\hat{o}}(w) \tag{4.18}$$

for each $w \in \mathcal{W}^{N'}$, where $\sigma \in \Sigma_{E_{N'}}$ is such that $w \in K^\sigma$.

Remark 4.5.4 *The P-value (Feltkamp et al. (1994b), Branzei et al. (2004)), that will be studied in depth in Section 4.6, is both an Obligation rule and a Construct & Charge rule. In fact, by definition, $P(w) := \phi^{\hat{o}^*}(w)$ for each $w \in \mathcal{W}^{N'}$, and by Remark 4.5.2 $\phi^{\hat{o}^*}(w) = \phi^{\sigma,\hat{o}^*}(w) = \chi^{\hat{\mathcal{C}},\sigma}(w)$ for each $w \in \mathcal{W}^{N'}$, where $\sigma \in \Sigma_{E_{N'}}$ is such that $w \in K^\sigma$, and $\chi^{\hat{\mathcal{C}},\sigma}$ is the Construct and Charge rule w.r.t. $\hat{\mathcal{C}}$ and σ.*

In an analogous way, the P^τ-values (Norde et al. (2004), Branzei et al. (2004)), with $\tau \in \Sigma_N$, introduced in Section 3.3, are also both Obligation rules and Construct & Charge rules. In fact, by Remark 3.3.1 and relation (3.4), $P^\tau(w) = \phi^{\hat{o}^\tau}(w)$ for each $w \in \mathcal{W}^{N'}$, and by Remark 4.5.3 $\phi^{\hat{o}^\tau}(w) = \phi^{\sigma,\hat{o}^\tau}(w) = \chi^{\mathcal{C}^\tau,\sigma}(w)$ for each $w \in \mathcal{W}^{N'}$, where $\sigma \in \Sigma_{E_{N'}}$ is such that $w \in K^\sigma$ and $\chi^{\mathcal{C}^\tau,\sigma}$ is the Construct and Charge rule w.r.t. $\mathcal{C}^\tau$ and σ.

Next two examples provide an illustration of two obligation rules.

Example 4.5.5 Consider the mcst situation $< N', w >$ with $N' = \{0, 1, 2, 3\}$ and w of Example 2.1.1. The contribution matrix $D^{\sigma,\hat{o}^*}$ is

$$D^{\sigma,\hat{o}^*} = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{3} & 0 & 0 \end{pmatrix}$$

and $w^\sigma = (12, 12, 20, 24, 24, 26)^t$.
Then, $P(w) = \phi^{\hat{o}^*}(w) = D^{\sigma,\hat{o}^*} w^\sigma = (16, 16, 16)^t$.

■

Example 4.5.6 Consider the mcst situation $< N', w >$ with $N' = \{0, 1, 2, 3\}$ and w of Example 2.1.1. Let $\tau \in \Sigma_N$ be such that $\tau^{-1}(1) = 2$, $\tau^{-1}(2) = 3$ and $\tau^{-1}(3) = 1$. The contribution matrix $D^{\sigma,\hat{o}^\tau}$ is

$$D^{\sigma,\hat{o}^\tau} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

and $w^\sigma = (12, 12, 20, 24, 24, 26)^t$.
Then, $P^\tau(w) = \phi^{\hat{o}^\tau}(w) = D^{\sigma,\hat{o}^\tau} w^\sigma = (12, 12, 24)^t$.

■

Now, we want to make clear that we have chosen the name "Obligation rule" because such rules deal with "(remaining) obligations" of players for the cost of edges along a step-wise connection procedure. Let $\hat{o}$ be an obligation map on $\Theta(N')$ and let $w \in \mathcal{W}^{N'}$. According to the corresponding Obligation rule $\phi^{\hat{o}}$, each player $i \in N$ is committed in paying, according to some specific protocol, fractions of edges summing up to 1 along a step-wise process. More precisely, an Obligation rule allocates the cost of an edge which forms in each step k, $k \in \{1, \ldots, |E_{N'}|\}$, of the Kruskal's algorithm to some players in N according to the k-th column of the contribution matrix $D^{\sigma,\hat{o}}$, with $\sigma \in \Sigma_{E_{N'}}$ such that $w \in K^\sigma$. By Proposition 4.5.1, after each step k, the total quantity of fractions of edges that each player $i \in N$ still has to pay is given by

$$1 - \sum_{j=1}^{k} D_{ij}^{\sigma,\hat{o}} = \hat{o}_i(\pi^{\sigma,k}). \tag{4.19}$$

Since equation (4.19) defines the sum of remaining fractions of edges that players are obliged to pay after step k up to the end of the connection procedure, we call $\hat{o}_i(\pi^{\sigma,k})$ the (remaining) *obligation* for player i at step k.

We collect some interesting properties of Obligation rules in Proposition 4.5.3.

Proposition 4.5.3 *The Obligation rules are efficient, satisfy the carrier property and form a convex set .*

Proof Let $\hat{o}$ be an obligation map on $\Theta(N')$, let $w \in \mathcal{W}^{N'}$ and let $\sigma \in \Sigma_{E_{N'}}$

be such that $w \in K^\sigma$.

i) From (4.15) and (4.18) it follows

$$\phi_i^{\hat{o}}(w) = \sum_{k=1}^{n} \bar{D}_{ik}^{\sigma,\hat{o}} w(\sigma(\rho^\sigma(k))),$$

for each $i \in N$, implying that

$$\sum_{i \in N} \phi_i^{\hat{o}}(w) = \sum_{k=1}^{n} w(\sigma(\rho^\sigma(k))) \sum_{i \in N} \bar{D}_{ik}^{\sigma,\hat{o}} = \sum_{k=1}^{n} w(\sigma(\rho^\sigma(k))) = w(\Gamma),$$

where the second equality follows from Proposition 4.5.1 and where Γ is a spanning network on N' of minimal cost. So, efficiency is proved.

ii) Let $i \in N$ be a player who is (w, N')-connected to the source 0. There exists $r \in \{1, \ldots, |E_{N'}|\}$ such that i is connected to 0 in $F^{\sigma,r}$ but not in $F^{\sigma,r-1}$ and $w(\sigma(r)) = 0$. Moreover, by the definition of an obligation map, $\hat{o}_i(\pi^{\sigma,k}) = 0$ for $k \in \{r, \ldots, |E_{N'}|\}$. It follows by (4.18) that $\phi_i^{\hat{o}}(w) = 0$ and then it is proved that $\phi^{\hat{o}}$ satisfies the carrier property.

iii) Let $\hat{o}^\bullet, \hat{o}^\circ$ and $\hat{o}^\alpha$, with $\alpha \in [0,1]$, be as in Remark 4.5.1. Then,

$$\begin{aligned}
&\alpha\phi^{\hat{o}^\bullet}(w) + (1-\alpha)\phi^{\hat{o}^\circ}(w) \\
&= \alpha D^{\sigma,\hat{o}^\bullet} w^\sigma + (1-\alpha) D^{\sigma,\hat{o}^\circ} w^\sigma \\
&= \left(\alpha D^{\sigma,\hat{o}^\bullet} + (1-\alpha) D^{\sigma,\hat{o}^\circ}\right) w^\sigma \\
&= D^{\sigma,\hat{o}^\alpha} w^\sigma = \phi^{\hat{o}^\alpha}(w)
\end{aligned}$$

for every $w \in \mathcal{W}^{N'}$ and $\sigma \in \Sigma_{E_{N'}}$ such that $w \in K^\sigma$, where the third equality follows from Remark 4.5.1 and the definition of $D^{\sigma,\hat{o}^\alpha}$. Then, it is proved that the set of Obligation rules is a convex set.

■

4.6 The P-value

A special Obligation rule is the P-value, studied in Branzei *et al.* (2004). It turns out that the P-value equals the Equal Remaining Obligations (ERO) rule suggested by Jos Potters (which explains the name P-value) and which is introduced in Feltkamp *et al.* (1994a).

Definition 4.6.1 The P-value is the map $P : \mathcal{W}^{N'} \to \mathbb{R}^N$, defined by

$$P(w) = \phi^{\hat{o}^*}(w) \tag{4.20}$$

for each $w \in \mathcal{W}^{N'}$ and where $\phi^{\hat{o}^*}$ is the Obligation rule w.r.t. the obligation map $\hat{o}^*$ of Example 4.5.1.

Example 4.5.5 provides an illustration of the P-value.

Remark 4.6.1 Let $\sigma \in \Sigma_{E_{N'}}$. For each $k \in \{1, \ldots, |E_{N'}|\}$, consider the simple mcst situation $e^{\sigma,k}$. Then, for $k > 1$, each edge $e \in F^{\sigma,k-1}$ has cost $e^{\sigma,k}(e) = 0$. Therefore, if i and j in N' are connected in $< N', F^{\sigma,k-1} >$, then they are also in the same $(e^{\sigma,k}, N')$-component. Conversely, if i and j are in the same $(e^{\sigma,k}, N')$-component, then they are also connected in $< N', F^{\sigma,k-1} >$ and as a consequence, by equation (4.13), $\hat{o}^*_i(\pi^{\sigma,k-1}) = \hat{o}^*_j(\pi^{\sigma,k-1})$.

Using the linearity of P (or $\phi^{\hat{o}^*}$) on K^σ, an alternative way of calculating $P(w)$, which will be useful in the following, is as linear combination of $P(e^{\sigma,k})$, $k \in \{1, \ldots, |E_{N'}|\}$, where $\sigma \in \Sigma_{E_{N'}}$ is such that $w \in K^\sigma$ (see relation (2.2)). In formula,

$$P(w) = w(\sigma(1))P(e^{\sigma,1}) + \textstyle\sum_{k=2}^{|E_{N'}|} \big(w(\sigma(k)) - w(\sigma(k-1))\big) P(e^{\sigma,k}). \tag{4.21}$$

Note that for each mcst situation $e^{\sigma,k} \in K^\sigma$, $k \in \{1, \ldots, |E_{N'}|\}$, we have

$$P(e^{\sigma,k}) = \sum_{r=1}^{n} e^{\sigma,k}(\sigma(\rho^\sigma(r))) \left(\hat{o}^*(\pi^{\sigma,\rho^\sigma(r-1)}) - \hat{o}^*(\pi^{\sigma,\rho^\sigma(r)})\right) = \hat{o}^*(\pi^{\sigma,k-1}), \tag{4.22}$$

where the second equality follows from the fact that $\hat{o}^*(\pi^{\sigma,\rho^\sigma(n)})$ is the zero vector. As we said at the beginning of this section, the P-value coincides with the *Equal Remaining Obligations* (ERO) rule. The ERO-rule has been introduced in Feltkamp *et al.* (1994) via an extension of Kruskal's algorithm. According to the ERO-rule, at each stage $k \in \{0, 1, \ldots, |E_{N'}|\}$ of the algorithm, each player $i \in N$ pays exactly the difference between *remaining obligations*, *i.e.* $\hat{o}^*_i(\pi^{\sigma,k-1}) - \hat{o}^*_i(\pi^{\sigma,k})$ for each $i \in N$, as shown in Theorem 4.3 of Feltkamp *et al.* (1994). An axiomatic characterization of the ERO-rule using the properties of NE (*Non-Emptiness*), FSC (*Free-for-Source-Component*), LOC (*Local*), Eff (*Efficiency*), ET (*Equal Treatment*) and IPCons (*Inversely Proportional Consistency*) is given there. In Section 6.2 we provide an alternative axiomatic characterization.

We end this section with a proposition that enlightens the connection between the P-value and the P^τ-values, $\tau \in \Sigma_{E_{N'}}$, according to Remark 4.5.4.

Proposition 4.6.1 *Let $w \in \mathcal{W}^{N'}$. Then,*

$$P(w) = \frac{1}{n!} \sum_{\tau \in \Sigma_N} P^\tau(w). \tag{4.23}$$

Proof By Remark 4.5.4 and relation (4.16) we only have to prove that

$$\bar{D}^{\sigma,\hat{o}^*} = \frac{1}{n!} \sum_{\tau \in \Sigma_N} \bar{D}^{\sigma,\hat{o}^\tau}. \tag{4.24}$$

Let $\sigma \in \Sigma_{E_{N'}}$ be such that $w \in K^\sigma$.

To prove (4.24), note that for each $i \in N$, the edge $\sigma(\rho^\sigma(i))$ connects two disconnected subsets of vertices $S, T \in \pi^{\sigma,\rho^\sigma(i-1)}$. Then, for each player $j \in N \setminus (S \cup T)$, if any, $\frac{1}{n!}\sum_{\tau \in \Sigma_N} \bar{D}^{\sigma,\hat{o}^\tau}_{ji} = \bar{D}^{\sigma,\hat{o}^*}_{ji} = 0$.

On the other hand, for players in $S \cup T$, we have two possibilities regarding the position of the source w.r.t. the sets S and T:

i) The source 0 belongs neither to S nor to T implying that for each $j \in T$ and for each $\tau \in \Sigma_N$

$$\hat{o}^\tau_j(\pi^{\sigma,\rho^\sigma(i-1)}) - \hat{o}^\tau_j(\pi^{\sigma,\rho^\sigma(i)}) = \begin{cases} 1 & \text{if } \tau^{-1}(j) = \min\{\tau^{-1}(k) | k \in T\} \text{ and} \\ & \quad \tau^{-1}(j) \neq \min\{\tau^{-1}(k) | k \in S \cup T\}; \\ & \\ 0 & \text{otherwise.} \end{cases}$$

The fraction of orderings $\tau \in \Sigma_N$ such that $\tau(\min\{\tau^{-1}(k) | k \in S \cup T\}) \in S$ is equal to $\frac{|S|}{|S \cup T|} = \frac{|S|}{|S|+|T|}$ whereas the fraction of orderings $\tau \in \Sigma_N$ such that $\tau^{-1}(j) = \min\{\tau^{-1}(k) | k \in T\}$ is equal to $\frac{1}{|T|}$. Then, it follows that for each $j \in T$

$$\begin{aligned}
&\tfrac{1}{n!}\textstyle\sum_{\tau\in\Sigma_N}\bar{D}_{ji}^{\sigma,\hat{o}^\tau}\\
&=\tfrac{1}{n!}\textstyle\sum_{\tau\in\Sigma_N}\left(\hat{o}_j^\tau(\pi^{\sigma,\rho^\sigma(i-1)})-\hat{o}_j^\tau(\pi^{\sigma,\rho^\sigma(i)})\right)\\
&=\tfrac{|S|}{|S\cup T|}\tfrac{1}{|T|}=\tfrac{1}{|T|}-\tfrac{1}{|S\cup T|}\\
&=\hat{o}_j^*(\pi^{\sigma,\rho^\sigma(i-1)})-\hat{o}_j^*(\pi^{\sigma,\rho^\sigma(i)})\\
&=\bar{D}_{ji}^{\sigma,\hat{o}^*}.
\end{aligned}$$

Similar arguments hold for each $j \in S$ too.

ii) The source 0 belongs either to S or to T. Without loss of generality, suppose $0 \in S$. Then, for each $j \in S$

$$\begin{aligned}
&\tfrac{1}{n!}\textstyle\sum_{\tau\in\Sigma_N}\bar{D}_{ji}^{\sigma,\hat{o}^\tau}\\
&=\tfrac{1}{n!}\textstyle\sum_{\tau\in\Sigma_N}\left(\hat{o}_j^\tau(\pi^{\sigma,\rho^\sigma(i-1)})-\hat{o}_j^\tau(\pi^{\sigma,\rho^\sigma(i)})\right)\\
&=0\\
&=\hat{o}_j^*(\pi^{\sigma,\rho^\sigma(i-1)})-\hat{o}_j^*(\pi^{\sigma,\rho^\sigma(i)})\\
&=\bar{D}_{ji}^{\sigma,\hat{o}^*}.
\end{aligned}$$

On the other hand, for each $j \in T$

$$\begin{aligned}
&\tfrac{1}{n!}\textstyle\sum_{\tau\in\Sigma_N}\bar{D}_{ji}^{\sigma,\hat{o}^\tau}\\
&=\tfrac{1}{n!}\textstyle\sum_{\tau\in\Sigma_N}\left(\hat{o}_j^\tau(\pi^{\sigma,\rho^\sigma(i-1)})-\hat{o}_j^\tau(\pi^{\sigma,\rho^\sigma(i)})\right)\\
&=\tfrac{1}{|T|}\\
&=\hat{o}_j^*(\pi^{\sigma,\rho^\sigma(i-1)})-\hat{o}_j^*(\pi^{\sigma,\rho^\sigma(i)})\\
&=\bar{D}_{ji}^{\sigma,\hat{o}^*}.
\end{aligned}$$

A similar argument holds if $0 \in T$.

Hence, (4.24) is proved implying that $P(w) = \frac{1}{n!}\sum_{\tau\in\Sigma_N} P^\tau(w)$. ■

4.7 Conservative Construct & Charge rules

The main result in this section is derived from the relation between Obligation rules and conservative Construct & Charge rules introduced in the previous sections.

Remark 4.7.1 *Let $\hat{o}$ be an obligation map on $\Theta(N')$ and let $\mathcal{C} = \{C^1, \ldots, C^n\}$ be a set of functions with $C^j : \{\mathbf{a}_{|j} : \boldsymbol{a} \in \mathcal{E}_{N'}\} \to \Delta(N)$, for each $j \in N$, such that*

$$C^j(\mathbf{a}_{|j}) = \hat{o}(\Pi(\mathbf{a}_{|j-1})) - \hat{o}(\Pi(\mathbf{a}_{|j})) \tag{4.25}$$

for each $\boldsymbol{a} \in \mathcal{E}_{N'}$ and $j \in N$. It is easy to see, via relation (4.12), that $\mathcal{C}$ satisfies the connection property and the involvement property. By Proposition 4.5.1, it follows that $\mathcal{C}$ satisfies the total aggregation property as well. As a consequence, $\mathcal{C}$ is charge system on N.
Note that, by relation (4.16), for all $\sigma \in \Sigma_{E_{N'}}$ and $w \in K^\sigma$

$$\begin{aligned} \phi^{\sigma,\hat{o}}(w) &= \textstyle\sum_{r=1}^{n} w(\sigma(\rho^\sigma(r))) \left(\hat{o}(\pi^{\sigma,\rho^\sigma(r-1)}) - \hat{o}(\pi^{\sigma,\rho^\sigma(r)})\right) \\ &= \textstyle\sum_{r=1}^{n} w(\sigma(\rho^\sigma(r)))C^r(\sigma(\rho^\sigma(1)), \ldots, \sigma(\rho^\sigma(r))) = \chi^{\mathcal{C},\sigma}(w). \end{aligned} \tag{4.26}$$

Relation (4.26) shows that the class of Obligation rules is a subclass of the class of CC-rules. The inclusion in the other way is not true, as indicated in Remark 4.4.2, and proved by the fact that the CC-rule w.r.t. the charge system introduced in Example 4.2.5 cannot be defined via relation (4.9). Another counterexample is the Proportional rule, *i.e.* the CC-rule w.r.t. the charge system introduced in Example 4.2.2, that cannot be defined via relation (4.9).

So, it makes sense to study for conservative charge systems the following property.

Definition 4.7.1 Let $\mathcal{C} = \{C^1, \ldots, C^n\}$ be a charge system on N. We say that $\mathcal{C}$ has the *obligation property* if there exists an obligation map $\hat{o}$ on $\Theta(N')$ such that

$$C^j(\mathbf{a}_{|j}) = \hat{o}(\Pi(\mathbf{a}_{|j-1})) - \hat{o}(\Pi(\mathbf{a}_{|j}))$$

for each $\mathbf{a} \in \mathcal{E}_{N'}$ and $j \in N$.

Theorem 4.7.1 *Let $\mathcal{C} = \{C^1, \ldots, C^n\}$ be a charge system on N with the obligation property. Then, $\mathcal{C}$ has the patch property.*

Proof Consider an obligation map $\hat{o}$ on $\Theta(N')$ such that

$$C^j(\mathbf{a}_{|j}) = \hat{o}(\Pi(\mathbf{a}_{|j-1})) - \hat{o}(\Pi(\mathbf{a}_{|j}))$$

for each $\mathbf{a} \in \mathcal{E}_{N'}$ and $j \in N$. The assertion that $\mathcal{C}$ has the patch property follows directly by Definition 4.5.1 and relation (4.26) on the obligation rule $\phi^{\hat{o}}$. ∎

In the following theorem, we give a sufficient condition for charge systems to satisfy the obligation property.

Theorem 4.7.2 *Let $\mathcal{C} = \{C^1, \ldots, C^n\}$ be a conservative charge system on N. Then, $\mathcal{C}$ has the obligation property.*

Proof Let $P^{\mathcal{C}}(S)$ be the potential of S with respect to the conservative charge system $\mathcal{C}$ for each $S \in 2^N \setminus \{\emptyset\}$. Consider the map $o^{\mathcal{C}} : 2^N \setminus \{\emptyset\} \to \mathbb{R}^N_+$ defined by

$$o^{\mathcal{C}}(S) = e^S - P^{\mathcal{C}}(S) \tag{4.27}$$

for each $S \in 2^N \setminus \{\emptyset\}$, where $e^S \in \mathbb{R}^N_+$ is such that $e^S_i = 1$ for each $i \in S$ and $e^S_i = 0$ for each $i \in N \setminus S$. Note that for each $j \in N$, we have

$$\begin{aligned}
&\hat{o}^{\mathcal{C}}(\Pi(\mathbf{a}_{|j-1})) - \hat{o}^{\mathcal{C}}(\Pi(\mathbf{a}_{|j}))\\
&= \textstyle\sum_{S\in\Pi(\mathbf{a}_{|j-1}),0\notin S} o^{\mathcal{C}}(S) - \sum_{S\in\Pi(\mathbf{a}_{|j}),0\notin S} o^{\mathcal{C}}(S)\\
&= \textstyle\sum_{S\in\Pi(\mathbf{a}_{|j-1}),0\notin S} \left(e^S - P^{\mathcal{C}}(S)\right) - \sum_{S\in\Pi(\mathbf{a}_{|j}),0\notin S} \left(e^S - P^{\mathcal{C}}(S)\right)\\
&= \textstyle\sum_{S\in\Pi(\mathbf{a}_{|j-1})} \left(e^{S\setminus\{0\}} - P^{\mathcal{C}}(S)\right) - \sum_{S\in\Pi(\mathbf{a}_{|j})} \left(e^{S\setminus\{0\}} - P^{\mathcal{C}}(S)\right)\\
&= \textstyle\sum_{S\in\Pi(\mathbf{a}_{|j})} P^{\mathcal{C}}(S) - \sum_{S\in\Pi(\mathbf{a}_{|j-1})} P^{\mathcal{C}}(S)\\
&+ \textstyle\sum_{S\in\Pi(\mathbf{a}_{|j-1})} e^{S\setminus\{0\}} - \sum_{S\in\Pi(\mathbf{a}_{|j})} e^{S\setminus\{0\}}\\
&= \textstyle\sum_{S\in\Pi(\mathbf{a}_{|j})} P^{\mathcal{C}}(S) - \sum_{S\in\Pi(\mathbf{a}_{|j-1})} P^{\mathcal{C}}(S)\\
&= A^{\mathcal{C}}(\mathbf{a}_{|j}) - A^{\mathcal{C}}(\mathbf{a}_{|j-1})\\
&= \textstyle\sum_{k=1}^{j} C^k(\mathbf{a}_{|k}) - \sum_{k=1}^{j-1} C^k(\mathbf{a}_{|k}) = C^j(\mathbf{a}_{|j}),
\end{aligned}$$

where the third equality follows from Lemma 4.3.1.(c.1), the fifth equality follows from the fact that $\sum_{S\in\theta} e^{S\setminus\{0\}} = e^N$ for each $\theta \in \Theta(N')$ and the sixth equality from Proposition 4.3.1.

We want to prove that $o^{\mathcal{C}}$ is an obligation function, *i.e.* $o^{\mathcal{C}}$ satisfies the properties (4.10) and (4.11).

By definition, it follows directly that $o^{\mathcal{C}}_i(S) = 0$ for each $i \in N \setminus S$ and $o^{\mathcal{C}}_i(S) \geq 0$ for each $i \in S$ and for each $S \in 2^N \setminus \{\emptyset\}$. Moreover, from (c.2) of Lemma 4.3.1, it follows that

$$\sum_{i\in N} o^{\mathcal{C}}_i(S) = \sum_{i\in S} \Big(1 - P^{\mathcal{C}}_i(S)\Big) = |S| - (|S| - 1) = 1,$$

for each $S \in 2^N \setminus \{\emptyset\}$, implying that property (4.10) holds.

Finally, by (c.3) in Lemma 4.3.1, we have that for each $S \subseteq T \subseteq N$, $S \neq \emptyset$, and each $i \in S$

$$o_i^{\mathcal{C}}(S) = 1 - P_i^{\mathcal{C}}(S) \geq 1 - P_i^{\mathcal{C}}(T) = o_i^{\mathcal{C}}(T), \tag{4.28}$$

which proves that property (4.11) holds, too. ■

The next theorem is the main result in this section.

Theorem 4.7.3 *For each charge system $\mathcal{C} = \{C^1, \ldots, C^n\}$ on N the following statements are equivalent:*

i) $\mathcal{C}$ is a conservative charge system;

ii) $\mathcal{C}$ satisfies the patch property;

iii) $\mathcal{C}$ satisfies the obligation property.

Proof Equivalence of i), ii) and iii) follows from Theorems 4.4.1, 4.7.1 and 4.7.2. Specifically, by Theorem 4.4.1, if $\mathcal{C}$ has the patch property then $\mathcal{C}$ is a conservative charge system. By Theorem 4.7.2, if $\mathcal{C}$ is a conservative charge system then $\mathcal{C}$ satisfies the obligation property. Finally, by Theorem 4.7.1, if $\mathcal{C}$ satisfies the obligation property then $\mathcal{C}$ has the patch property. ■ From Theorem 4.7.3 and Remark 4.7.1 we conclude that the class of conservative CC-rules coincides with the class of Obligation rules.

Remark 4.7.2 *As we already observed in Remark 4.5.4, since the P-value and the P^τ-values, with $\tau \in \Sigma_N$, are Obligation rules, one can obtain the corresponding charge systems using relation (4.25). It is easy to check, for example, that the obligation map of Example 4.5.1, which defines the P-value as in Section 4.6, may be obtained by relation (4.25) on the charge system $\hat{\mathcal{C}}$ of Example 4.2.3.*

Chapter 5

Monotonicity properties for cost allocation rules

5.1 Introduction

In this chapter, we study some properties of Obligation rules in mcst situations where the cardinality of the set of agents can vary in time, and also increasing or decreasing of connection costs may occur. In Section 5.2, we show that Obligation rules are cost monotonic and induce population monotonic allocation schemes. Note that the concept of cost monotonicity defined in Section 5.2 and introduced in Tijs *et al.* (2006a) is stronger than the concept of cost monotonicity introduced in Dutta and Kar (2004), because we simply impose that if some connection costs go down, then no agents will pay more (as in the strong cost monotonicity property used by Bergañtinos and Vidal-Puga (2004)).

The irreducible core (that we will call the Bird core) is central in Sections 5.3 and 5.4. There, we will give a new "tree free" way to introduce the Bird core by constructing for each mcst-problem a related problem, where the weight function is a non-Archimedean semimetric. Moreover, we introduce a related concept of cost monotonicity for multisolutions in mcst situations which generalizes the concept of cost monotonicity for mcst solutions introduced in Section 5.2. The relations between stable cost monotonic rules and the Bird core are also discussed in Section 5.4.

Section 5.2 is based on Tijs, Branzei, Moretti, Norde (2006a); sections 5.3 and 5.4 are based on Tijs, Moretti, Branzei, Norde (2006b).

5.2 Cost monotonicity for solutions and pmas

In this section we will discuss some nice monotonicity properties of the Obligation rules. First, we provide the definition of cost monotonic solutions for mcst situations.

Definition 5.2.1 A solution $F : \mathcal{W}^{N'} \to I\!R^N$ is a *cost monotonic* solution if for all mcst situations $w, \hat{w} \in \mathcal{W}^{N'}$ such that $w(\hat{e}) \leq \hat{w}(\hat{e})$ for one edge $\hat{e} \in E_{N'}$ and $w(e) = \hat{w}(e)$ for each $e \in E_{N'} \setminus \{\hat{e}\}$, it holds that $F(w) \leq F(\hat{w})$.

We prove in Theorem 5.2.2 that Obligation rules are cost monotonic; the main step is the following lemma.

Lemma 5.2.1 *Let $\hat{o}$ be an obligation map on $\Theta(N')$ and let $w \in \mathcal{W}^{N'}$. Let $\hat{e} \in E_{N'}$ and let $h > w(\hat{e})$ be such that there is no $e \in E_{N'}$ with $w(\hat{e}) < w(e) < h$. Define $\tilde{w} \in \mathcal{W}^{N'}$ by $\tilde{w}(e) := w(e)$ if $e \in E_{N'} \setminus \{\hat{e}\}$ and $\tilde{w}(\hat{e}) = h$. Then: $\phi^{\hat{o}}(\tilde{w}) \geq \phi^{\hat{o}}(w)$.*

Proof Let $Z := \{e \in E_{N'} | w(e) = w(\hat{e})\}$ be the set of edges that have the same cost as $\hat{e}$. Let $\sigma \in \Sigma_{E_{N'}}$ be such that $w \in K^{\sigma}$. Without loss of generality we may assume that $\sigma^{-1}(\hat{e}) = \max\{\sigma^{-1}(e) | e \in Z\}$, *i.e.* σ ranks the edges of Z with $\hat{e}$ last. By construction we also have $\tilde{w} \in K^{\sigma}$ and hence

$$\phi^{\hat{o}}(\tilde{w}) = D^{\sigma,\hat{o}}\tilde{w}^{\sigma} \geq D^{\sigma,\hat{o}}w^{\sigma} = \phi^{\hat{o}}(w),$$

where at the inequality we used the fact that $\tilde{w}^{\sigma} \geq w^{\sigma}$ and the fact that the matrix $D^{\sigma,\hat{o}}$ is non-negative. ■

Theorem 5.2.2 *Obligation rules are cost monotonic.*

Proof Let $\hat{o}$ be an obligation map on $\Theta(N')$ and let $\phi^{\hat{o}}$ be the Obligation rule w.r.t $\hat{o}$. Let $w, \hat{w} \in \mathcal{W}^{N'}$ and $\hat{e} \in E_{N'}$ be as in Definition 5.2.1.
Let $H := \{h \in I\!R |$ there is an edge $f \in E_{N'}$ s.t. $h = w(f) \in (w(\hat{e}), \hat{w}(\hat{e}))\}$. If $H = \emptyset$ then the statement follows directly from Lemma 5.2.1. If $H \neq \emptyset$ write $H = \{h_1, \ldots, h_k\}$ with $h_1 < \ldots < h_k$.

Consider the sequence of precisely $k+2$ mcst situations $w_0, \ldots, w_{k+1} \in \mathcal{W}^{N'}$ such that $w_0 = w$, $w_{k+1} = \hat{w}$ and for each $r \in \{1, \ldots, k\}$, $w_r(e) = w(e)$ for each $e \in E_{N'} \setminus \{\hat{e}\}$ and $w_r(\hat{e}) = h_r$.

Applying Lemma 5.2.1 for each $r \in \{1, \ldots, k\}$, with w_{r-1} in the role of w and w_r in the role of $\tilde{w}$, it follows that

$$\phi^{\hat{o}}(\hat{w}) = \phi^{\hat{o}}(w_{k+1}) \geq \phi^{\hat{o}}(w_k) \geq \ldots \geq \phi^{\hat{o}}(w_0) = \phi^{\hat{o}}(w),$$

which finally proves cost monotonicity of Obligation rules. ■

The following theorem shows that Obligation rules induce a pmas for the corresponding mcst games.

Before introducing the theorem, we need to introduce some further notations. Let o be an obligation function and $\hat{o}$ the corresponding obligation map. Let $S \subseteq N$, let o_S denote the restriction of o to $2^S \setminus \{\emptyset\}$ and let $\hat{o}_S$ denote the corresponding obligation map, *i.e.*

$$\hat{o}_S(\theta) = \sum_{T \in \theta, 0 \notin T} o_S(T)$$

for every $\theta \in \Theta(S \cup \{0\})$.

Recall also that if $w \in \mathcal{W}^{N'}$, then the Obligation rule $\phi^{\hat{o}_S}$ w.r.t the obligation map $\hat{o}_S$ and applied to $w_{|S'}$, *i.e.* the restriction of the weight function w to $E_{S'} \subseteq E_{N'}$ as defined in Section 2.1, provides a vector in $I\!R^S$ according to Definition 4.5.3 w.r.t. the set of nodes S'.

Theorem 5.2.3 *Let $\hat{o}$ be an obligation map on $\Theta(N')$, let $\phi^{\hat{o}}$ be the Obligation rule w.r.t $\hat{o}$, and let $w \in \mathcal{W}^{N'}$. Then, the table $[\phi^{\hat{o}_S}(w_{|S'})]_{S \in 2^N \setminus \{\emptyset\}}$ is a pmas for the mcst game (N, c_w).*

Proof Given $S \subset T \subseteq N$, define $< T', \hat{w} >$ with $T' = T \cup \{0\}$ and

$$\hat{w}(\{i,j\}) = \begin{cases} w(\{i,j\}) & \text{if } i,j \in S' \\ \\ w(\{i,j\}) + \lambda_S & \text{otherwise} \end{cases} \tag{5.1}$$

where $\lambda_S = 1 + \max\{w(\{i,j\}) | i,j \in S'\}$.

Then, in $< T', \hat{w} >$ each edge with at least one node not in S' is more expensive than in $< T', w_{|T'} >$.

Further, let $\bar{\sigma} \in \Sigma_{E_{T'}}$ be such that $\hat{w} \in K^{\bar{\sigma}}$ and let $\sigma^{S'} \in \Sigma_{E_{S'}}$ be such that $\sigma^{S'}(i) = \bar{\sigma}(i)$ for each $i \in \{1, \ldots, |E_{S'}|\}$. Then, by (5.1) it follows that $w_{|S'} \in K^{\sigma^{S'}}$.

Note that for each $i \in S$

$$\phi_i^{\hat{o}_T}(\hat{w}) = \phi_i^{\hat{o}_S}(w_{|S'}). \tag{5.2}$$

This follows from the fact that in $< S', w_{|S'} >$ the edges with at least one node not in S' are discarded and in $< T', \hat{w} >$ the edges with at least one node not in S' are allowed but they are too expensive. The result is that applying the Kruskal procedure on $< T', \hat{w} >$ w.r.t. $\bar{\sigma}$ the players in S' are already connected to 0 before one of the edges with nodes not in S' is considered. So, by definition of an obligation map, we have that the contribution matrix $D^{\bar{\sigma}, \hat{o}_T}$ with $|T|$ rows and $|E_{T'}|$ columns is of the form

$$D^{\bar{\sigma}, \hat{o}_T} = \begin{array}{c} \begin{bmatrix} D^{\sigma^{S'}, \hat{o}_S} & N^1 \\ \\ N^2 & R \end{bmatrix} \end{array} \begin{array}{l} \dashrightarrow \text{players in } S \\ \\ \dashrightarrow \text{players in } T \setminus S, \end{array}$$

where the four submatrices $D^{\sigma^{S'}, \hat{o}_S}, N^1, N^2$ and R are such that:

- $D^{\sigma^{S'}, \hat{o}_S}$ is the contribution matrix w.r.t. to $\sigma^{S'}$ and to $\hat{o}_S$ with $|S|$ rows and $|E_{S'}|$ columns;
- N^1 is the null matrix with $|S|$ rows and $|E_{T'}| - |E_{S'}|$ columns;
- N^2 is the null matrix with $|T| - |S|$ rows and $|E_{S'}|$ columns;
- R is a real valued matrix with $|T| - |S|$ rows and $|E_{T'}| - |E_{S'}|$ columns obtained according to the definition of the contribution matrix $D^{\bar{\sigma}, \hat{o}_T}$.

Hence, for each $i \in S$, $\phi_i^{\hat{o}_T}(\hat{w}) = (D^{\sigma^{S'}, \hat{o}_S} \hat{w}_{|S'}^{\sigma^{S'}})_i = \phi_i^{\hat{o}_S}(\hat{w}_{|S'}) = \phi_i^{\hat{o}_S}(w_{|S'})$, which yields equation (5.2). [Here $(D^{\sigma^{S'}, \hat{o}_S} \hat{w}_{|S'}^{\sigma^S})_i$ is the i-th component of the vector $D^{\sigma^{S'}, \hat{o}_S} \hat{w}_{|S'}^{\sigma^{S'}}$.]

Recall that Obligation rules are cost monotonic. Since $\hat{w}(e) \geq w_{|T'}(e)$ for each $e \in E_{T'}$, we have

$$\phi_i^{\hat{o}_T}(\hat{w}) \geq \phi_i^{\hat{o}_T}(w_{|T'}), \qquad \text{for each } i \in T. \tag{5.3}$$

From (5.2) and (5.3) we obtain

$$\phi_i^{\hat{o}_S}(w_{|S'}) \geq \phi_i^{\hat{o}_T}(w_{|T'}) \quad \text{ for each } i \in S. \tag{5.4}$$

From (5.4) and the efficiency property it follows that $[\phi^{\hat{o}_S}(w_{|S'})]_{S \in 2^N \setminus \{\emptyset\}}$ is a pmas for the mcst game (N, c_w). ■

From Theorem 5.2.3 and the definition of a pmas, it follows that Obligation rules provide cost allocations which are core elements of the game (N, c_w).

5.3 Minimal mcst situations

Let $w \in \mathcal{W}^{N'}$. For each path $\mathbf{p} = (i_0, i_1, \ldots, i_k)$ from i to j in the graph $< N', E_{N'} >$ we denote the set of its edges by $E(\mathbf{p})$, that is $E(\mathbf{p}) = \{\{i_0, i_1\}, \{i_1, i_2\}, \ldots, \{i_{k-1}, i_k\}\}$. Moreover, we call $\max_{e \in E(\mathbf{p})} w(e)$ the *top of the path* $\mathbf{p}$ and denote it by $t(\mathbf{p})$. We denote by $\mathcal{P}_{ij}^{N'}$ the set of all paths without cycles from i to j in the graph $< N', E_{N'} >$.

Now, we define the key concept of this section, namely the reduced weight function.

Definition 5.3.1 Let $w \in \mathcal{W}^{N'}$. The *reduced weight function* $\overline{w}$ is given by

$$\overline{w}(i,j) = \min_{\mathbf{p} \in \mathcal{P}_{ij}^{N'}} \max_{e \in E(\mathbf{p})} w(e) = \min_{\mathbf{p} \in \mathcal{P}_{ij}^{N'}} t(\mathbf{p}) \tag{5.5}$$

for each $i, j \in N'$, $i \neq j$.

Now, extending $\overline{w}$ by putting $\overline{w}(i,i) = 0$ for each $i \in N'$, we obtain a non-negative function on the set of all pairs of elements in N'. The obtained reduced weight function $\overline{w}$ is a semimetric on N' with the sharp triangle inequality, *i.e.* a *non-Archimedean* (NA-)*semimetric*. In formula, for each $i, j, k \in N'$

$$\begin{aligned}
&\overline{w}(i,j) \geq 0 \text{ and } \overline{w}(i,i) = 0 \text{ (non-negativity);}\\
&\overline{w}(i,j) = \overline{w}(j,i) \text{ (symmetry);}\\
&\overline{w}(i,k) \leq \max\{\overline{w}(i,j), \overline{w}(j,k)\} \text{ (sharp triangle inequality).}
\end{aligned}$$

The proof is left to the reader. If $w > 0$, then $\overline{w}$ is a non-Archimedean metric on the set N'.

For the reduced weight function $\overline{w}$ we have a special property related to triangles, as the next lemma shows.

Proposition 5.3.1 (The isoscele triangle property) *Let $\overline{w}$ be the reduced weight function corresponding to $w \in \mathcal{W}^{N'}$ and $i, j, k \in N'$ such that $\overline{w}(i,j) \leq \overline{w}(i,k)$ and $\overline{w}(i,j) \leq \overline{w}(k,j)$. Then, $\overline{w}(i,k) = \overline{w}(j,k)$.*

Proof By the sharp triangle inequality $\overline{w}(i,k) \leq \max\{\overline{w}(i,j), \overline{w}(j,k)\} = \overline{w}(j,k)$ and $\overline{w}(j,k) \leq \max\{\overline{w}(j,i), \overline{w}(i,k)\} = \overline{w}(i,k)$.

So, $\overline{w}(i,k) = \overline{w}(j,k)$. ■

This property for NA-semimetrics will be useful in proving that there are many minimum cost spanning trees for $(N', \overline{w})$, as we will see in Theorem 5.3.2.

In the sequel we simply refer to $\overline{w}$ as the mcst situation which assigns to each edge $\{i,j\} \in E_{N'}$ the reduced weight value as defined in equality (5.5). Further, we will denote by $\overline{\mathcal{W}}^{N'} \subset \mathcal{W}^{N'}$ the set of all mcst situations which assign to each edge $\{i,j\} \in E_{N'}$ the distance $\overline{w}(i,j)$ provided by a reduced weight $\overline{w}$ on N'.

Example 5.3.1 Consider the mcst situation $< N', w >$ with $N' = \{0,1,2,3\}$ and w as depicted in Figure 5.1. Note that $w \in K^{\sigma}$, with $\sigma(1) = \{1,2\}$, $\sigma(2) = \{0,1\}$, $\sigma(3) = \{1,3\}$, $\sigma(4) = \{0,3\}$, $\sigma(5) = \{0,2\}$, $\sigma(6) = \{2,3\}$.
The corresponding mcst situation $\overline{w}$ is depicted in Figure 5.2.

■

One main result in this section, Proposition 5.3.2, concerns an interesting relation which can be established between the mcst situation $\overline{w}$ and a *minimal*

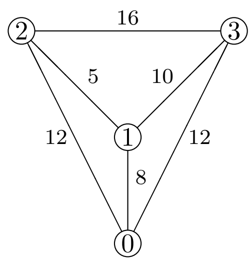


Figure 5.1: An mcst situation with three agents.

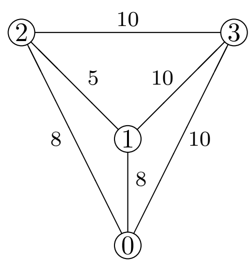


Figure 5.2: The mcst situation $\overline{w}$ corresponding to w.

mcst situation w^Γ as defined by Bird (1976), where Γ is an mcst for N' in w. Given an mcst situation $w \in \mathcal{W}^{N'}$ and an mcst Γ for N' in w, the minimal mcst situation w^Γ is defined (cf. Bird (1976)) by

$$w^\Gamma(\{i,j\}) = \max_{e \in E(\mathbf{p}_{ij}^\Gamma)} w(e) = t(\mathbf{p}_{ij}^\Gamma), \tag{5.6}$$

where $\mathbf{p}_{ij}^\Gamma \in \mathcal{P}_{ij}^{N'}$ is the unique path in Γ from i to j.

Proposition 5.3.2 *Let $w \in \mathcal{W}^{N'}$ and $i, j \in N'$. Let Γ be an mcst for N' in w and $\mathbf{p}_{ij}^\Gamma$ be the unique path in Γ from i to j. Then,*

$$t(\mathbf{p}_{ij}^\Gamma) = \min_{\mathbf{p} \in \mathcal{P}_{ij}^{N'}} t(\mathbf{p}). \tag{5.7}$$

Proof Let $\mathbf{p}^* \in \arg\min_{\mathbf{p} \in \mathcal{P}_{ij}^{N'}} t(\mathbf{p})$ and let e^* be an edge on $\mathbf{p}^*$ such that $t(\mathbf{p}^*) = w(e^*)$. Let $\hat{e} = \{m, n\}$ be an edge on $\mathbf{p}_{ij}^\Gamma$ with $w(\hat{e}) = t(\mathbf{p}_{ij}^\Gamma)$.
We have to prove that $w(\hat{e}) = w(e^*)$. If so, then it follows immediately that $\min_{\mathbf{p} \in \mathcal{P}_{ij}^{N'}} t(\mathbf{p}) = w(e^*) = w(\hat{e}) = t(\mathbf{p}_{ij}^\Gamma)$.

If $e^* = \hat{e}$ then of course $w(e^*) = w(\hat{e})$. Otherwise, first note that by definition of e^*

$$w(\hat{e}) \geq w(e^*). \tag{5.8}$$

Let S_m be the set of all nodes $r \in N'$ such that n is not on the path from m to r in $< N', \Gamma >$; let S_n be the set of nodes $r \in N'$ such that m is not on the path from n to r in $< N', \Gamma >$, *i.e.*

$$S_m = \{r \in N' | \nexists e \in E(\mathbf{p}^{\Gamma}_{mr}) \text{ with } n \in e\}$$

and

$$S_n = \{r \in N' | \nexists e \in E(\mathbf{p}^{\Gamma}_{nr}) \text{ with } m \in e\}.$$

Note that $\{S_n, S_m\}$ is a partition of N' and nodes in S_n are connected in $< N', \Gamma >$ to nodes in S_m via edge $\{m, n\}$. Moreover, by the definition of a path without cycles, i, j must belong to different sets of the partition $\{S_n, S_m\}$. So, without loss of generality we suppose that $i \in S_m$ and $j \in S_n$.

Consider the set of edges $E^+ = \{\{t, v\} | t \in S_m, v \in S_n\}$. Then,

$$w(\{m, n\}) = w(\hat{e}) \leq w(e), \text{ for each } e \in E^+. \tag{5.9}$$

In order to prove inequality (5.9), suppose on the contrary that $w(\{m, n\}) > w(e)$ for some $e \in E^+$. Then, the graph $\Gamma^+ = (\Gamma \setminus \{\hat{e}\}) \cup \{e\}$ would be a spanning network in N' cheaper than Γ, which yields a contradiction.

By the definition of a path, for each $\mathbf{p} \in \mathcal{P}^{N'}_{ij}$ there exists at least one edge $e \in E^+$ such that e is on the path $\mathbf{p}$. By inequality (5.9), it follows that $t(\mathbf{p}) \geq w(e) \geq w(\hat{e})$. This implies that $w(e^*) = \min_{\mathbf{p} \in \mathcal{P}^{N'}_{ij}} t(\mathbf{p}) \geq w(\hat{e})$. Together with inequality (5.8) we have finally $w(e^*) = w(\hat{e})$. ■

As a direct consequence of Proposition 5.3.2 we have that the mcst situation $\overline{w}$ coincides, for each mcst Γ for w, with the minimal mcst situation w^{Γ} introduced by Bird (1976). So, $w^{\Gamma} = w^{\check{\Gamma}}$ for each pair of mcst $\Gamma, \check{\Gamma}$, a fact which is already known (cf. Aarts (1994), Feltkamp (1995), Feltkamp *et al.* (1994)), but with a complicated proof. Let $w \in \mathcal{W}^{N'}$ and let Γ be an mcst for w. Let $\tau \in \Sigma_N$. We say that Γ and τ *fit* (or, also, that τ fits with Γ) if $E^{\Gamma}_{[\tau(1)]'}$, $E^{\Gamma}_{[\tau(2)]'}, \ldots,$

$E^{\Gamma}_{[\tau(|N|)]'}$ are spanning networks on sets of nodes $[\tau(1)]'$, $[\tau(2)]', \ldots,$ $[\tau(|N|)]'$, respectively.

Example 5.3.2 In Figure 5.3 is depicted an mcst, denoted by Γ, for the mcst situation $\overline{w}$ of Figure 5.2. Consider $\tau_1, \tau_2 \in \Sigma_N$ such that $\tau_1(1) = 1$, $\tau_1(2) = 2$,

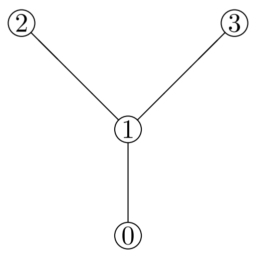


Figure 5.3: An mcst Γ for the mcst situation $\overline{w}$ of Figure 5.2.

$\tau_1(3) = 3$ and $\tau_2(1) = 1$, $\tau_2(2) = 3$, $\tau_2(3) = 2$. Note that both τ_1 and τ_2 fit with Γ but none of the other four elements of Σ_N fits with Γ.

■

Remark 5.3.1 Let $w \in \mathcal{W}^{N'}$, let Γ be an mcst for w and let $\tau \in \Sigma_N$ be an order such that Γ and τ fit. Then,

$$\sum_{e \in E^{\Gamma}_{[\tau(r)]'}} w(e) = c_w([\tau(r)]) \tag{5.10}$$

for each $r \in \{1, \ldots, |N|\}$. So, $E^{\Gamma}_{[\tau(r)]'}$ is an mcst for the mcst situation $< [\tau(r)]'$, $w_{|[\tau(r)]'} >$.

Remark 5.3.2 Let $w \in \mathcal{W}^{N'}$, let Γ be an mcst for w and let $\tau \in \Sigma_N$ be an order such that Γ and τ fit. The marginal vector $m^{\tau}(c_w)$ of the mcst game c_w coincides with the Bird allocation in w corresponding to Γ and therefore $m^{\tau}(c_w) \in \mathcal{C}(c_w)$, as is proved in Granot and Huberman (1981).

Remark 5.3.3 For each $\sigma \in \Sigma_{E_{N'}}$ there exists a tree Γ which is an mcst for every $w \in K^{\sigma}$; further, there exists a $\tau \in \Sigma_N$ such that Γ and τ fit.

The considerations in Remarks 5.3.1-5.3.3 together with the next lemma prelude to Theorem 5.3.2.

Lemma 5.3.1 *Let $w \in \overline{\mathcal{W}}^{N'}$, let Γ be an mcst for w and let $\tau \in \Sigma_N$ be such that Γ and τ fit. Let $r \in \{1, \dots, |N|-1\}$ and let $\check{\tau} \in \Sigma_N$ be such that $\check{\tau}(r) = \tau(r+1)$, $\check{\tau}(r+1) = \tau(r)$ and $\check{\tau}(i) = \tau(i)$ for each $i \in \{1, \dots, |N|\} \setminus \{r, r+1\}$ (i.e. $\check{\tau}$ is obtained from τ by a neighbor switch of $\tau(r)$ and $\tau(r+1)$). Then, there is an mcst $\check{\Gamma}$ for w such that $\check{\tau}$ and $\check{\Gamma}$ fit.*

Proof If $\tau(r)$ is not the immediate predecessor of $\tau(r+1)$ in Γ then take $\check{\Gamma} = \Gamma$ and then $\check{\tau}$ and $\check{\Gamma}$ fit.
If $\tau(r)$ is the immediate predecessor of $\tau(r+1)$ in Γ, then let $k \in [\tau(r-1)]'$ be the immediate predecessor of $\tau(r)$ in Γ.

First, note that

$$w(\{k, \tau(r+1)\}) \geq w(\{k, \tau(r)\}) \tag{5.11}$$

and

$$w(\{k, \tau(r+1)\}) \geq w(\{\tau(r), \tau(r+1)\}) \tag{5.12}$$

because Γ is an mcst for w.

Consider two cases:

c.1) $w(\{k, \tau(r)\}) \leq w(\{\tau(r), \tau(r+1)\})$. Take $\check{\Gamma} = (\Gamma \setminus \{\{\tau(r), \tau(r+1)\}\}) \cup \{\{k, \tau(r+1)\}\}$. By inequality (5.11) and the isoscele triangle property $w(\{k, \tau(r+1)\}) = w(\{\tau(r), \tau(r+1)\})$, and then $\check{\Gamma}$ is an mcst in w and $\check{\Gamma}$ and $\check{\tau}$ fit.

c.2) $w(\{\tau(r), \tau(r+1)\}) < w(\{k, \tau(r)\})$. Take $\check{\Gamma} = (\Gamma \setminus \{\{k, \tau(r)\}\}) \cup \{\{k, \tau(r+1)\}\}$. By inequality (5.12) and the isoscele triangle property $w(\{k, \tau(r)\}) = w(\{k, \tau(r+1)\})$ and then $\check{\Gamma}$ is an mcst in w and $\check{\Gamma}$ and $\check{\tau}$ fit.

■

Theorem 5.3.2 *Let $w \in \overline{\mathcal{W}}^{N'}$. Then,*

i) for each $\tau \in \Sigma_N$ there exists an mcst Γ such that Γ and τ fit.

ii) Let c_w be the mcst game corresponding to w. Then, $m^\tau(c_w) \in \mathcal{C}(c_w)$ for all $\tau \in \Sigma_N$ and c_w is a concave game.

Proof

i) Let $\hat{\Gamma}$ be an mcst for w. Then, there is at least one $\hat{\tau} \in \Sigma_N$ such that $\hat{\Gamma}$ and $\hat{\tau}$ fit. Further, each τ can be obtained from $\hat{\tau}$ by a suitable sequence of neighbor switches and so, by applying Lemma 5.3.1 repeatedly, we complete the proof of assertion *i*).

ii) Let Γ be an mcst in N' for w and let $\tau \in \Sigma_N$ be such that Γ and τ fit. By Remark 5.3.2, it follows that $m^\tau(c_w)$ coincides with the Bird allocation corresponding to Γ. Hence, again by Remark 5.3.2, $m^\tau(c_w) \in \mathcal{C}(c_w)$. Finally, by the Ichiishi theorem (Ichiishi (1981)) telling that a game is concave iff all marginal vectors are in the core of the game, it follows that c_w is a concave game.

■

Let $w \in \mathcal{W}^{N'}$. We call the core of the mcst game $c_{\overline{w}}$ the *Bird core* of the mcst game c_w and denote it by $\mathcal{BC}(w)$. By Theorem 5.3.2 it directly follows that the Bird core $\mathcal{BC}(w)$ of the mcst game c_w is the convex hull of all the Bird allocations corresponding to the minimum cost spanning trees for $\overline{w}$. Note also that $\mathcal{BC}(w) \subseteq \mathcal{C}(c_w)$, since $c_{\overline{w}}(S) \leq c_w(S)$ for each $S \in 2^N \setminus \{\emptyset\}$ and $c_{\overline{w}}(N) = c_w(N)$ (cf. Feltkamp (1995)).

Example 5.3.3 Consider the mcst situation w of Figure 5.1 and the corresponding reduced mcst situation $\overline{w}$ of Figure 5.2. Then,

	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
c_w	8	12	12	13	18	24	23
$c_{\overline{w}}$	8	8	10	13	18	18	23

There are six minimum cost spanning trees for $\overline{w}$. Three of them lead to the Bird allocation $(8,5,10)$ and the other three to the Bird allocation $(5,8,10)$. Further, $m^\tau(c_{\overline{w}}) = (8,5,10)$ for $\tau \in \Sigma_N$ with $(\tau(1),\tau(2),\tau(3)) \in \{(1,2,3),$ $(1,3,2),(3,1,2)\}$ and $m^\tau(c_{\overline{w}}) = (5,8,10)$ for $\tau \in \Sigma_N$ with $(\tau(1),\tau(2),\tau(3)) \in$

$\{(2,1,3),\ (2,3,1),(3,2,1)\}$. The Bird core $\mathcal{BC}(w)$ is the convex hull of the marginal vectors of the game $c_{\overline{w}}$, that is $\mathcal{BC}(w) = \text{conv}\{(8,5,10),\ (5,8,10)\} \subset \mathcal{C}(c_w)$.

■

5.4 Cost monotonicity for multisolutions

In Section 5.2 we have proved that the class of Obligation rules is a class of solutions for mcst situations which are cost monotonic, *i.e.* if the costs of some edges increase, then no agent will pay less.

In this section we introduce a related concept of cost monotonicity for multisolutions on mcst situations. We call a correspondence $G : \mathcal{W}^{N'} \twoheadrightarrow I\!R^N$ assigning to every mcst situation w a set of cost allocations in $I\!R^N$ a *multisolution*. For instance, by Theorem 4.7.3, non-conservative CC-rules are not solutions for mcst situations, but they are multisolutions.

Definition 5.4.1 A multisolution $M : \mathcal{W}^{N'} \twoheadrightarrow I\!R^N$ is a *cost monotonic* multisolution if for all mcst situations $w, w' \in \mathcal{W}^{N'}$ such that $w(e) \leq w'(e)$ for each $e \in E_{N'}$, it holds that

$$M(w) \subseteq compr^-(M(w')) \quad \text{and} \quad M(w') \subseteq compr^+(M(w)),$$

where $compr^-(B) = \{x \in I\!R^N | \exists b \in B \text{ s.t. } x_i \leq b_i\ \forall i \in N\}$ and $compr^+(B) = \{x \in I\!R^N | \exists b \in B \text{ s.t. } b_i \leq x_i\ \forall i \in N\}$, for each $B \subset I\!R^N$.

Cost monotonicity for multi-solutions is not satisfied in general by non-conservative CC-rules, as it is shown in Example 5.4.1, dealing with specific mcst situations where the optimal tree is unique.

Example 5.4.1 Consider the mcst situation $< N', w >$ with $N' = \{0,1,2,3\}$ and w as depicted in Figure 5.4 (left side). Note that there exists a unique $\sigma \in \Sigma_{N'}$ with $w \in K^\sigma$, where σ is such that $\sigma(1) = \{1,2\}$, $\sigma(2) = \{1,3\}$, $\sigma(3) = \{2,3\}$, $\sigma(4) = \{1,0\}$, $\sigma(5) = \{2,0\}$, $\sigma(6) = \{3,0\}$. We apply Definition

Figure 5.4: Two mcst situations w (left side) and w' (right side).

4.4.1 to the charge systems $\tilde{\mathcal{C}}$ introduced in Example 4.2.2 to calculate the allocations provided by the corresponding CC-rules on $< N', w >$. We have

$$\begin{aligned}&\chi^{\tilde{\mathcal{C}},\sigma}(w)\\&= 12 * (\tfrac{1}{2}, \tfrac{1}{2}, 0)^t + 16 * (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2})^t + 24 * (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2})^t\\&= (16, 16, 20)^t.\end{aligned}$$

Now, consider the mcst situation $< N', w' >$ with w' as depicted in Figure 5.4 (right side), where $w'(e) = w(e)$ for all $e \in E_{N'} \setminus \{1,2\}$ and $w'(\{1,2\}) > w(\{1,2\})$. Note that also for this mcst situation there exists a unique $\sigma' \in \Sigma_{N'}$ with $w' \in K^{\sigma}$, where σ' is such that $\sigma'(1) = \{1,3\}$, $\sigma'(2) = \{1,2\}$, $\sigma'(3) = \{2,3\}$, $\sigma'(4) = \{1,0\}$, $\sigma'(5) = \{2,0\}$, $\sigma'(6) = \{3,0\}$. We have

$$\begin{aligned}&\chi^{\tilde{\mathcal{C}},\sigma'}(w')\\&= 16 * (\tfrac{1}{2}, 0, \tfrac{1}{2})^t + 18 * (\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4})^t + 24 * (\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4})^t\\&= (18.5, 21, 18.5)^t.\end{aligned}$$

Agent 3 is better off in w', where the cost of edge $\{1,2\}$ is larger. So, $\{\chi^{\tilde{\mathcal{C}},\sigma'}(w')\}$ is not a subset of *compr*$^{+}(\{\chi^{\tilde{\mathcal{C}},\sigma}(w)\})$.

■

Before discussing properties of the Bird core as multisolution for mcst situations, we introduce the following propositions dealing with mcst situations originated from NA-semimetrics.

Proposition 5.4.1 *Let $w \in \overline{\mathcal{W}}^{N'}$, let Γ be an mcst for w and $\tau \in \Sigma_N$ be such that Γ and τ fit. Then,*

$$m^{\tau}_{\tau(j)}(c_w) = \min_{k \in (\tau(j))'} w(k, \tau(j)),$$

for each $j \in \{2, \dots, |N|\}$.

Proof Let $j \in \{2, \dots, |N|\}$. Note that by Remark 5.3.1

$$m^{\tau}_{\tau(j)}(c_w) = c_w([\tau(j)]) - c_w((\tau(j))) = \sum_{e \in E^{\Gamma}_{[\tau(j)]'}} w(e) - \sum_{e \in E^{\Gamma}_{(\tau(j))'}} w(e). \quad (5.13)$$

Since Γ and τ fit, we have $E^{\Gamma}_{[\tau(j)]'} \setminus E^{\Gamma}_{(\tau(j))'} = \{\{\tau(j), s\}\}$, for some $s \in (\tau(j))'$. Because $E^{\Gamma}_{[\tau(j)]'}$ is an mcst for $w_{|[\tau(j)]'}$, we have $s \in \arg\min_{k \in (\tau(j))'} w(\{k, \tau(j)\})$. So,

$$\sum_{e \in E^{\Gamma}_{[\tau(j)]'}} w(e) - \sum_{e \in E^{\Gamma}_{(\tau(j))'}} w(e) = \min_{k \in (\tau(j))'} w(k, \tau(j)). \quad (5.14)$$

From (5.13) and (5.14) follows the proposition. ■

Proposition 5.4.2 *Let $w, w' \in \overline{\mathcal{W}}^{N'}$ be NA-semimetric mcst situations such that $w(e) \leq w'(e)$ for each $e \in E_{N'}$. Then, it holds that*

$$m^{\tau}(c_w) \leq m^{\tau}(c_{w'}) \text{ for each } \tau \in \Sigma_N.$$

Proof Let $\tau \in \Sigma_N$. By Theorem 5.3.2 there exist two mcst's Γ and Γ' for w and w', respectively, such that they both fit with τ. First, note that

$$m^{\tau}_{\tau(1)}(c_w) = w(0, \tau(1)) \leq w'(0, \tau(1)) = m^{\tau}_{\tau(1)}(c_{w'}).$$

Further,

$$\begin{aligned} m^{\tau}_{\tau(j)}(c_w) &= \min\nolimits_{k \in (\tau(j))'} w(k, \tau(j)) \\ &\leq \min\nolimits_{k \in (\tau(j))'} w'(k, \tau(j)) \\ &= m^{\tau}_{\tau(j)}(c_{w'}), \end{aligned}$$

for each $j \in \{2, \dots, |N|\}$, where the first and the second equality follow by Proposition 5.4.1 and the inequality follows from $w(e) \leq w'(e)$ for each $e \in E_{N'}$.

■

Theorem 5.4.1 *The correspondence $\mathcal{BC}$ is a cost monotonic multisolution.*

Proof Let $w, w' \in \mathcal{W}^{N'}$ be such that $w(e) \leq w'(e)$ for each $e \in E_{N'}$. By Theorem 5.3.2 and properties of concave games, $\mathcal{BC}(w)$ is a convex set whose extreme points are the marginal vectors of the game $c_{\overline{w}}$, *i.e.* each element of $\mathcal{BC}(w)$ is a convex combination of marginal vectors of the game $c_{\overline{w}}$. Let $x \in \mathcal{BC}(w)$. There exist numbers α^τ, with $\tau \in \Sigma_N$, $0 \leq \alpha^\tau \leq 1$, $\sum_{\tau \in \Sigma_N} \alpha^\tau = 1$ and

$$x = \sum_{\tau \in \Sigma_N} \alpha^\tau \ m^\tau(c_{\overline{w}}). \tag{5.15}$$

Hence,

$$\begin{array}{rcl} x & = & \sum_{\tau \in \Sigma_N} \alpha^\tau \ m^\tau(c_{\overline{w}}) \\ & \leq & \sum_{\tau \in \Sigma_N} \alpha^\tau \ m^\tau(c_{\overline{w'}}) \\ & = & x' \in \mathcal{BC}(w'), \end{array} \tag{5.16}$$

where the inequality follows by Proposition 5.4.2 and the fact that $\overline{w}(e) \leq \overline{w}'(e)$ for each $e \in E_{N'}$ and the second equality by Theorem 5.3.2, implying that $\mathcal{BC}(w) \subseteq compr^-(\mathcal{BC}(w'))$. Using a similar argument the other way around in relations (5.16), it follows that $\mathcal{BC}(w') \subseteq compr^+(\mathcal{BC}(w))$, which concludes the proof. ■

To connect the cost monotonicity of the Bird core with cost monotonicity of Obligation rules, we need Proposition 5.4.3.

Proposition 5.4.3 *Let $F : \mathcal{W}^{N'} \to I\!R^N$ be a cost monotonic and efficient solution. Then,*

i) $F(\overline{w}) = F(w)$ for every $w \in \mathcal{W}^{N'}$;

ii) If F is also stable (i.e. $F(w') \in \mathcal{C}(c_{w'})$ for every $w' \in \mathcal{W}^{N'}$), then $F(w) \in \mathcal{BC}(w)$ for every $w \in \mathcal{W}^{N'}$.

Proof Let $w \in \mathcal{W}^{N'}$. First, note that by Definition 5.3.1,

$$\overline{w}(e) \leq w(e) \text{ for each } e \in E_{N'}. \tag{5.17}$$

Let Γ be an mcst for w.

i) By inequality (5.17) and cost monotonicity of F, $F(\overline{w}) \leq F(w)$. On the other hand Γ is an mcst for $\overline{w}$ too, and by efficiency of F

$$\sum_{i \in N} F_i(\overline{w}) = \sum_{i \in N} F_i(w) = w(\Gamma).$$

So, $F(\overline{w}) = F(w)$.

ii) By inequality (5.17),

$$c_{\overline{w}}(S) \leq c_w(S) \text{ for all } S \subseteq N,$$

and by Definition 5.3.1

$$c_{\overline{w}}(N) = c_w(N) = w(\Gamma).$$

Then, by stability of F, $F(\overline{w}) \in \mathcal{C}(c_{\overline{w}}) = \mathcal{BC}(w) \subseteq \mathcal{C}(c_w)$ and by result (i) $F(w) \in \mathcal{BC}(w)$ too.

■

Remark 5.4.1 Proposition 5.4.3 can be extended to multisolutions which are cost monotonic and efficient (Property 6.3.1 in Section 6.3) multisolutions. From this follows that $\mathcal{BC}$ is the "largest" cost monotonic stable multisolution.

Remark 5.4.2 In Section 5.2 we have proved that Obligation rules are both cost monotonic and stable solutions for mcst situations. So, by Proposition 5.4.3, it follows that for each $w \in \mathcal{W}^{N'}$, the set $\mathcal{F}(w) = \{\phi(w) \mid \phi \text{ is an Obligation rule}\}$ is a subset of the Bird core $\mathcal{BC}(w)$ and $\mathcal{F}(w) = \mathcal{F}(\overline{w})$.

Chapter 6

Additivity-based characterizations for cost allocation protocols

6.1 Introduction

In Section 4.5 we have introduced the definition of Obligation rules on Kruskal cones and the related notion of contribution matrix w.r.t. an obligation map $\hat{o}$ and an ordering $\sigma \in \Sigma_{E_{N'}}$ of the edges. As a consequence, Obligation rules are additive on each Kruskal cone in the space of mcst situations with a fixed number of users, *i.e.* the allocation vector provided by an Obligation rule on the mcst situation $w + w'$ is equal to the sum of allocation vectors provided by the same Obligation rule on each single mcst situation w and w', for each w, w' in the Kruskal cone K^σ. In this chapter, we show that the *Cone-wise Positive Linearity* (CPL), defined by Property 6.2.4 and reformulated for multi-solutions by Property 6.3.4, is a fundamental property for the axiomatic characterizations presented in this chapter. In fact, the CPL property is satisfied by every Obligation rule. In particular, the CPL property plays an important role for the axiomatic characterization of a special Obligation rule, the P-value (Branzei *et* *al.* (2006b)). More surprisingly, the CPL property extended to multi-solutions plays an important role also in the axiomatic characterization of the Bird core (Tijs *et al.* (2006b)).

In Section 6.2 we give an axiomatic characterization of the P-value. An axiomatic characterization of the Bird core is given in Section 6.3. Finally, in Section 6.4 the additivity property of Obligation rules is used to characterize these solutions for mcst situations using a value-theoretic approach based on sharing values for cost games.

Section 6.2 is based on Branzei, Moretti, Norde, Tijs (2004); section 6.3 is based on Tijs, Moretti, Branzei, Norde (2006b); section 6.4 is based on Moretti, Tijs, Branzei, Norde (2005).

6.2 An axiomatic characterization of the P-value

Recall that a *solution* for mcst situations is a map $F : \mathcal{W}^{N'} \to I\!R^N$ assigning to every mcst situation w a unique cost allocation in $I\!R^N$. Some interesting properties for solutions of mcst situations are the following:

Property 6.2.1 The solution F is *efficient* (EFF) if for each $w \in \mathcal{W}^{N'}$

$$\sum_{i \in N} F_i(w) = w(\Gamma),$$

where Γ is a minimum cost spanning network on N'.

Property 6.2.2 The solution F has the *Equal Treatment* (ET) property if for each $w \in \mathcal{W}^{N'}$ and for each $i, j \in N$ with $C_i(w) = C_j(w)$,

$$F_i(w) = F_j(w).$$

Property 6.2.3 The solution F has the *upper bounded contribution* (UBC) property if for each $w \in \mathcal{W}^{N'}$ and every (w, N')-component $C \neq \{0\}$

$$\sum_{i \in C \setminus \{0\}} F_i(w) \leq \min_{i \in C \setminus \{0\}} w(\{i, 0\}).$$

Property 6.2.4 The solution F has the *Cone-wise Positive Linearity* (CPL) property if for each $\sigma \in \Sigma_{E_{N'}}$, for each pair of mcst situations $w, \widehat{w} \in K^\sigma$ and for each pair $\alpha, \widehat{\alpha} \geq 0$, we have

$$F(\alpha w + \widehat{\alpha}\widehat{w}) = \alpha F(w) + \widehat{\alpha} F(\widehat{w}).$$

Proposition 6.2.1 *The P-value satisfies the properties EFF, ET, UBC and CPL.*

Proof Let $w \in \mathcal{W}^{N'}$ and let $\sigma \in \Sigma_{E_{N'}}$ be such that $w \in K^\sigma$. Then, the following considerations hold:

i) Let $\sigma(t_1)$, $\sigma(t_2), \ldots, \sigma(t_n)$, be the n edges of the mcst Γ corresponding to Kruskal order σ. These edges correspond to non-zero columns in $D^{\sigma,\hat{o}^*}$ and then the sum of coordinates of each column equals 1. Hence,

$$P(w) = D^{\sigma,\hat{o}^*} w^\sigma = \sum_{r=1}^{n} w(\sigma(t_r)) D^{\sigma,\hat{o}^*} e^{t_r},$$

$$\sum_{i \in N} P_i(w) = \sum_{r=1}^{n} w(\sigma(t_r)) \sum_{i \in N} \left(D^{\sigma,\hat{o}^*} e^{t_r}\right)_i = \sum_{r=1}^{n} w(\sigma(t_r)) = w(\Gamma),$$

which proves the EFF property.

ii) Note that if w is the zero function then it trivially follows that $P_i(w) = P_j(w)$ for each $i, j \in N$.

Consider $w \neq 0$ and define $k = min\{j | w(\sigma(j)) > 0\}$. Then, w^σ is of the form $(0, \ldots, 0, w(\sigma(k)), \ldots,\ w(\sigma(|E_{N'}|)))^t$, and for each $i \in N$

$$P_i(w) = \left(D^{\sigma,\hat{o}^*} w^\sigma\right)_i = \sum_{r=k}^{|E_{N'}|} (\hat{o}_i^*(\pi^{\sigma,r-1}) - \hat{o}_i^*(\pi^{\sigma,r})) w(\sigma(r)). \tag{6.1}$$

Let C be a (w, N')-component and consider two users $i, j \in C$. By Remark 4.6.1 this means that i and j are connected in the graph $< N', F^{\sigma,k-1} >$ and so also in $< N', F^{\sigma,r} >$ for every $r \in \{k, \ldots, |E_{N'}|\}$. Then, for each $r \in \{k, \ldots, |E_{N'}|\}$

$$\hat{o}_i^*(\pi^{\sigma,r-1}) - \hat{o}_i^*(\pi^{\sigma,r}) = \hat{o}_j^*(\pi^{\sigma,r-1}) - \hat{o}_j^*(\pi^{\sigma,r}).$$

Hence, by (6.1), $P_i(w) {=} P_j(w)$, which proves the ET property.

iii) If w is the zero function then it directly follows that $\sum_{i\in S} P_i(w) = 0 = \min_{i\in S} w(\{i,0\})$, for each $S \in 2^N \setminus \{\emptyset\}$.

Consider $w \neq 0$ and let $C \neq \{0\}$ be a (w, N')-component. Note that there exists $m \in \{1,\ldots,|E_{N'}|\}$ such that $\sigma(m) \subseteq C \cup \{0\}$ and $w(\sigma(m)) = \min_{i\in C\setminus\{0\}} w(\{i,0\})$. Define $k = min\{j|w(\sigma(j)) > 0\}$. If $m < k$, then $0 \in C$ and since nodes in $C\setminus\{0\}$ pay nothing according to $P(w)$, the UBC property holds. Instead, if $m \geq k$ then

$$\begin{aligned}
&\textstyle\sum_{i\in C\setminus\{0\}} P_i(w) = \\
&= \textstyle\sum_{i\in C\setminus\{0\}} \sum_{r=k}^{m} w(\sigma(r))(\hat{o}_i^*(\pi^{\sigma,r-1}) - \hat{o}_i^*(\pi^{\sigma,r})) \leq \\
&\leq w(\sigma(m)) \textstyle\sum_{i\in C\setminus\{0\}} \sum_{r=k}^{m} (\hat{o}_i^*(\pi^{\sigma,r-1}) - \hat{o}_i^*(\pi^{\sigma,r})) = \\
&= w(\sigma(m)) \textstyle\sum_{i\in C\setminus\{0\}} \hat{o}_i^*(\pi^{\sigma,k-1}) = \\
&= w(\sigma(m))
\end{aligned} \tag{6.2}$$

where the first equality follows from $\hat{o}_i^*(\pi^{\sigma,u}) = 0$ for all $u \in \{m,\ldots,|E_{N'}|\}$ and for each $i \in C$, and in the last one we use the fact that all nodes in $C\setminus\{0\}$ are connected in the graph $< N', F^{\sigma,k-1} >$. Note that (6.2) proves the UBC property.

iv) The CPL property follows trivially from the definition of P.

■

Theorem 6.2.1 *The P-value is the unique solution which satisfies the properties EFF, ET, UBC and CPL on the class $\mathcal{W}^{N'}$ of mcst situations.*

Proof We already know by Proposition 6.2.1 that the P-value satisfies the four properties EFF, ET, UBC and CPL. To prove the uniqueness consider a map $\psi : \mathcal{W}^{N'} \to I\!R^N$ satisfying EFF, ET, UBC and CPL.

Let $\sigma \in \Sigma_{E_{N'}}$ and $k \in \{1,\ldots,|E_{N'}|\}$. First, we will show that for mcst situation $e^{\sigma,k} \in K^\sigma$, $\psi(e^{\sigma,k}) = P(e^{\sigma,k})$. By UBC, for each $(e^{\sigma,k}, N')$-component $C \neq \{0\}$

$$\sum_{i\in C\setminus\{0\}} \psi_i(e^{\sigma,k}) \leq \min_{i\in C\setminus\{0\}} e^{\sigma,k}(\{i,0\}) = \begin{cases} 0 & \text{if } 0 \in C \\ \\ 1 & \text{if } 0 \notin C \end{cases} \tag{6.3}$$

implying that

$$\sum_{i\in N}\psi_i(e^{\sigma,k})=\sum_{C\in\mathcal{C}(e^{\sigma,k})}\sum_{j\in C\setminus\{0\}}\psi_j(e^{\sigma,k})\le|\mathcal{C}(e^{\sigma,k})|-1=e^{\sigma,k}(\Gamma),$$

where Γ is a minimum spanning network on N' for mcst situation $e^{\sigma,k}$. By EFF, we have $\sum_{i\in N}\psi_i(e^{\sigma,k})=e^{\sigma,k}(\Gamma)$, and then the inequalities in equation (6.3) are equalities. Finally, by ET, we find that

$$\psi_i(e^{\sigma,k})\quad=\begin{cases}0 & \text{if } 0\in C_i(e^{\sigma,k})\\ \\ \frac{1}{|C_i(e^{\sigma,k})|} & \text{if } 0\notin C_i(e^{\sigma,k})\end{cases}=\hat{o}_i^*(\pi^{\sigma,k-1})=P_i(e^{\sigma,k})\tag{6.4}$$

for each $i\in N$, where the last equality follows by relation (4.22). Note that we only used EFF, ET and UBC properties to get relation (6.4) for mcst situation $e^{\sigma,k}$. Now, we use the CPL property to show that for any mcst situation $w\in\mathcal{W}^{N'}$, $\psi(w)=P(w)$. Let $\sigma\in\Sigma_{E_{N'}}$ be such that $w\in K^\sigma$. From the CPL property of ψ and relation (2.2) it follows

$$\psi(w)=w(\sigma(1))\psi(e^{\sigma,1})+\sum_{k=2}^{|E_{N'}|}\big(w(\sigma(k))-w(\sigma(k-1))\big)\psi(e^{\sigma,k}).\tag{6.5}$$

Further, from (4.21), (6.4) and (6.5) we obtain $\psi(w)=P(w)$. ■

To prove the logical independence of the four properties we need to consider some other solutions on $\mathcal{W}^{N'}$:

i) z, such that $z_i(w)=0$ for each $i\in N$ and mcst situation w;

ii) P^τ, with $\tau\in\Sigma_N$;

iii) ϵ, such that $\epsilon_i(w)=\frac{w(\Gamma)}{|N|}$ for each $i\in N$, where Γ is a minimum spanning network on N' for mcst situation w;

iv) $\mathcal{D}$, such that (w,N')-components "pay" proportionally to their "distance" from the source, *i.e.* such that for each $i\in N$

$$\mathcal{D}_i(w)=\begin{cases}\frac{1}{|C_i(w)|}\frac{\min_{j\in C_i(w)}w(\{j,0\})}{\sum_{C\in\mathcal{C}(w)}\min_{j\in C\setminus\{0\}}w(\{j,0\})}\,w(\Gamma) & \text{if } 0\notin C_i(w)\\ \\ 0 & \text{if } 0\in C_i(w),\end{cases}$$

where Γ is a minimum spanning network on N' for mcst situation w.

Proposition 6.2.2 *The axioms EFF, ET, UBC and CPL are logically independent.*

Proof The logical independence of the four properties follows from the following table.

	EFF	ET	UBC	CPL
z	no	yes	yes	yes
P^τ	yes	no	yes	yes
ϵ	yes	yes	no	yes
$\mathcal{D}$	yes	yes	yes	no

It is trivial to show that z satisfies axioms ET, UBC and CPL but not EFF. Being an Obligation rule, P^τ satisfies the CPL, EFF and UBC, as it will be proved in Proposition 6.2.3. In order to show that P^τ does not satisfy the ET property, consider the mcst situation $< N', w >$ with $N' = \{0, 1, 2\}$ and w as depicted in Figure 6.1, and let $\tau \in \Sigma_{\{1,2\}}$ be such that $\tau(1) = 1$ and $\tau(2) = 2$. Then, $P_1^\tau(w) = 0$ and $P_2^\tau(w) = 1$.

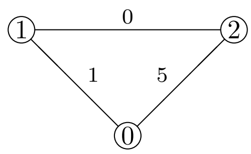


Figure 6.1: The mcst situation $< \{0, 1, 2\}, w >$.

To prove the third row, it is easy to see that ϵ satisfies EFF, ET and CPL. To see that ϵ does not satisfy the UBC property, consider again the mcst situation $< N', w >$ with $N' = \{0, 1, 2\}$ and w as depicted in Figure 6.1. Then, $\epsilon(w) = (\frac{3}{2}, \frac{3}{2})$, *i.e.* player 1 in the (w, N')-component $C = \{1\}$ pays more than $\min_{i \in C\setminus\{0\}} w(\{i, 0\}) = 1$.

For the proof of the last row, note that $\mathcal{D}$ trivially satisfies EFF and ET. Let $w \in \mathcal{W}^{N'}$ and let $\sigma \in \Sigma_{E_{N'}}$ be such that $w \in K^\sigma$. The UBC property follows from the fact that for each component $C \in \mathcal{C}(w)$, if $0 \notin C(w)$ then

$$\min_{j\in C} w(\{j, 0\}) = \frac{\min_{j\in C(w)} w(\{j,0\})}{\sum_{C\in\mathcal{C}(w)} \min_{j\in C\setminus\{0\}} w(\{j,0\})} \sum_{C\in\mathcal{C}(w)} \min_{j\in C\setminus\{0\}} w(\{j, 0\}) \geq$$

$$\geq \frac{\min_{j\in C(w)} w(\{j,0\})}{\sum_{C\in\mathcal{C}(w)} \min_{j\in C\setminus\{0\}} w(\{j,0\})} w(\Gamma) = \sum_{j\in C} \mathcal{D}_j(w).$$

In order to prove that $\mathcal{D}$ does not satisfy the CPL property, consider the two mcst situations $< N', w' >$ and $< N', w'' >$, with $N' = \{0, 1, 2\}$ and w', w'' as depicted in Figure 6.2.

Figure 6.2: Two mcst situations in the same Kruskal cone.

Note that $w', w'' \in K^{\sigma}$ with $\sigma(1) = \{0, 2\}$, $\sigma(2) = \{1, 2\}$ and $\sigma(3) = \{0, 1\}$. Then, $\mathcal{D}(w') = (\frac{1}{1} \times \frac{4}{4} \times 1, 0) = (1, 0)$ and $\mathcal{D}(w'') = (\frac{1}{1} \times \frac{20}{21} \times 3, \frac{1}{1} \times \frac{1}{21} \times 3) = (\frac{60}{21}, \frac{3}{21})$.

Differently, the sum of the two mcst situations $w' + w''$ is the mcst situation $< N', w' + w'' >$ with $w' + w''$ depicted in Figure 6.3. Finally, $\mathcal{D}(w' + w'') =$

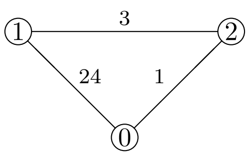


Figure 6.3: The mcst situation $< \{0, 1, 2\}, w' + w'' >$.

$(\frac{1}{1} \times \frac{24}{25} \times 4, \frac{1}{1} \times \frac{1}{25} \times 4) = (\frac{96}{25}, \frac{4}{25}) \neq (\frac{81}{21}, \frac{3}{21}) = \mathcal{D}(w') + \mathcal{D}(w'')$. ■

We conclude this section with Proposition 6.2.3 claiming that every Obligation rule satisfies three of the four properties presented above.

Proposition 6.2.3 *Obligation rules satisfy the properties EFF, UBC and CPL.*

Proof We already proved in Proposition 4.5.3 that Obligation rules are efficient.

As we already observed, the CPL property follows trivially from the definition of Obligation rule, since the contribution matrix w.r.t. an obligation map $\hat{o}$ and an ordering $\sigma \in \Sigma_{E_{N'}}$ is the same for each $w \in K^{\sigma}$.

The same arguments used to prove that the P-value satisfies the UBC property works also for every Obligation rule. More precisely, let $\hat{o}$ be an obligation map on $\Theta(N')$, let $w \in \mathcal{W}^{N'}$ and let $\sigma \in \Sigma_{E_{N'}}$ be such that $w \in K^{\sigma}$.

If w is the zero function then for the Obligation rule $\phi^{\hat{o}}$ it directly follows that $\sum_{i \in S} \phi_i^{\hat{o}}(w) = 0 = \min_{i \in S} w(\{i, 0\})$, for each $S \in 2^N \setminus \{\emptyset\}$.

Consider $w \neq 0$ and let $C \neq \{0\}$ be a (w, N')-component. Note that there exists an $m \in \{1, \ldots, |E_{N'}|\}$ such that $\sigma(m) \subseteq C \cup \{0\}$ and $w(\sigma(m)) = \min_{i \in C \setminus \{0\}} w(\{i, 0\})$. Define $k = min\{j | w(\sigma(j)) > 0\}$. If $m < k$, then $0 \in C$ since nodes in $C \setminus \{0\}$ pay nothing according to $\phi^{\hat{o}}(w)$ and the UBC property holds. Instead, if $m \geq k$ then

$$\begin{aligned}
&\textstyle\sum_{i \in C \setminus \{0\}} \phi_i^{\hat{o}}(w) = \\
&= \textstyle\sum_{i \in C \setminus \{0\}} \sum_{r=k}^{m} w(\sigma(r))(\hat{o}_i(\pi^{\sigma, r-1}) - \hat{o}_i(\pi^{\sigma, r})) \leq \\
&\leq w(\sigma(m)) \textstyle\sum_{i \in C \setminus \{0\}} \sum_{r=k}^{m} (\hat{o}_i(\pi^{\sigma, r-1}) - \hat{o}_i(\pi^{\sigma, r})) = \\
&= w(\sigma(m)) \textstyle\sum_{i \in C \setminus \{0\}} \hat{o}_i(\pi^{\sigma, k-1}) = \\
&= w(\sigma(m))
\end{aligned} \tag{6.6}$$

where in the first equality we use that $\hat{o}_i(\pi^{\sigma, u}) = 0$ for all $u \in \{m, \ldots, |E_{N'}|\}$ and for each $i \in C$, and in the last one we use the fact that all nodes in $C \setminus \{0\}$ are connected in the graph $< N', F^{\sigma, k-1} >$. Note that relation (6.6) proves the UBC property. ■

6.3 An axiomatic characterization of the Bird core

In order to introduce an axiomatic characterization of the Bird core, we need to prove the following fact for NA-semimetric mcst situations.

Lemma 6.3.1 *Let $w, w' \in \mathcal{W}^{N'}$ and let $\sigma \in \Sigma_{E_{N'}}$ be such that $w, w' \in K^{\sigma}$. Let $\alpha, \alpha' \geq 0$. Then, $\alpha\overline{w}, \alpha'\overline{w'}, \overline{\alpha w + \alpha' w'} \in K^{\hat{\sigma}}$ for some $\hat{\sigma} \in \Sigma_{E_{N'}}$.*

Proof By relation (5.5), for each edge $e \in E_{N'}$, there is an edge $\bar{e} \in E_{N'}$ such that $\overline{w}(e) = w(\bar{e})$: given that $e = \{i, j\}$, $\bar{e}$ is such that $w(\bar{e}) = \min_{P \in \mathcal{P}_{ij}^{N'}} t(\mathbf{p})$. Note that for each w_1 in the same cone K^{σ} as w we have $\overline{w_1}(e) = w_1(\bar{e})$. This implies that for all pairs of edges $e_1, e_2 \in E_{N'}$

$$\overline{w}(e_1) \leq \overline{w}(e_2) \Leftrightarrow w(\bar{e}_1) \leq w(\bar{e}_2) \Leftrightarrow \overline{w_1}(e_1) \leq \overline{w_1}(e_2).$$

So, for each $\bar{\sigma} \in \Sigma_{E_{N'}}$ we have:

$$\overline{w} \in K^{\bar{\sigma}} \Leftrightarrow \overline{w_1} \in K^{\bar{\sigma}}.$$

Using this fact, respectively, for αw, $\alpha' w'$ and $\alpha w + \alpha' w' \in K^{\sigma}$ in the role of w_1, we obtain

$$\overline{w} \in K^{\bar{\sigma}} \Leftrightarrow \alpha\overline{w}, \alpha'\overline{w'}, \overline{\alpha w + \alpha' w'} \in K^{\bar{\sigma}},$$

for each $\bar{\sigma} \in \Sigma_{E_{N'}}$. ■

Proposition 6.3.1 *Let $w, w' \in \mathcal{W}^{N'}$ and let $\sigma \in \Sigma_{E_{N'}}$ be such that $w, w' \in K^{\sigma}$. Let $\alpha, \alpha' \geq 0$. Then,*

i) $\overline{\alpha w + \alpha' w'} = \alpha\overline{w} + \alpha'\overline{w'}$;

ii) $c_{\overline{\alpha w + \alpha' w'}} = \alpha c_{\overline{w}} + \alpha' c_{\overline{w'}}$.

[*The NA-semimetric mcst situations $\overline{w}, \overline{w'}, \overline{\alpha w + \alpha' w'}$ are obtained via reduction of the weight functions $w, w', \alpha w + \alpha' w'$, respectively.*]

Proof

i) Let $e = \{i, j\} \in E_{N'}$. We have

$$\begin{aligned}
\overline{\alpha w + \alpha' w'}(e) &= (\alpha w + \alpha' w')(\hat{e}) \\
&= \alpha w(\hat{e}) + \alpha' w'(\hat{e}) \\
&= \alpha\overline{w}(e) + \alpha'\overline{w'}(e),
\end{aligned}$$

where $\hat{e} \in E_{N'}$ is such that $w(\hat{e}) = \min_{\mathbf{p}\in\mathcal{P}_{ij}^{N'}} \max_{a\in E(\mathbf{p})} \big((\alpha w+\alpha' w')(a)\big)$, and where the second equality follows from the fact that w, w' and $(\alpha w + \alpha' w')$ all belong to K^{σ};

ii) Note that, by Lemma 6.3.1, $\alpha\overline{w}, \alpha'\overline{w'}, \overline{\alpha w + \alpha' w'} \in K^{\bar{\sigma}}$ for some $\bar{\sigma} \in \Sigma_{E_{N'}}$. For each $S \in 2^N \setminus \{\emptyset\}$, there is, according to Remark 5.3.3, a common mcst Γ_S for $\alpha\overline{w}, \alpha'\overline{w'}$ and $\overline{\alpha w + \alpha' w'}$. Hence,

$$\begin{aligned} \alpha c_{\overline{w}}(S) + \alpha' c_{\overline{w'}}(S) &= \textstyle\sum_{e\in\Gamma_S} \alpha\overline{w}(e) + \sum_{e\in\Gamma_S} \alpha'\overline{w'}(e) \\ &= \textstyle\sum_{e\in\Gamma_S} \big(\alpha\overline{w'}(e) + \alpha'\overline{w}(e)\big) \\ &= \textstyle\sum_{e\in\Gamma_S} \big(\overline{\alpha w + \alpha' w'}(e)\big) \\ &= c_{\overline{\alpha w+\alpha' w'}}(S), \end{aligned}$$

where the third equality follows by (i).

■

Some interesting properties for multisolutions on the class of mcst situations are the following.

Property 6.3.1 The multisolution G is *efficient*$^\bigstar$ (EFF$^\bigstar$) if for each $w \in \mathcal{W}^{N'}$ and for each $x \in G(w)$

$$\sum_{i\in N} x_i = w(\Gamma),$$

where Γ is a minimum cost spanning network for w on N'.

Property 6.3.2 The multisolution G has the *positive* (POS) property if for each $w \in \mathcal{W}^{N'}$ and for each $x \in G(w)$

$$x_i \geq 0$$

for each $i \in N$.

Property 6.3.3 The multisolution G has the *Upper Bounded Contribution*$^\bigstar$ (UBC$^\bigstar$) property if for each $w \in \mathcal{W}^{N'}$ and every (w, N')-component $C \neq \{0\}$

$$\sum_{i\in C\setminus\{0\}} x_i \leq \min_{i\in C\setminus\{0\}} w(\{i, 0\})$$

for each $x \in G(w)$.

Property 6.3.4 The multisolution G has the *Cone-wise Positive Linearity*$^\bigstar$ (CPL$^\bigstar$) property if for each $\sigma \in \Sigma_{E_{N'}}$, for each pair of mcst situations $w, \widehat{w} \in K^\sigma$ and for each pair $\alpha, \widehat{\alpha} \geq 0$, we have

$$G(\alpha w + \widehat{\alpha}\widehat{w}) = \alpha G(w) + \widehat{\alpha} G(\widehat{w}).$$

[Here we denote by $\alpha G(w) + \widehat{\alpha} G(\widehat{w})$ the set $\{\alpha x + \widehat{\alpha}\widehat{x} | x \in G(w), \widehat{x} \in G(\widehat{w}\}$.]

Proposition 6.3.2 *The Bird core $\mathcal{BC}$ satisfies the properties EFF$^\bigstar$, POS, UBC$^\bigstar$ and CPL$^\bigstar$.*

Proof Let $w \in \mathcal{W}^{N'}$ and let $\sigma \in \Sigma_{E_{N'}}$ be such that $w \in K^\sigma$. Since $\mathcal{BC}(w) = \mathcal{C}(c_{\overline{w}})$, the following considerations hold:

i) For each allocation $x \in \mathcal{BC}(w)$, $\sum_{i \in N} x_i = w(\Gamma)$ for some mcst Γ for $\overline{w}$ by the efficiency property of the core of the game $c_{\overline{w}}$. So, $\mathcal{BC}$ has the EFF$^\bigstar$ property.

ii) For each allocation $x \in \mathcal{BC}(w)$, $x_i \geq 0$ for each $i \in N$ since the Bird core is the convex hull of all Bird allocations in the mcst $\overline{w}$, which are vectors in $\mathbb{R}^N_+$. So, $\mathcal{BC}$ has the POS property.

iii) For each (w, N')-component $C \neq \{0\}$ and each $x \in \mathcal{BC}(w)$

$$\sum_{i \in C \setminus \{0\}} x_i \leq c_{\overline{w}}(C \setminus \{0\}) = \min_{i \in C \setminus \{0\}} w(\{i, 0\})$$

by coalitional rationality of the core of the game $c_{\overline{w}}$. So, $\mathcal{BC}$ has the UBC$^\bigstar$ property.

iv) Let $\sigma \in \Sigma_{E_{N'}}$, let $w, w' \in \mathcal{W}^{N'}$ be such that $w, w' \in K^\sigma$ and let $\alpha, \alpha' \geq 0$. Since the core is additive on the class of concave games (see Dragan *et al.* (1989)), we have

$$\mathcal{BC}(\alpha w + \alpha' w') = \mathcal{C}(c_{\overline{\alpha w + \alpha' w'}}) = \alpha \mathcal{C}(c_{\overline{w}}) + \alpha' \mathcal{C}(c_{\overline{w'}}) = \alpha \mathcal{BC}(w) + \alpha' \mathcal{BC}(w').$$

Hence, $\mathcal{BC}$ has the CPL$^\bigstar$ property.

■

Inspired by the axiomatic characterization of the P-value (Branzei *et al.* (2004)) we provide the following theorem.

Theorem 6.3.2 *The Bird core $\mathcal{BC}$ is the largest multisolution which satisfies EFF$^\bigstar$, POS, UBC$^\bigstar$ and CPL$^\bigstar$, i.e. for each multisolution F which satisfies EFF$^\bigstar$, POS, UBC$^\bigstar$ and CPL$^\bigstar$, we have $F(w) \subseteq \mathcal{BC}(w)$, for each $w \in \mathcal{W}^{N'}$.*

Proof We already know by Proposition 6.3.2 that the Bird core $\mathcal{BC}$ satisfies the four properties EFF$^\bigstar$, POS, UBC$^\bigstar$ and CPL$^\bigstar$.
Let $\Psi : \mathcal{W}^{N'} \twoheadrightarrow \mathbb{R}^N$ be a multisolution satisfying EFF$^\bigstar$, POS, UBC$^\bigstar$ and CPL$^\bigstar$. Let $w \in \mathcal{W}^{N'}$ and $\sigma \in \Sigma_{E_{N'}}$ be such that $w \in K^\sigma$. We have to prove that $\Psi(w) \subseteq \mathcal{BC}(w)$.
First, note that by the CPL$^\bigstar$ property of Ψ

$$\Big(w(\sigma(1))\Psi(e^{\sigma,1}) + \sum_{k=2}^{|E_{N'}|} \big(w(\sigma(k)) - w(\sigma(k-1))\big)\Psi(e^{\sigma,k})\Big) = \Psi(w). \tag{6.7}$$

Let $x \in \Psi(w)$. According to (6.7) there exists $x^{e^{\sigma,k}} \in \Psi(e^{\sigma,k})$ for each $k \in \{1, \dots, |E_{N'}|\}$ such that

$$x = w(\sigma(1))x^{e^{\sigma,1}} + \sum_{k=2}^{|E_{N'}|} \big(w(\sigma(k)) - w(\sigma(k-1))\big)x^{e^{\sigma,k}}.$$

By the UBC$^\bigstar$ property, for each $k \in \{1, \dots, |E_{N'}|\}$ and for each $(e^{\sigma,k}, N')$-component $C \neq \{0\}$ we have

$$\sum_{i \in C\setminus\{0\}} x_i^{e^{\sigma,k}} \leq \min_{i \in C\setminus\{0\}} e^{\sigma,k}(\{i,0\}) = \begin{cases} 0 & \text{if } 0 \in C \\ \\ 1 & \text{if } 0 \notin C \end{cases} \tag{6.8}$$

implying that

$$\sum_{i \in N} x_i^{e^{\sigma,k}} = \sum_{C \in \mathcal{C}(e^{\sigma,k})} \sum_{j \in C\setminus\{0\}} x_j^{e^{\sigma,k}} \leq |\mathcal{C}(e^{\sigma,k})| - 1 = e^{\sigma,k}(\Gamma),$$

where Γ is a minimum spanning network on N' for the simple mcst situation $e^{\sigma,k}$.
By the EFF$^\bigstar$ property, we have $\sum_{i \in N} x_i^{e^{\sigma,k}} = e^{\sigma,k}(\Gamma)$, and then inequalities in relation (6.8) are equalities, that is

$$\sum_{i \in C\setminus\{0\}} x_i^{e^{\sigma,k}} = \begin{cases} 0 & \text{if } 0 \in C \\ \\ 1 & \text{if } 0 \notin C. \end{cases} \tag{6.9}$$

Now, consider the game $c_{\overline{e^{\sigma,k}}}$ corresponding to the simple mcst situation $\overline{e^{\sigma,k}}$. Note that for each $S \in 2^N \setminus \{\emptyset\}$,

$$c_{\overline{e^{\sigma,k}}}(S) = \Big|\{C \mid C \text{ is a } (e^{\sigma,k}, N') - \text{component}, C \cap S \neq \emptyset, 0 \notin C\}\Big|,$$

which is the number of $(e^{\sigma,k}, N')$-components not connected to 0 in $e^{\sigma,k}$ with at least one node in the player set S.

By (6.9) and the POS property, it follows that $\sum_{i\in S} x_i^{e^{\sigma,k}} \leq c_{\overline{e^{\sigma,k}}}(S)$ and together with the EFF$^\bigstar$ property we have $x^{e^{\sigma,k}} \in \mathcal{C}(c_{\overline{e^{\sigma,k}}}) = \mathcal{BC}(e^{\sigma,k})$. Moreover, from Proposition 6.3.1 it follows

$$x = \Big(w(\sigma(1))x^{e^{\sigma,1}} + \sum_{k=2}^{|E_{N'}|} \big(w(\sigma(k)) - w(\sigma(k-1))\big)x^{e^{\sigma,k}}\Big) \in \mathcal{C}(c_{\overline{w}}) = \mathcal{BC}(w). \tag{6.10}$$

Keeping into account relation (6.7), we have $\Psi(w) \subseteq \mathcal{BC}(w)$. ■

6.4 Sharing values for mcst games

In this section the set of Obligation rules, and, consequently, the set of conservative CC-rules, will be considered from a value-theoretic point of view.

First, we introduce some notions. The *dual unanimity game* (N, u_S^*) on $S \subseteq N$ is the game described by $u_S^*(T) = 1$ if $S \cap T \neq \emptyset$ and $u_S^*(T) = 0$, otherwise. It is well-known that the dual unanimity games form a basis of the linear space $\mathcal{G}^N$ implying that every cost game (N, c) can be written as a linear combination of dual unanimity games in a unique way, *i.e.* $c = \sum_{S\subseteq N, S\neq\emptyset} \alpha_S(c)u_S^*$. The coefficients $(\alpha_S(c))_{S\in 2^N\setminus\{\emptyset\}}$ are called *dual unanimity coefficients* of the cost game (N, c).

A *sharing system* is a map $q : 2^N \setminus \{\emptyset\} \to \mathbb{R}_+^N$ such that $q(S) \in \Delta(S)$, for every nonempty coalition S. With every sharing system q one can associate a *sharing value* m^q, defined for every $c \in \mathcal{G}^N$ and every $i \in N$ by

$$m_i^q(c) = \sum_{S\subseteq N: i\in S} q_i(S)\alpha_S(c) \tag{6.11}$$

where $\alpha_S(c)$ is the dual unanimity coefficient of S in the game c.

The most well-known value in the theory of cost games is the *Shapley value*, introduced by Shapley (1953). This value can be described in several ways. In this setting the definition of the Shapley value using dual unanimity games fits better. In formula

$$\phi_i(c) = \sum_{S \subseteq N: i \in S} \frac{\alpha_S(c)}{|S|}. \tag{6.12}$$

for each $i \in N$. Note that relation (6.12) can be obtained by relation (6.11) with $q_i(S) = \frac{1}{|S|}$, for each $i \in N$ and $S \subseteq N$ such that $i \in S$.

Further, with every obligation function o one can associate a special sharing value m^o.

The following lemmas are helpful in relating sharing values with Obligation rules.

Lemma 6.4.1 *Let $w \in \mathcal{W}^{N'}$ and let $\sigma \in \Sigma_{E_{N'}}$ be such that $w \in K^\sigma$. Then,*

i) $\overline{w} = \overline{w}(\sigma(1))\overline{e^{\sigma,1}} + \sum_{k=2}^{|E_{N'}|} \big(\overline{w}(\sigma(k)) - \overline{w}(\sigma(k-1))\big)\overline{e^{\sigma,k}}$,

ii) $c_{\overline{w}} = \overline{w}(\sigma(1))c_{\overline{e^{\sigma,1}}} + \sum_{k=2}^{|E_{N'}|} \big(\overline{w}(\sigma(k)) - \overline{w}(\sigma(k-1))\big)c_{\overline{e^{\sigma,k}}}$,

where $\overline{e^{\sigma,k}} \in K^\sigma$, $k \in \{1, 2, \ldots, |E_{N'}|\}$, is the the minimal mcst situation on $e^{\sigma,k} \in K^\sigma$.

Proof The proof follows from relation (2.2) and by Proposition 6.3.1. ■

Let o be an obligation function and let $\hat{o}$ be the corresponding obligation map on $\Theta(N')$. Let $w \in \mathcal{W}^{N'}$. From relation (2.2) and the definition of Obligation rule via relation (4.14), it follows that $\phi^{\hat{o}}(w)$ can be calculated as linear combination of $\phi^{\hat{o}}(e^{\sigma,k})$, $k \in \{1, \ldots, |E_{N'}|\}$. More precisely, let $\sigma \in \Sigma_{E_{N'}}$ be such that $w \in K^\sigma$, then

$$\phi^{\hat{o}}(w) = w(\sigma(1))\phi^{\hat{o}}(e^{\sigma,1}) + \textstyle\sum_{k=2}^{|E_{N'}|} \big(w(\sigma(k)) - w(\sigma(k-1))\big)\phi^{\hat{o}}(e^{\sigma,k}), \tag{6.13}$$

where for $k \in \{1, \ldots, |E_{N'}|\}$ and $e^{\sigma,k} \in K^\sigma$

$$\phi^{\hat{o}}(e^{\sigma,k}) = \textstyle\sum_{r=k+1}^{|E_{N'}|} \big(\hat{o}(\pi^{\sigma,r-1}) - \hat{o}(\pi^{\sigma,r})\big) = \hat{o}(\pi^{\sigma,k}) = \sum_{V \in \pi^{\sigma,k}: 0 \notin V} o(V). \tag{6.14}$$

Further, from Proposition 4.5.3 (efficiency of Obligation rules), Theorems 5.2.2 (cost monotonicity of Obligation rules) and Proposition 5.4.3.i it follows that for every $w \in \mathcal{W}^{N'}$

$$\phi^{\hat{o}}(w) = \phi^{\hat{o}}(\overline{w}). \tag{6.15}$$

Now, we introduce the following lemma.

Lemma 6.4.2 *Let $\sigma \in \Sigma_{E_{N'}}$, and let $e^{\sigma,k} \in K^\sigma$, $k \in \{1, \ldots, |E_{N'}|\}$. Let $\hat{o}$ be an obligation map on $\Theta(N')$. Then,*

i) $c_{\overline{e^{\sigma,k}}} = \sum_{V \in \pi^{\sigma,k}: 0 \notin V} u^*_V$,

ii) $m^o(c_{\overline{e^{\sigma,k}}}) = \phi^{\hat{o}}(\overline{e^{\sigma,k}})$,

where $\overline{e^{\sigma,k}} \in K^\sigma$, $k \in \{1, 2, \ldots, |E_{N'}|\}$, *is the the minimal mcst situation on* $e^{\sigma,k} \in K^\sigma$.

Proof First, note that by Lemma 6.3.1, $\overline{e^{\sigma,k}} \in K^\sigma$.

i) follows from the fact that for each $S \in 2^N \setminus \{\emptyset\}$,

$$c_{\overline{e^{\sigma,k}}}(S) = |\{V \ : \ V \text{ is a } (\overline{e^{\sigma,k}}, N') - \text{component}, V \cap S \neq \emptyset, 0 \notin V\}|,$$

where the $(\overline{e^{\sigma,k}}, N')$-components are precisely the elements of the partition $\pi^{\sigma,k}$;

ii)

$$\begin{array}{l} m^o(c_{\overline{e^{\sigma,k}}}) = m^o(\sum_{V \in \pi^{\sigma,k}: 0 \notin V} u^*_V) \\ = \sum_{V \in \pi^{\sigma,k}: 0 \notin V} m^o(u^*_V) \\ = \sum_{V \in \pi^{\sigma,k}: 0 \notin V} o(V) = \phi^{\hat{o}}(\overline{e^{\sigma,k}}), \end{array} \tag{6.16}$$

where the first equality follows by part i) of Proof, the second equality follows from linearity of m^o, the third equality follows from relation (6.11) and the last equality follows from relations (6.14) and (6.15).

■

Finally, we introduce the main result of this section.

Theorem 6.4.3 *Let $w \in \mathcal{W}^{N'}$ and let $\sigma \in \Sigma_{E_{N'}}$ be such that $w \in K^\sigma$. Let $\hat{o}$ be an obligation map on $\Theta(N')$. Then,*

$$m^o(c_{\overline{w}}) = \phi^{\hat{o}}(\overline{w}).$$

Proof Note that

$$\begin{aligned}
&m^o(c_{\overline{w}}) \\
&= \overline{w}(\sigma(1))m^o(c_{\overline{e^{\sigma,1}}}) + \textstyle\sum_{k=2}^{|E_{N'}|} \big(\overline{w}(\sigma(k)) - \overline{w}(\sigma(k-1))\big)m^o(c_{\overline{e^{\sigma,k}}}) \\
&= \overline{w}(\sigma(1))\phi^{\hat{o}}(\overline{e^{\sigma,1}}) + \textstyle\sum_{k=2}^{|E_{N'}|} \big(\overline{w}(\sigma(k)) - \overline{w}(\sigma(k-1))\big)\phi^{\hat{o}}(\overline{e^{\sigma,k}}) \\
&= \phi^{\hat{o}}(\overline{w}),
\end{aligned}$$

where the first equality follows from Lemma 6.4.1.ii and the linearity of m^o, the second equality from Lemma 6.4.2.ii, and the third equality follows from relation (6.13). ■

Corollary 6.4.4 *The P-value on $\overline{w}$ equals the Shapley value on $c_{\overline{w}}$.*

Proof Consider the charge system of Example 4.2.3. As we already said in Remark 4.7.2, such a charge system leads to a conservative CC-rule which corresponds to the P-value (Branzei *et al.* (2004)). The obligation function o^* obtained from the charge system $\hat{\mathcal{C}}$ of Example 4.2.3 via relation (4.27) is such that $o^*(S) = \frac{e^S}{|S|}$ for each $S \in 2^N \setminus \{\emptyset\}$, where e^S is the $|N|$-vector such that $e^S_i = 1$ if $i \in S$ and $e^S_i = 0$ if $i \in N \setminus S$. Then, from relation (6.12) it follows directly that $m^{o^*}(c_{\overline{w}})$ is the Shapley value of the game $c_{\overline{w}}$. ■

Chapter 7

Variants of mcst games

7.1 Introduction

In this section we consider some variants of minimum cost spanning tree games. One variant, presented in Section 7.2, is the class of *monotonic minimum cost spanning tree games* which are characterized by the fact that coalitions are allowed to use networks which contain nodes outside the coalition (Steiner trees). Two other variants are obtained by considering directed weighted graphs. Here the aim of coalitions is to construct a directed network such that every player in the coalition is connected with the source via a directed path. This approach leads to the class of *directed minimum cost spanning tree games* and *monotonic directed minimum cost spanning tree games*, both presented in Section 7.3. For any of these new classes of games we present an example that does not have a pmas. Finally, in Section 7.4, a special subclass of directed minimum cost spanning tree problems, introduced in Moretti *et al.* (2002), is studied, which shows up in considering the problem of connecting houses in mountains with a water purifier. It is shown that the games corresponding to these problems always have a pmas. Sections 7.2 and 7.3 are based on Section 6 in Norde, Moretti, Tijs (2004) and Section 7.4 is based on Moretti, Norde, Pham Do, Tijs (2002).

7.2 Monotonic mcst games

First, we consider the class of monotonic minimum cost spanning tree games.

Definition 7.2.1 Let $< N', w >$ be a complete weighted graph. The *monotonic minimum cost spanning tree game* (N, c^{mon}), corresponding to $< N', w >$, is defined by

$$c^{\text{mon}}(S) = \min\{w(\Gamma) : \Gamma \text{ is a spanning network for some coalition } T \supseteq S\}$$

for every $S \in 2^N \backslash \{\emptyset\}$.

In the following example we present a monotonic minimum cost spanning tree game without a pmas.

Example 7.2.1 Consider the complete weighted graph $< N', w >$ with $N' = \{0, 1, 2, 3, 4, 5, 6\}$ and cost function w as depicted in Figure 7.1. All edges which are depicted have cost 1, whereas all other edges have cost 10. A minimum

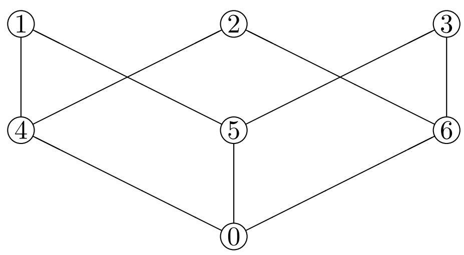


Figure 7.1: The cost function of Example 7.2.1.

cost spanning tree for $S = \{1, 2, 3\}$ is $\{\{0, 4\}, \{0, 5\}, \{1, 4\}, \{2, 4\}, \{3, 5\}\}$. So, $c^{\text{mon}}(123) = 5$. A minimum cost spanning tree for $S = \{1, 2\}$ is $\{\{0, 4\}, \{1, 4\}, \{2, 4\}\}$. So, $c^{\text{mon}}(12) = 3$. In a similar way one gets $c^{\text{mon}}(13) = c^{\text{mon}}(23) = 3$. If

c^{mon} has a pmas $\{x_{S,i}\}_{S\in 2^N\setminus\{\emptyset\}, i\in S}$ then

$$\begin{aligned}
10 &= 2c^{\text{mon}}(123) \\
&= 2(x_{123,1}+x_{123,2}+x_{123,3}) \\
&\leq x_{12,1}+x_{13,1}+x_{12,2}+x_{23,2}+x_{13,3}+x_{23,3} \\
&= x_{12,1}+x_{12,2}+x_{13,1}+x_{13,3}+x_{23,2}+x_{23,3} \\
&= c^{\text{mon}}(12)+c^{\text{mon}}(13)+c^{\text{mon}}(23) \\
&= 9,
\end{aligned}$$

which yields a contradiction. In fact, we have shown that c^{mon} is not even totally balanced. Megiddo (1978) already noted that monotonic minimum cost spanning tree games do not have to be totally balanced.

■

7.3 Directed mcst games

In order to provide the definition of directed minimum cost spanning tree games and monotonic directed minimum cost spanning tree games we need some more terminology. A *complete directed weighted graph* is a tuple $< N', w >$ where

i) $N' = \{0, 1, \ldots, n\}$;

ii) $w : D \to I\!R_+$, where $D = \{(i,j) : i, j \in N', i \neq j\}$.

Elements of D are called *directed arcs*. A *directed path* from i to j in network $\Gamma \subseteq D$ is a sequence of nodes $i = i_0, i_1, \ldots, i_k = j$ such that $(i_s, i_{s+1}) \in \Gamma$ for every $s \in \{0, \ldots, k-1\}$. Network Γ is a spanning network for S ($S \subseteq N$) if for every $(i,j) \in \Gamma$ we have $\{i,j\} \subseteq S \cup \{0\}$ and if for every $i \in S$ there is a directed path in Γ from i to 0.

Definition 7.3.1 Let $< N', w >$ be a complete directed weighted graph. The *directed minimum cost spanning tree game* (N, c), corresponding to $< N', w >$, is defined by

$$c(S) = \min\{w(\Gamma) : \Gamma \text{ is a spanning network for } S\}$$

for every $S \in 2^N \backslash \{\emptyset\}$, whereas the *monotonic directed minimum cost spanning tree game* (N, c^{mon}), corresponding to $< N', w >$, is defined by

$$c^{\text{mon}}(S) = \min\{w(\Gamma) : \Gamma \text{ is a spanning network for some coalition } T \supseteq S\}$$

for every $S \in 2^N \backslash \{\emptyset\}$.

We conclude this section with two examples which show that directed minimum cost spanning tree games and monotonic directed minimum cost spanning tree games do not necessarily have a pmas.

Example 7.3.1 Consider the complete directed weighted graph $< N', w >$ with $N' = \{0, 1, 2, 3, 4, 5, 6\}$ and cost function w as depicted in Figure 7.2. All directed arcs which are depicted have cost 0 whereas all other directed arcs have cost 1. Let (N, c) be the directed minimum cost spanning tree game,

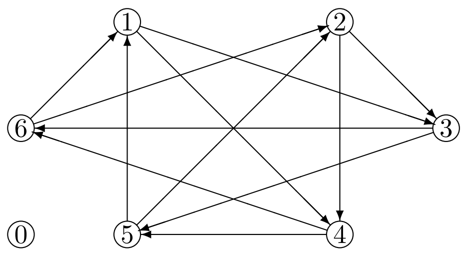


Figure 7.2: The cost function of Example 7.3.1.

corresponding to $< N', w >$, and suppose that $x = \{x_{S,i}\}_{S \in 2^N \backslash \{\emptyset\}, i \in S}$ is a pmas for c. A minimum cost spanning network for N is obtained by taking all directed arcs with cost 0 and directed arc $(1, 0)$. So, $c(123456) = 1$. Now, consider $S = \{1, 3, 4\}$. We have

$$\begin{aligned} 2 = c(134) &= x_{134,1} + x_{134,3} + x_{134,4} \\ &\leq x_{14,1} + x_{3,3} + x_{14,4} \\ &= x_{14,1} + x_{14,4} + x_{3,3} \\ &= c(14) + c(3) \\ &= 1 + 1 = 2, \end{aligned}$$

and hence $x_{134,3} = x_{3,3} = c(3) = 1$. Since also $c(13) = c(4) = 1$ we get in a similar way that $x_{134,4} = 1$. Therefore, $x_{134,1} = c(134) - x_{134,3} - x_{134,4} = 0$

and hence, by population monotonicity, $x_{N,1} \leq x_{134,1} = 0$. By considering respectively coalitions $\{2,3,4\}$, $\{3,5,6\}$, $\{4,5,6\}$, $\{5,1,2\}$ and $\{6,1,2\}$ we get via analogous arguments that the numbers $x_{N,2}, \ldots, x_{N,6}$ are all nonpositive. This contradicts however that $\sum_{i \in N} x_{N,i} = c(N) = 1$.

■

Example 7.3.2 Consider the complete directed weighted graph $< N', w >$ with $N' = \{0,1,2,3,4,5,6\}$ and cost function w as depicted in Figure 7.3. All directed arcs which are depicted have cost 0, whereas all other directed arcs have cost 1. Let c^{mon} be the monotonic directed minimum cost spanning tree

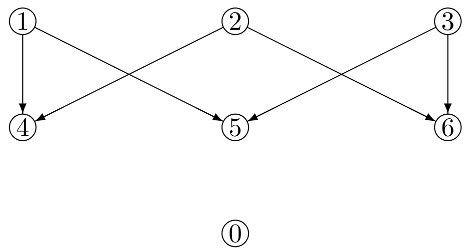


Figure 7.3: The cost function of Example 7.3.2.

game corresponding to $< N', w >$. A minimum cost spanning network for $S = \{1,2,3\}$ is $\{(1,4),(2,4),(3,5),(4,0),(5,0)\}$ so $c^{\text{mon}}(123) = 2$. A minimum cost spanning network for $S = \{1,2\}$ is $\{(1,4),(2,4),(4,0)\}$ so $c^{\text{mon}}(12) = 1$. In a similar way one gets $c^{\text{mon}}(13) = c^{\text{mon}}(23) = 1$. Since $2c^{\text{mon}}(123) > c^{\text{mon}}(12) + c^{\text{mon}}(13) + c^{\text{mon}}(23)$ we conclude, in a similar way as in Example 7.2.1, that c^{mon} has no pmas.

■

7.4 A connection situation on mountains

Consider a group of persons whose houses lie on mountains which surround a valley or part of a coast. Their houses are not yet connected to a drainage where one has to empty their sewage. Obviously, sewage has to be collected downhill, in a water purifier in the valley or along the coast. Each one wants to connect his house with a drain pipe to the water purifier. However, it is possible but not necessary for everyone to be connected directly with the water purifier, being connected via others is sufficient. Assuming that pipes are large enough one pipe can serve more than one person. Only connections from houses to strictly lower ones are allowed but, sometimes, connection from higher houses to lower houses is impossible (*e.g.* because of a natural reef between the two houses). A formal description of this kind of situations on mountains is given below.

Consider a tuple given by $< N, \{0\}, A, w >$, where $N = \{1, 2, \ldots, n\}$ corresponds to the set of agents (houses), 0 corresponds to the water purifier and where $< N \cup \{0\}, A >$ is a rooted directed graph with $N \cup \{0\}$ as set of vertices, $A \subset N \times (N \cup \{0\})$ as set of arcs and 0 as the root. We assume also that the following conditions M.1 and M.2 hold:

M.1 (Direct connection possibility) For each $k \in N$, $(k, 0) \in A$.

M.2 (No cycles) For each $s \in I\!N$ and $v_1, v_2, \ldots, v_s \in N \cup \{0\}$ such that $(v_1, v_2) \in A$, $(v_2, v_3) \in A$, …, $(v_{s-1}, v_s) \in A$ we have $(v_s, v_1) \notin A$.

Further, $w : A \to I\!R_+$ is a non-negative function on the set of arcs. We call such a tuple $< N, \{0\}, A, w >$ with the properties M.1 and M.2 a *mountain situation*.

Given a mountain situation $< N, \{0\}, A, w >$ there exists an intrinsic height function $h : N \cup \{0\} \to I\!N \cup \{0\}$ such that $(i, j) \in A$ implies $h(i) > h(j)$. One defines h as follows: for $i \in N \cup \{0\}$, $h(i)$ is the length of a longest path from i to 0.

To avoid too many technicalities we will assume in the following that $< N, \{0\}, A, w >$ does not only satisfy M.1 and M.2, but also M.3:

M.3 (Genericity condition) For each $k \in N$ and all $i, j \in N \cup \{0\}, i \neq j$: $(k, i) \in A, (k, j) \in A \Rightarrow w(k, i) \neq w(k, j)$.

Condition M.3 gives us the possibility to speak of the cheapest connection point $b_S(k)$ of k in S, with $S \in 2^N \setminus \{\emptyset\}$:

$$b_S(k) = \text{argmin}_{l \in S \cup \{0\} : (k,l) \in A} w(k, l).$$

Condition M.3 is useful to avoid too many technicalities. We invite the reader to adjust the results of this section for situations where M.3 does not hold.

Two interesting questions related to such a mountain situation are:

Q.1 How to find a 0-connecting subtree $< N \cup \{0\}, T >$ of $< N \cup \{0\}, A >$, *i.e.* a subtree connecting each $i \in N$ with 0, with minimum cost?

Q.2 How to allocate the connection costs in such a tree among the agents?

Given a mountain situation $< N, \{0\}, A, w >$, the next theorem answers question Q.1 showing that there is a unique optimal tree (with minimum cost), connecting all players in N with the root 0. This tree corresponds to the situation where each agent $k \in N$ connects himself with his best connection point $b_N(k) \in N \cup \{0\}$.

Theorem 7.4.1 *Let* $< N, \{0\}, A, w >$ *be a mountain situation and let* $T = \{(k, b_N(k)) \,|\, k \in N\}$. *Then*

(i) $< N \cup \{0\}, T >$ *is a 0-connecting subtree of* $< N \cup \{0\}, A >$.

(ii) The tree $< N \cup \{0\}, T >$ *is the unique 0-connecting subtree with minimum cost.*

Proof (i) Since $T \subset A$, clearly T does not contain cycles. That T is a tree connecting each point $i \in N$ via a path with 0 follows from the claim that for each $s \in \{1, \ldots, L\}$, where $L = \max\{h(i) \,|\, i \in N \cup \{0\}\}$, the next property $P(s)$ holds:

> $P(s)$: for each $k \in N$ with $h(k) = s$ there is a $t(k) \in \mathbb{N}$ and a sequence $v_0, v_1, \ldots, v_{t(k)}$ such that $v_0 = k$, $v_{r+1} = b_N(v_r)$ for $r = 0, 1, \ldots, t(k) - 1$, and $v_{t(k)} = 0$.

We prove the claim by induction to s. $P(1)$ holds because for each $k \in N$ with $h(k) = 1$ we take $t(k) = 1$, $v_0 = k$ and $v_1 = 0$. Suppose now that $P(s)$ holds for each $s < m$ with $m \in \{2, \ldots, L\}$. Let $k \in N$ be such that $h(k) = m$. Then $h(b_N(k)) < m$. If $h(b_N(k)) = 0$, then $b_N(k) = 0$ and we take $t(k) = 1$, $v_0 = k$, $v_1 = 0$. Suppose $h(b_N(k)) \neq 0$. Then, by the induction hypothesis, there is a $t(b_N(k))$ and a sequence $v_0, v_1, \ldots, v_{t(b_N(k))}$ determining a path in A from $b_N(k)$

to 0 with $v_{r+1} = b_N(v_r)$ for $r \in \{0, 1, \ldots, t(b_N(k)) - 1\}$. Then $w_0, w_1, \ldots, w_{t(k)}$ is a desired path for k, where $t(k) = t(b_N(k)) + 1$, $w_0 = k$, $w_i = v_{i-1}$ for $i \in \{1, \ldots, t(k)\}$. So, $P(m)$ holds.

(ii) Let $< N \cup \{0\}, G >$ be a 0-connecting tree unequal to $< N \cup \{0\}, T >$. Then for each point $k \in N$, there is a $\pi(k) \in N \cup \{0\}$ such that $(k, \pi(k)) \in G$. Moreover, since $G \neq T$ we can choose $\pi : N \to N \cup \{0\}$ such that there is a $k^* \in N$ with $\pi(k^*) \neq b_N(k^*)$, implying $w(k^*, \pi(k^*)) > w(k^*, b_N(k^*))$ by M.3. Then

$$\sum_{(i,j) \in G} w(i, j) \geq \sum_{k \in N} w(k, \pi(k)) > \sum_{k \in N} w(k, b_N(k)).$$

So, $< N \cup \{0\}, G >$ is not optimal. ■

Example 7.4.1 Figure 7.4 corresponds to a mountain situation $< N, \{0\}, A, w >$, where $N = \{1, 2, 3\}$, $A = \{(1, 0), (2, 0), (2, 1), (3, 0), (3, 1), (3, 2)\}$ and $w(i, j) = 10i - 5j$ for each $(i, j) \in A$. Then the intrinsic height function h is described by $h(i) = i$ for each $i \in N$. Since $b_N(1) = 0$, $b_N(2) = 1$, $b_N(3) = 2$, the tree $< N \cup \{0\}, T >$ with $T = \{(1, 0), (2, 1), (3, 2)\}$ is an optimal 0-connecting tree with cost $10 + 15 + 20 = 45$.

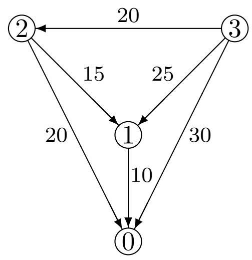


Figure 7.4: The mountain situation of Example 7.4.1

■

7.4.1 Cooperative cost games on mountain situations

To provide an answer to question Q.2 we need to introduce the corresponding cooperative cost games. For further use, we first recall bounds for core elements for a cooperative cost game (N, c):

$$M_i(N,c) \leq x_i \leq c(\{i\}) \text{ for all } x \in \text{core}(N,c), i \in N. \tag{7.1}$$

Here $M_i(N,c) = c(N) - c(N\backslash\{i\})$, is the marginal contribution to the costs of N by player $i \in N$. Note that the second inequality in (7.1) is one of the stability conditions $\sum_{i \in S} x_i \leq c(S)$, $S \in 2^N$, of core allocations. For the first inequality in (7.1) note that

$$x_i = \sum_{k=1}^{n} x_k - \sum_{k \in N\backslash\{i\}} x_k = c(N) - \sum_{k \in N\backslash\{i\}} x_k \geq c(N) - c(N\backslash\{i\}) = M_i(N,c),$$

where the second equality follows from the fact that core allocations are efficient and the inequality from the stability condition $\sum_{i \in S} x_i \leq c(S)$ of core allocations, with $N\backslash\{i\}$ in the role of S.

Let $< N, \{0\}, A, w >$ be a mountain situation. Then the corresponding cooperative cost game $(N, \hat{c})$ is given by $\hat{c}(\emptyset) = 0$ and for $T \in 2^N\backslash\{\emptyset\}$ the cost $\hat{c}(T)$ of coalition T is the cost of the optimal 0-connecting tree in the mountain problem $< T, \{0\}, A(T), w_T >$, where $A(T) = \{(i,j) \in A \,|\, i \in T, j \in T \cup \{0\}\}$, and $w_T : A(T) \to I\!R_+$ is the restriction of $w : A \to I\!R_+$ to $A(T)$. Note that for each $T \in 2^N\backslash\{\emptyset\}$,

$$\hat{c}(T) = \sum_{k \in T} w(k, b_T(k)).$$

Take the allocation $B(N, \{0\}, A, w) \in I\!R^N$ with $B_k(N, \{0\}, A, w) = w(k, b_N(k))$, corresponding to the situation where each player i pays his cheapest connection in $N \cup \{0\}$. Then $B(N, \{0\}, A, w)$ is a core element of $(N, \hat{c})$, since

$$\hat{c}(N) = \sum_{k \in N} w(k, b_N(k)) = \sum_{k \in N} B_k(N, \{0\}, A, w)$$

by Theorem 7.4.1, and further

$$\hat{c}(T) = \sum_{k \in T} w(k, b_T(k)) \geq \sum_{k \in T} w(k, b_N(k)) = \sum_{k \in T} B_k(N, \{0\}, A, w)$$

for each $T \in 2^N\backslash\{\emptyset\}$. The core element $B(N, \{0\}, A, w)$ corresponds to the situation where the player $b_N(k)$ to which k connects himself does not ask a compensation for this service to k. But there are other interesting core allocations in general, corresponding to situations where compensation plays a role.

In the description of these core elements the second cheapest connection point of k in $T \cup \{0\}$,

$$s_T(k) = \begin{cases} \text{argmin}_{l \in (T\cup\{0\})\setminus\{b_T(k)\}:(k,l)\in A} w(k,l) & \text{if } b_T(k) \neq 0 \\ 0 & \text{if } b_T(k) = 0, \end{cases}$$

plays a role.

Suppose player k wants to connect to $b_N(k) \neq 0$ and player $b_N(k)$ wants to ask a price $p_k \geq 0$ from k for connecting k. Which price p_k can $b_N(k)$ ask for his service to k such that k connects with $b_N(k)$ and does not go, *e.g.*, to the second best connection point $s_N(k)$ for a connection? The price should be an element of the closed interval $[0, w(k, s_N(k)) - w(k, b_N(k))]$. A price p_k larger than $w(k, s_N(k)) - w(k, b_N(k))$ can lead to a connection to $s_N(k)$, and if $s_N(k) \neq 0$ even to a positive compensation for $s_N(k)$, *e.g.* $\frac{1}{2}(p_k - w(k, s_N(k)) + w(k, b_N(k)))$ and then both players k and $s_N(k)$ are better off. The allocations $(x_1, \ldots, x_n)$ corresponding to such competitive prices in the given closed interval turn out to be just the core allocations of the k-connection game (N, c_k) to be introduced now.

The k-connection game (N, c_k) is the cooperative cost game with $c_k(S) = 0$ if $k \notin S$ and $c_k(S) = w(k, b_S(k))$ otherwise. Note that, if $b_N(k) \neq 0$, then $M_{b_N(k)}(N, c_k) = c_k(N) - c_k(N\setminus\{b_N(k)\}) = w(k, b_N(k)) - w(k, s_N(k))$.

Theorem 7.4.2 *Let $(N, c_1), \ldots, (N, c_n)$ be the connection games corresponding to the mountain situation $< N, \{0\}, A, w >$ and let $(N, \hat{c})$ be the corresponding cost game. Then*

(i) $c = \sum_{k=1}^{n} c_k$,

(ii) *core*$(N, \hat{c}) \supset P(N, c)$ *where* $P(N, c) = \sum_{k=1}^{n}$ *core*(N, c_k),

(iii) *for every* $T \in 2^N\setminus\{\emptyset\}$ *we have core*$(T, c_k) = \{0\}$ *if* $k \notin T$,

core$(T, c_k) =$
$\{w(k, b_T(k))e^k - (e^{b_T(k)} - e^k)p \mid 0 \leq p \leq w(k, s_T(k)) - w(k, b_T(k))\}$

if $k \in T, b_T(k) \neq 0$, *and*
core$(T, c_k) = \{w(k, 0)e^k\}$ *if* $k \in T, b_T(k) = 0$.

[Here $e^k \in I\!R^T$ is the k-th standard basis vector with k-th coordinate 1 and the other coordinates 0.]

Proof (i) is a direct consequence of the definitions of $c, c_1, \ldots, c_n$.
(ii) follows from (i) because core$(N, \cdot)$ is a superadditive correspondence.
(iii) Note that if $k \notin T$ then (T, c_k) is the zero game and hence core(T, c_k)= $\{0\}$. If $k \in T$ and $b_T(k) \neq 0$ then $M_i(T, c_k) = c_k(i) = 0$ if $i \in T \backslash \{k, b_T(k)\}$. For $x \in$ core(T, c_k) we have, by (7.1), $x_i = 0$ for each $i \in T \backslash \{k, b_T(k)\}$. Further, since core allocations are efficient, we have $x_k + x_{b_T(k)} = c_k(T) = w(k, b_T(k))$, and, by (7.1), $w(k, b_T(k)) - w(k, s_T(k)) = M_{b_T(k)}(T, c_k) \leq x_{b_T(k)} \leq 0$. This implies that

$$\begin{aligned} &\text{core}(T, c_k) \subset \\ &\quad \{w(k, b_T(k))e^k - (e^{b_T(k)} - e^k)p \,|\, 0 \leq p \leq w(k, s_T(k)) - w(k, b_T(k))\}. \end{aligned}$$

For the reverse inclusion, note that for $x^p = w(k, b_T(k))e^k - (e^{b_T(k)} - e^k)p$ with $0 \leq p \leq w(k, s_T(k)) - w(k, b_T(k))$ we have $x^p(T) = \sum_{i \in T} x_i^p = w(k, b_T(k)) = c_k(T)$ and for $S \subset T$:

$$\begin{array}{rcll} x^p(S) & = & w(k, b_T(k)) & \\ & = & c_k(S) & \text{if } \{k, b_T(k)\} \subset S \\ x^p(S) & = & 0 & \\ & = & c_k(S) & \text{if } \{k, b_T(k)\} \cap S = \emptyset \\ x^p(S) & = & w(k, b_T(k)) + p & \\ & \leq & w(k, s_T(k)) & \\ & \leq & c_k(S) & \text{if } k \in S, b_T(k) \notin S, \text{ and} \\ x^p(S) & = & -p & \\ & \leq & 0 & \\ & = & c_k(S) & \text{if } k \notin S, b_T(k) \in S. \end{array}$$

So, $x^p \in$ core(T, c_k).
If $k \in T$ and $b_T(k) = 0$ the statement can be proved in a similar way. ■

The subset $P(N, \hat{c})$ of core$(N, \hat{c})$ is the set of price supported core elements. In the next section we will show that elements x of $P(N, \hat{c})$ are pmas-extendable *i.e.* there exists a population monotonic allocation scheme $\{a_{T,i}\}_{T \in 2^N \backslash \{\emptyset\}, i \in T}$ such that $a_{N,i} = x_i$ for each $i \in N$.

Example 7.4.2 Consider again the mountain situation of Example 7.4.1. The cost game $(N, \hat{c})$ corresponding to this situation and the k-connection games are given in the next table:

$S =$	(1)	(2)	(3)	$(1,2)$	$(1,3)$	$(2,3)$	$(1,2,3)$
$c(S) =$	10	20	30	25	35	40	45
$c_1(S) =$	10	0	0	10	10	0	10
$c_2(S) =$	0	20	0	15	0	20	15
$c_3(S) =$	0	0	30	0	25	20	20

Note that $c = c_1+c_2+c_3$, $\text{core}(N, c_1) = \{(10, 0, 0)\}$, $\text{core}(N, c_2) = \text{conv}\{(0, 15, 0), (-5, 20, 0)\}$, and $\text{core}(N, c_3) = \text{conv}\{(0, 0, 20), (0, -5, 25)\}$.

Finally, note that $B(N, \{0\}, A, w) = (10, 15, 20)$.

■

7.4.2 Pmas on mountain situations

The scheme $A^0 = (w(i, b_T(i)))_{T \in 2^N \setminus \{\emptyset\}, i \in T}$ is an example of a pmas. To find other pmas-es it is interesting to note that if A^k is a pmas of the connection game (N, c_k), for each $k \in N$, then $\sum_{k=1}^{n} A^k$ is a pmas of $(N, \hat{c})$. This motivates us to concentrate on how to find a pmas of connection games.

If $k \notin T$ then (T, c_k) is the zero game and hence $\text{core}(T, c_k) = \{0\}$. If $k \in T$ then it follows from Theorem 7.4.2.iii that

$$\text{core}(T, c_k) = \{x_T^{\alpha_k} \in I\!R^T \mid \alpha_k \in [0, 1]\},$$

where

$$x_T^{\alpha_k} = w(k, b_T(k))e^k + \alpha(w(k, b_T(k)) - w(k, s_T(k)))(e^{b_T(k)} - e^k)$$

if $b_T(k) \neq 0$, and $x_T^{\alpha_k} = w(k, 0)e^k$ if $b_T(k) = 0$. Note that the core has a unique element if $b_T(k) = 0$. The next Theorem 7.4.4 shows that each core element $x_N^{\alpha_k}$ of (N, c_k) can be extended to a pmas, namely A^{α_k}. Here $A^{\alpha_k} =$

$(a^{\alpha_k}_{T,i})_{T\in 2^N\backslash\{\emptyset\},i\in T}$ is the allocation scheme, where, for every $T \in 2^N\backslash\{\emptyset\}$,

$$(a^{\alpha_k}_{T,i})_{i\in T} = \begin{cases} 0 & \text{if } k \notin T; \\ (x^{\alpha_k}_N)_{i\in T} & \text{if } k \in T \text{ and } b_N(k) \in T; \\ x^0_T & \text{if } k \in T \text{ and } b_N(k) \notin T. \end{cases}$$

This cost allocation scheme corresponds to the situation where $k \in T$ pays his connection cost $w(k, b_T(k))$ and also as compensation for $b_N(k)$ of α_k times the marginal contribution of $b_N(k)$ in T if $b_N(k) \in T$, and no compensation if $b_N(k) \notin T$. Note that 'column' k of A^0 equals 'column' k of the scheme $B(N, \{0\}, A, w)$. Note moreover, that in rows T with $k \notin T$ we have a core element since 0 is the unique core element of $< T, c_k >$, and in rows T with $k \in T$ and $b_N(k) \notin T$ we also have core elements. It follows from the following lemma that also the rows with $k \in T$ and $b_N(k) \in T$ contain core elements. So, A^{α_k} is a stable monotonic allocation scheme.

Lemma 7.4.3 *Let $T \in 2^N$ be such that $k \in T$ and $b_N(k) \in T$. Then $(a^{\alpha_k}_{T,i})_{i\in T} = (x^{\alpha_k}_N)_{i\in T} \in$ core(T, c_k).*

Proof The only thing to show is that $-\alpha(w(k, b_N(k)) - w(k, s_N(k))) \in [0, w(k, s_T(k)) - w(k, b_T(k))]$. Note that

$$\begin{aligned} 0 &\leq -\alpha(w(k, b_N(k)) - w(k, s_N(k))) \\ &= \alpha(w(k, s_N(k)) - w(k, b_N(k))) \\ &\leq w(k, s_N(k)) - w(k, b_N(k)) \\ &= w(k, s_N(k)) - w(k, b_T(k)) \\ &\leq w(k, s_T(k)) - w(k, b_T(k)). \end{aligned}$$

At the last equality we used the fact that $b_N(k) = b_T(k)$ and at the last inequality that

$$\begin{aligned} w(k, s_N(k)) &= \min\{w(k,i) \,|\, i \in (N \cup \{0\})\backslash\{b_N(k)\}, (k,i) \in A\} \\ &\leq \min\{w(k,i) \,|\, i \in (T \cup \{0\})\backslash\{b_T(k)\}, (k,i) \in A\} \\ &= w(k, s_T(k)). \end{aligned}$$

■

Theorem 7.4.4 *For each $\alpha \in [0, 1]$, $A^\alpha = (a^\alpha_{T,i})_{T\in 2^N\backslash\{\emptyset\},i\in T}$ is a pmas of (N, c_k).*

Proof By Lemma 7.4.3, we only have to prove that $a_{S,i} \geq a_{T,i}$ for all $S, T \in 2^N$ and $i \in N$ with $i \in S \subset T$. Take $i \in N$, $S, T \in 2^N$ such that $i \in S \subset T$. We consider 3 cases:
(i) Suppose that $i \in S \backslash \{k, b_N(k)\}$. Then $a^{\alpha}_{S,i} = 0 \geq 0 = a^{\alpha}_{T,i}$ since the column $(a_{U,i})_{U \in 2^N \backslash \{\emptyset\}, i \in U}$ is a zero column.
(ii) Suppose that $i = b_N(k) \in S \subset T$. Then $a^{\alpha}_{S,b_N(k)} = a^{\alpha}_{T,b_N(k)} = \alpha(w(k, b_N(k)) - w(k, s_N(k)))$ if $k \in S$, and $a^{\alpha}_{S,b_N(k)} = a^{\alpha}_{T,b_N(k)} = 0$ if $k \notin T$. If $k \notin S$ and $k \in T$ then $a^{\alpha}_{S,b_N(k)} = 0 \geq a^{\alpha}_{T,b_N(k)} = \alpha(w(k, b_N(k)) - w(k, s_N(k)))$.
(iii) Suppose that $i = k \in S \subset T$. Then $a^{\alpha}_{S,k} = a^{\alpha}_{T,k} = (x^{\alpha}_N)_k$ if $b_N(k) \in S$, and $a^{\alpha}_{S,k} = w(k, b_S(k)) \geq w(k, b_T(k)) = a^{\alpha}_{T,k}$ if $b_N(k) \notin T$. If $b_N(k) \notin S$ and $b_N(k) \in T$ then $a^{\alpha}_{S,k} = w(k, b_S(k)) \geq w(k, s_N(k)) \geq (1-\alpha)w(k, b_N(k)) + \alpha w(k, s_N(k)) = (x^{\alpha}_N)_k = a^{\alpha}_{T,k}$. ■

Theorem 7.4.5 *Each core element $x \in P(N, c)$ can be extended to a pmas of (N, c).*

Proof Since $P(N, c) = \sum_{k=1}^{n} \text{core}(N, c_k)$ in view of Theorem 7.4.2 one can find $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0, 1]$ such that $x = \sum_{k=1}^{n} x_N^{k,\alpha_k}$ with $x_N^{k,\alpha_k} \in \text{core}(N, c_k)$ for every $k \in \{1, \ldots, n\}$. Each x_N^{k,α_k} has a pmas extension A^{k,α_k} by Theorem 7.4.4. Then $A = \sum_{k=1}^{n} A^{k,\alpha_k}$ is a pmas of (N, c). ■

Example 7.4.3 Reconsider the situation of Example 7.4.1. Then $(10, 0, 0)$ is the unique core element of (N, c_1), the core element $(-2\frac{1}{2}, 17\frac{1}{2}, 0)$ of (N, c_2) is the midpoint of the core of (N, c_2), and $(0, -2\frac{1}{2}, 22\frac{1}{2})$ is the midpoint of the core of (N, c_3). So $x = (7\frac{1}{2}, 15, 22\frac{1}{2}) = (10, 0, 0) + (-2\frac{1}{2}, 17\frac{1}{2}, 0) + (0, -2\frac{1}{2}, 22\frac{1}{2}) \in P(N, c)$. Then $A^{1,\frac{1}{2}} + A^{2,\frac{1}{2}} + A^{3,\frac{1}{2}}$ is a pmas extending x. In matrix notation

$$A^{1,\frac{1}{2}} + A^{2,\frac{1}{2}} + A^{3,\frac{1}{2}} =$$

$$\begin{array}{c} \\ N \\ (12) \\ (13) \\ (23) \\ (1) \\ (2) \\ (3) \end{array}
\begin{array}{|ccc|} \hline 1 & 2 & 3 \\ \hline 10 & 0 & 0 \\ 10 & 0 & * \\ 10 & * & 0 \\ * & 0 & 0 \\ 10 & * & * \\ * & 0 & * \\ * & * & 0 \\ \hline \end{array}
+
\begin{array}{|ccc|} \hline 1 & 2 & 3 \\ \hline -2\frac{1}{2} & 17\frac{1}{2} & 0 \\ -2\frac{1}{2} & 17\frac{1}{2} & * \\ 0 & * & 0 \\ * & 20 & 0 \\ 0 & * & * \\ * & 20 & * \\ * & * & 0 \\ \hline \end{array}
+
\begin{array}{|ccc|} \hline 1 & 2 & 3 \\ \hline 0 & -2\frac{1}{2} & 22\frac{1}{2} \\ 0 & 0 & * \\ 0 & * & 25 \\ * & -2\frac{1}{2} & 22\frac{1}{2} \\ 0 & * & * \\ * & 0 & * \\ * & * & 30 \\ \hline \end{array}
=
\begin{array}{|ccc|} \hline 1 & 2 & 3 \\ \hline 7\frac{1}{2} & 15 & 22\frac{1}{2} \\ 7\frac{1}{2} & 17\frac{1}{2} & * \\ 10 & * & 25 \\ * & 17\frac{1}{2} & 22\frac{1}{2} \\ 10 & * & * \\ * & 20 & * \\ * & * & 30 \\ \hline \end{array},$$

a pmas extension of $(7\frac{1}{2}, 15, 22\frac{1}{2})$.

■

7.4.3 Bi-monotonic allocation schemes and cost monotonicity

A connection game (N, c_k) has the property that k is a veto player because $c_k(S) = 0$ for all S not containing k. For such games bi-monotonic allocation schemes (bi-mas) are introduced in Branzei *et al.* (2001) (see also Voorneveld *et al.* (2002)). A bi-mas for such a game with a veto player is a stable allocation scheme with the property that the veto player is weakly better off and the other players weakly worse off in larger coalitions. Let us be more specific. An allocation scheme $(b_{T,i})_{T\in 2^N\setminus\{0\}, i\in T}$ is a bi-monotonic allocation scheme for (N, c_k) if

$$\text{each row } (b_{T,i})_{i\in T} \in \text{core}(T, c_k), \tag{7.2}$$

and for all $S, T \in 2^N$ with $k \in S \subset T$

$$b_{T,k} \leq b_{S,k} \tag{7.3}$$

$$b_{T,i} \geq b_{S,i} \text{ for all } i \in S\backslash\{k\}. \tag{7.4}$$

It turns out that for connection games bi-monotonic allocation schemes exist. Moreover, each core element of (N, c_k) can be extended to a bi-mas, as Theorem 7.4.6 shows. For $\alpha \in [0,1]$, let $(b^\alpha_{T,i})_{T\in 2^N\setminus\{0\}, i\in T}$ be the allocation scheme with

$$(b^\alpha_{T,i})_{i\in T} = \begin{cases} x^\alpha_T & \text{if } k \in T, \\ 0 & \text{if } k \notin T. \end{cases}$$

Theorem 7.4.6 *For every $\alpha \in [0,1]$, $(b^\alpha_{T,i})_{T\in 2^N\setminus\{0\}, i\in T}$ is a bi-mas extending x^α_N.*

Proof (i) In view of Theorem 7.4.2 row T in $(b^\alpha_{T,i})_{T\in 2^N\setminus\{0\}, i\in T}$ is a core element for each $T \subset N$ and row N equals x^α_N. So, (7.2) holds.
(ii) To prove (7.3) note that for $S \subset T$ and $k \in S$ we have

$$w(k, b_S(k)) \geq w(k, b_T(k)), \tag{7.5}$$

$$w(k, s_S(k)) \geq w(k, s_T(k)). \tag{7.6}$$

Using (7.5) and (7.6) we obtain (7.3) as follows:

$$\begin{aligned} b^{\alpha}_{T,k} &= (1-\alpha)w(k, b_T(k)) + \alpha w(k, s_T(k)) \\ &\leq (1-\alpha)w(k, b_S(k)) + \alpha w(k, s_S(k)) \\ &= b^{\alpha}_{S,k}. \end{aligned}$$

(iii) To prove (7.4) for S, T with $i, k \in S \subset T$, $i \neq k$, we consider 3 cases: $i \neq b_S(k)$; $i = b_T(k)$; $i = b_S(k) \neq b_T(k)$.
If $i \neq b_S(k)$, then $i \neq b_T(k)$; so, $b^{\alpha}_{S,i} = b^{\alpha}_{T,i} = 0$.
If $i = b_T(k)$, then $i = b_S(k)$ and then

$$\begin{aligned} b^{\alpha}_{T,i} &= \alpha(w(k,i) - w(k, s_T(k))) \\ &\geq \alpha(w(k,i) - w(k, s_S(k))) \\ &= b^{\alpha}_{S,i}, \end{aligned}$$

where the inequality follows from (7.6).
If $i = b_S(k) \neq b_T(k)$, then $b^{\alpha}_{S,i} = \alpha(w(k, b_S(k)) - w(k, s_S(k))) \leq 0 = b^{\alpha}_{T,i}$. ■

Example 7.4.4 Take the game of Example 7.4.2. Then for $k = 3$ the bi-mas, corresponding to $\alpha = \frac{1}{2}$, is given by

	1	2	3
(123)	0	$-2\frac{1}{2}$	$22\frac{1}{2}$
(12)	0	0	$*$
(13)	$-2\frac{1}{2}$	$*$	$27\frac{1}{2}$
(23)	$*$	-5	25
(1)	0	$*$	$*$
(2)	$*$	0	$*$
(3)	$*$	$*$	30

■

Now, suppose a mountain situation $< N, \{0\}, A, w^1 >$ changes to the mountain situation $< N, \{0\}, A, w^2 >$, where $w^2(i,j) = w^1(i,j)$ for all $(i,j) \in A \backslash \{(k,l)\}$ and $w^2(k,l) > w^1(k,l)$. Consider the allocations $B^1 = B(N, \{0\}, A, w^1)$ and $B^2 = B(N, \{0\}, A, w^2)$. Then, obviously, $B^1_i = B^2_i$ for all $i \in N \backslash \{k\}$, and $B^1_k = w(k, b_N(k)) = B^2_k$ if $b_N(k) \neq l$, while $B^2_k > B^1_k$ if $b_N(k) = l$. So the allocation rule B is cost monotonic on the class of mountain situations. Allocation rules where compensations for connections play a role do not have this cost monotonicity property. The reason is that if an arc increases so much in costs that there is a change of best connection points, the new connection point profits from the compensation and is better off.

Example 7.4.5 Consider again the mountain situation of Example 7.4.1. Consider the allocation $B(N, \{0\}, A, w)$ and the allocation $E(N, \{0\}, A, w)$, where compensations of half of the marginal contribution take place. The allocation $B(N, \{0\}, A, w)$ equals $(10, 15, 20)$ and $E(N, \{0\}, A, w)$ equals $(7\frac{1}{2}, 15, 22\frac{1}{2})$. If we change the mountain situation such that the cost of $(3, 2)$ raises to 40 then we obtain as allocations $B(N, \{0\}, A, w)$ and $E(N, \{0\}, A, w)$, respectively, $(10, 15, 25)$ and $(5, 17\frac{1}{2}, 27\frac{1}{2})$. In the allocation $E(N, \{0\}, A, w)$ player 1 is better off because of compensations from two players.

$\blacksquare$

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Samenvatting

In dit proefschrift staat de toepassing van coöperatieve speltheorie centraal bij de analyse van kostentoewijzingsproblemen, die voortvloeien uit situaties waar netwerken geconstrueerd moeten worden. Zo'n situatie doet zich voor wanneer een aantal economische agenten rechtstreeks of indirect verbonden moet worden met één of andere voorziening (de bron). Als verbindingskosten tussen agenten hoog zijn dan zullen agenten de mogelijkheid van samenwerking onderzoeken om de kosten te drukken. In feite zal een groep agenten, die besluit om samen te werken, een netwerk vormen van minimale kosten. Zo'n netwerk wordt een *minimum opspannende boom (mob)* genoemd. Echter, indien zo'n mob gevonden is, dan is dat nog geen garantie dat deze ook daadwerkelijk wordt geïmplementeerd: de agenten moeten ook nog in staat zijn om de kosten van deze mob op één of andere manier te verdelen en daartoe moet het bijbehorende kostentoewijzingsprobleem opgelost worden. Dit kostentoewijzingsprobleem is bestudeerd in het fundamentele paper van Bird in 1976. Voor problemen waarbij een netwerk geconstrueerd dient te worden heeft Bird een bijbehorend coöperatief spel geformuleerd (een zogeheten mob spel), waarbij de spelers de economische agenten zijn en waarbij de waarde van een coalitie gelijk is aan de minimale kosten van een netwerk dat alle spelers in de coalitie rechtstreeks of indirect met de bron verbindt.

In dit proefschrift worden een aantal kostentoewijzingsmechanismen voorgesteld. Deze brengen de agenten fracties van de kosten van verbindingen in rekening, die gevormd worden tijdens het toepassen van het algoritme voor het vinden van een mob, dat geïntroduceerd werd door Kruskal in 1956. Deze mechanismen worden *Construct and Charge* regels genoemd. Zij kunnen makkelijk geïmplementeerd worden in praktische situaties, zijn flexibel voor veranderingen in de situatie, en voldoen aan de voorwaarde van continue controle door de betrokken agenten. Het blijkt dat een deel van deze mechanismen overeenkomt met de klasse van *Obligatie* regels. Aangetoond wordt dat Obligatie regels monotoon in de kosten zijn en een populatie monotoon toewijzingsschema induceren. Interessante regels onder de Obligatie regels zijn de P*-waarde* en de P^{τ}*-waarden*, voor iedere volgorde τ van de spelers. Andere karakteristieken van Obligatie regels zijn dat verschillende uitvoerbare volgordes van de verbindingen tot dezelfde kostentoewijzingen leiden en dat al deze toewijzingen in de Bird core zitten. Varianten van netwerkconstructieproblemen worden ook bekeken.

In Hoofdstuk 2 wordt de benodigde voorkennis en notatie geïntroduceerd. The concepten 'mob situatie' en 'mob spel' worden geformuleerd en geïllustreerd voor volledige grafen met een kostenstructuur. Deze voorbeelden zullen in het gehele proefschrift gebruikt worden om andere concepten te illustreren. Basisbegrippen uit de coöperatieve speltheorie, zoals 'core' en 'populatie monotoon toewijzingsschema's' worden ook behandeld en toegelicht via voorbeelden.

In Hoofdstuk 3 wordt het Subtraction Algoritme gepresenteerd. Dit algoritme berekent voor iedere mob situatie en iedere volgorde van de spelers een populatie monotoon toewijzingsschema. Als basis van dit algoritme dient een decompositie-stelling, die toont hoe ieder mob spel geschreven kan worden als een niet-negatieve combinatie van mob spelen behorende bij $0-1$ kostenfuncties (dit worden *simpele mob spelen* genoemd). Het blijkt dat het Subtraction Algoritme nauwe banden heeft met het fameuse algoritme van Kruskal voor het vinden van een mob. Bovendien wordt voor iedere volgorde τ van de spelers de P^{τ}-waarde geïntroduceerd als het toewijzingsmechanisme voor mob situaties die de kosten van de grote coalitie verdeelt via het Subtraction Algoritme, met τ als initialisatie.

In Hoofdstuk 4 wordt de collectie *Construct and Charge* (CC-) regels voor mob situaties geïntroduceerd. Deze regels worden gedefinieerd aan de hand van *belastingsystemen*, en specificeren speciale toewijzingsmechanismen aan de hand van het algoritme van Kruskal voor het vinden van een mob. Bovendien besteedt dit hoofdstuk speciale aandacht aan *Obligatie regels* voor mob situaties. Kenmerkend voor deze regels is dat zij voor iedere mob situatie een kostentoewijzing opleveren, die gezien kan worden als het product van een dubbelstochastische matrix met de kostenvector van verbindingen in een optimale boom, verkregen via het algoritme van Kruskal. Aangetoond wordt dat speciale belastingsystemen, de zogeheten *conservatieve* systemen, tot een collectie van CC-regels leidt dat samenvalt met de collectie Obligatie regels. Een interessante eigenschap van deze regels is dat verschillende uitvoerbare volgordes van de verbindingen tot dezelfde kostentoewijzingen leiden. Eigenschappen van speciale Obligatie regels, zoals de Potters-regel (P-waarde) en de P^τ-waardes, eerder geïntroduceerd in Hoofdstuk 3, worden ook besproken. Het blijkt dat de P-waarde samenvalt met de Equal Remaining Obligations (ERO) regel, voorgesteld door Jos Potters (hetgeen ook de naam van de regel verklaart). Bovendien blijkt dat de P-waarde het gemiddelde is van de P^τ-waarden.

In Hoofdstuk 5 wordt eerst aangetoond dat Obligatie regels monotoon in de kosten zijn en bovendien leiden tot een populatie monotoon toewijzingsschema. Daarna wordt een nieuwe manier voor het definiëren van de irreducibele core gepresenteerd, gebaseerd op een niet-Archimedische halfmetriek. De Bird core blijkt interessante monotoniciteits- en additiviteitskenmerken te hebben en ieder stabiel toewijzingsmechanisme dat monotoon in de kosten is, blijkt een selectie van de Bird core op te leveren.

In Hoofdstuk 6 wordt een axiomatische karakterisering van de P-waarde gegeven, waarbij kegelwijze positieve lineariteit van de P-waarde een fundamentele eigenschap blijkt te zijn en waarbij ook de ontbinding van een mob situatie in simpele mob situaties een rol speelt. Gebruik makend van de additiviteitseigenschap wordt ook een karakterisering van de Bird core gegeven. Een waardetheoretische interpretatie van Obligatie regels, waarbij gebruik gemaakt wordt van 'sharing values' voor kostenspelen wordt ook bediscussieerd.

In Hoofdstuk 7 wordt aangetoond dat voor varianten van de klassieke mob spelen een populatie monotoon toewijzingsschema niet noodzakelijkerwijs hoeft te bestaan. In het bijzonder behandelt dit hoofdstuk monotone mob situaties en gerichte mob situaties. Gerichte mob situaties van een speciale soort worden bekeken, namelijk die situaties die te voorschijn komen wanneer het probleem bekeken wordt voor het verbinden van eenheden (huizen) in de bergen met een waterzuiveringsinstallatie. Voor dergelijke situaties wordt een simpele methode beschreven om een populatie monotoon toewijzingsschema te bepalen. Het blijkt dat de cores van gerelateerde kostentoewijzingsproblemen een simpele structuur hebben en dat ieder core element uitgebreid kan worden tot een populatie monotoon toewijzingsschema en ook tot een bi-monotoon toewijzingsschema.