VINES

for Jack on his 75th birthday

Adrian Bondy

with

Stephen Locke

VINES

What on earth is a vine?

VINES

NOT *The* Vines

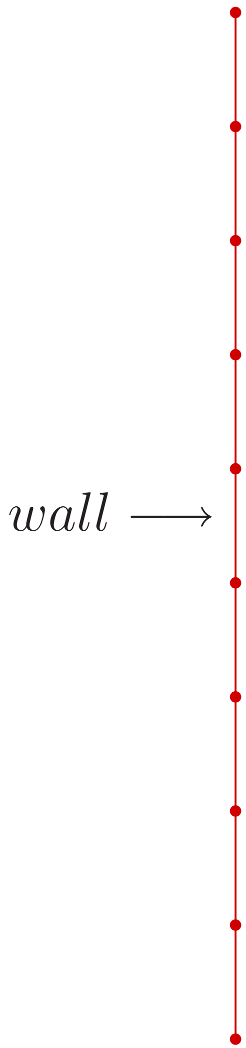
VINES

NOT *these* vines

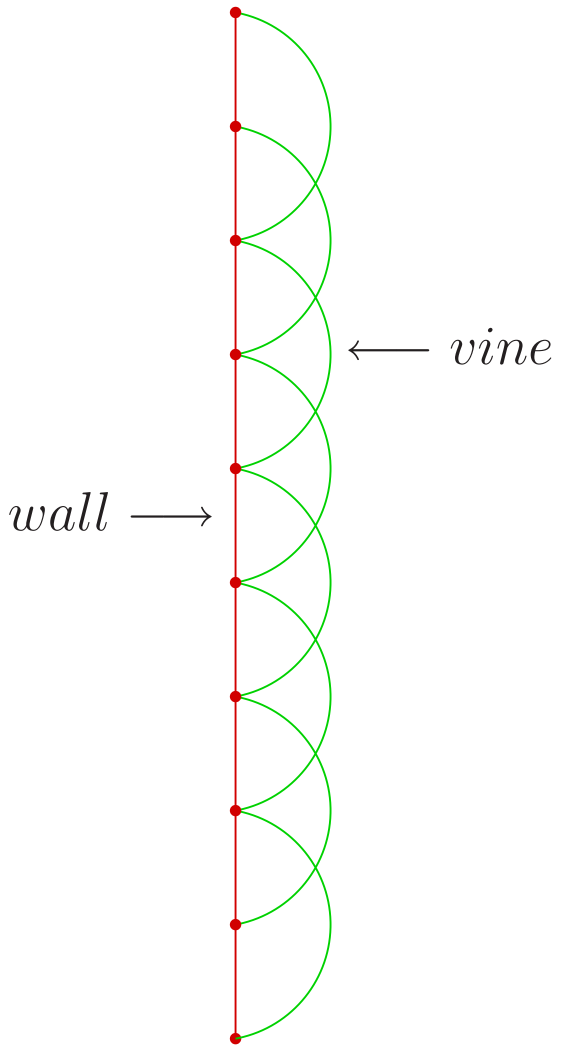
VINES

NOT *these* vines

these vines:



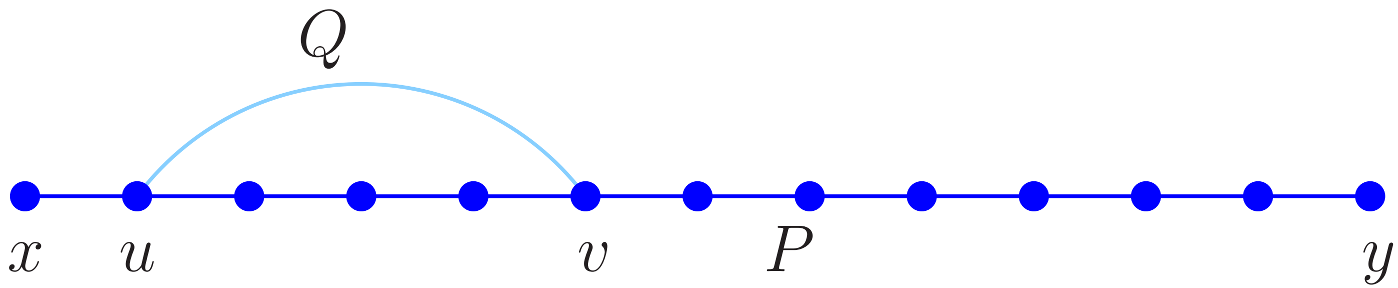
these vines:



OVERLAPPING PATHS

xPy: path P from x to y

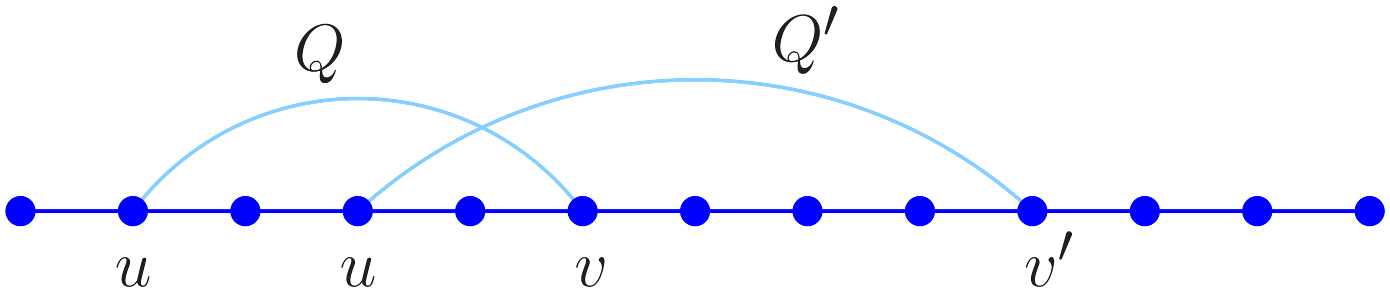
ear of P: path uQv such that $Q \cap P = \{u, v\}$



OVERLAPPING PATHS

Ears uQv and $u'Q'v'$ overlap on P if:

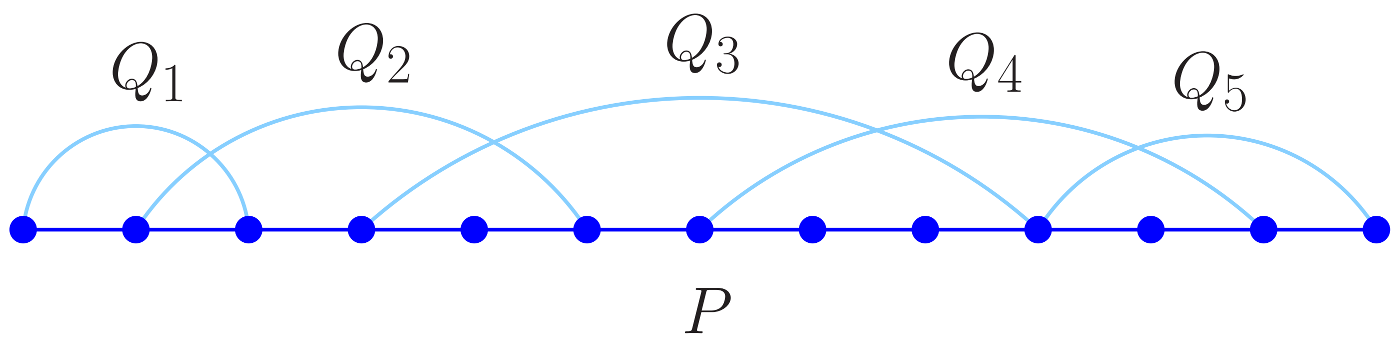
- $Q \cap Q' = \emptyset$
- $u \prec u' \prec v \prec v'$ or $u' \prec u \prec v' \prec v$



VINES

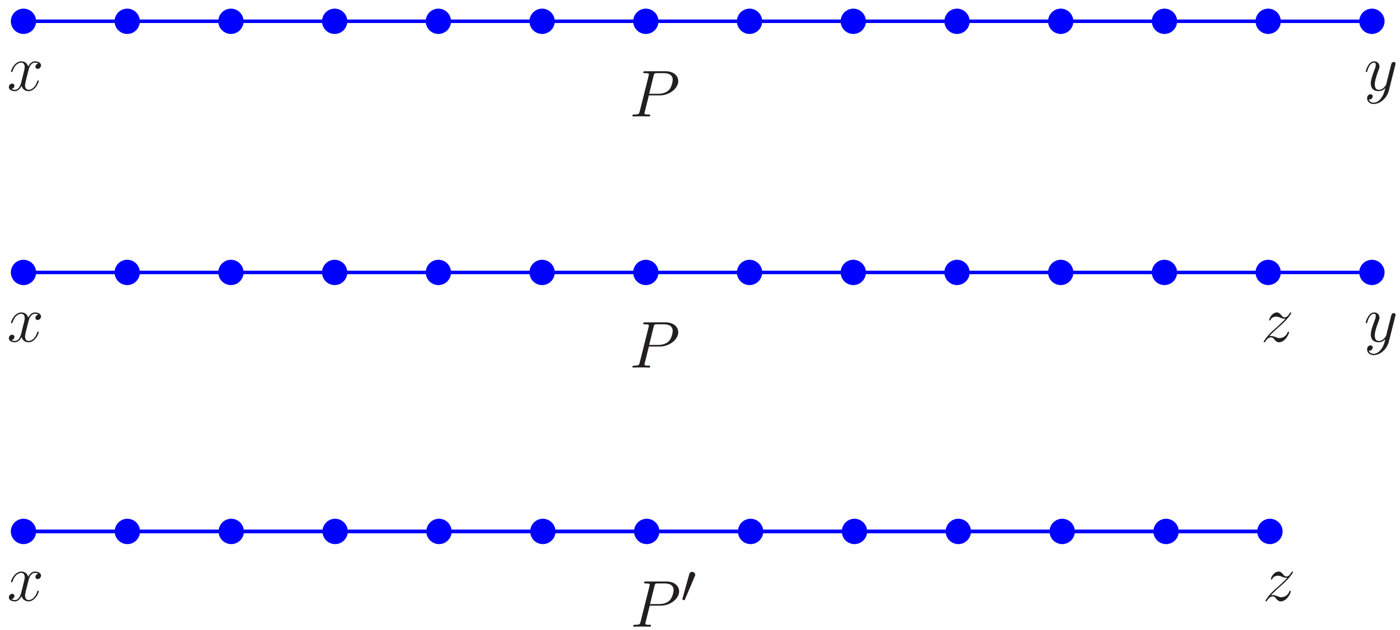
vine on a path P: sequence of ears $\mathcal{Q} := (Q_1, Q_2, \ldots, Q_r)$ where:

- Q_1 starts at the first vertex of P
- Q_r ends at the last vertex of P
- consecutive ears overlap
- nonconsecutive ears do not overlap

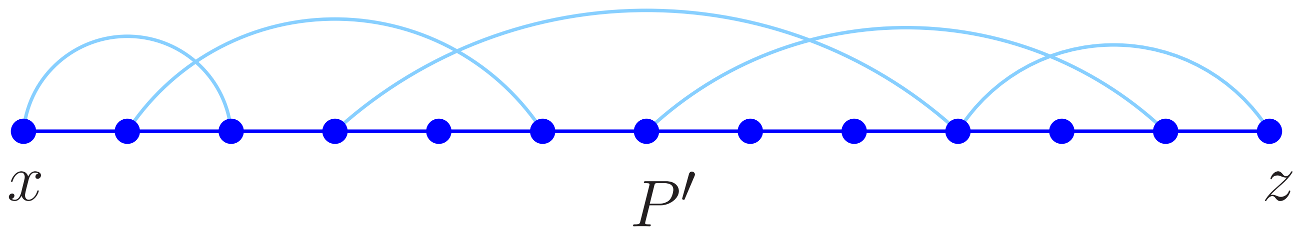


If P is a path in a 2-connected graph, there is a vine on P

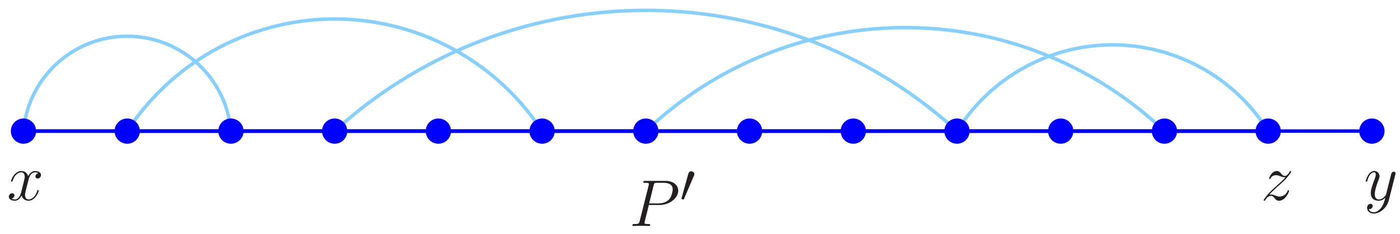
<u>Proof</u>. Induction on the length of P:



VINES



VINES



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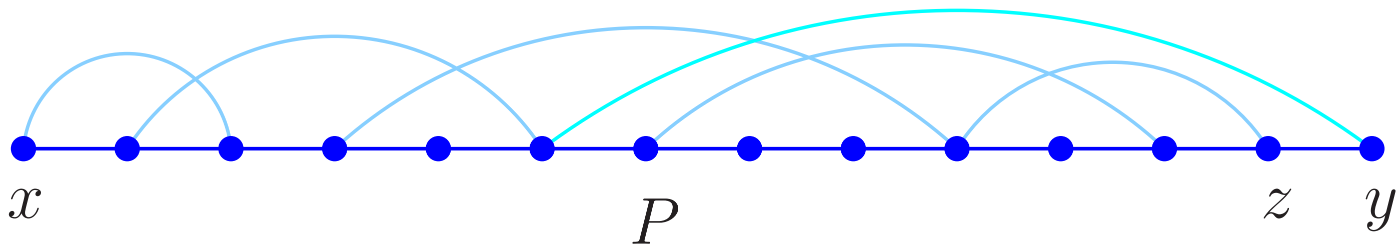

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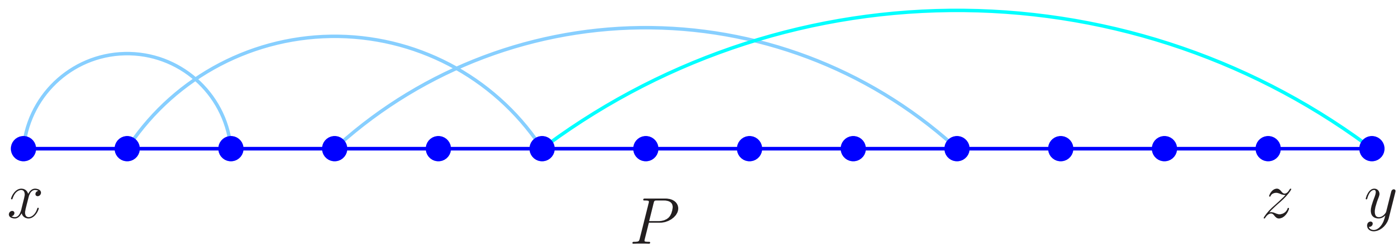

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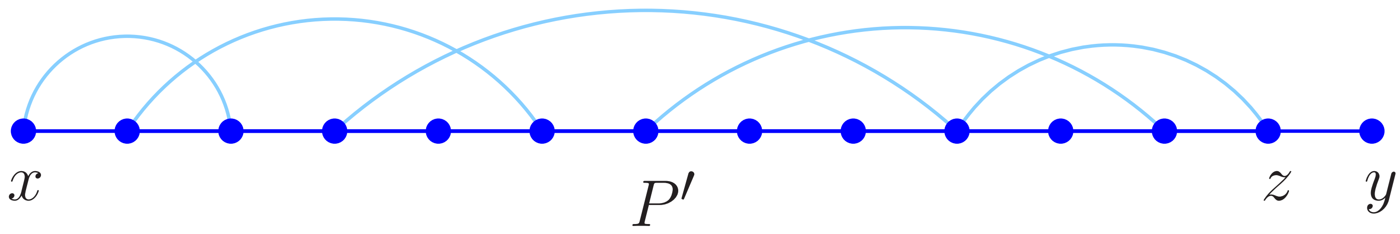

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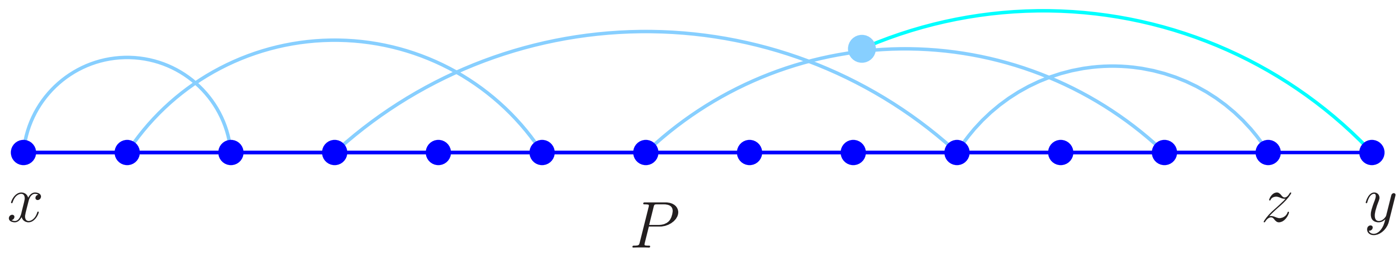

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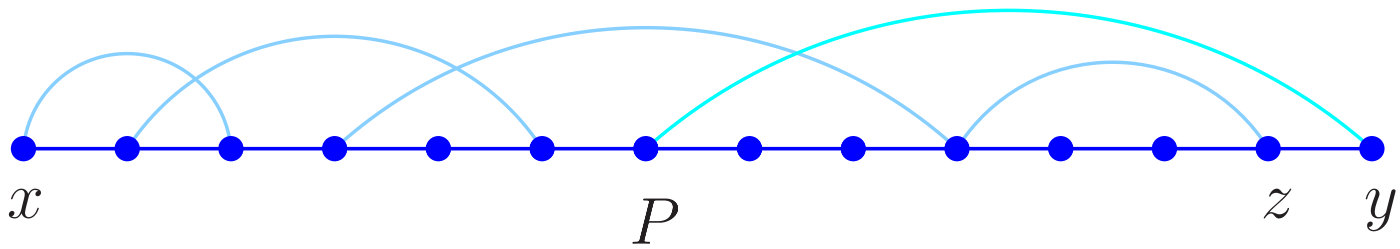

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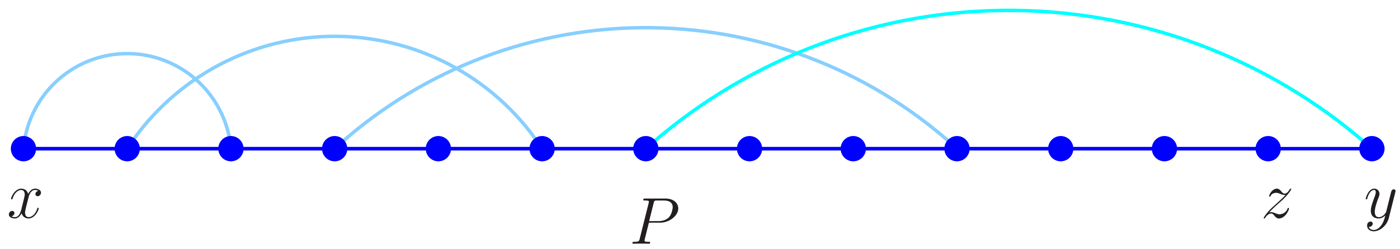
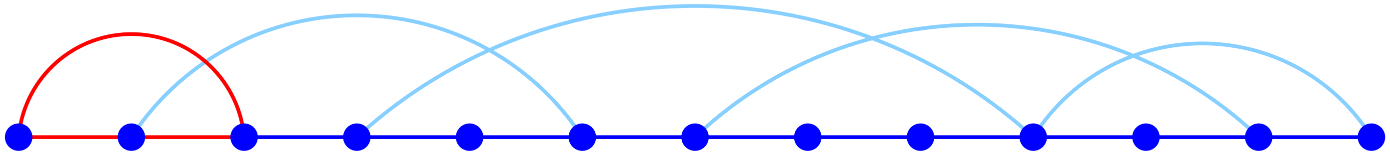


CYCLES IN VINES

Each ear defines a cycle:



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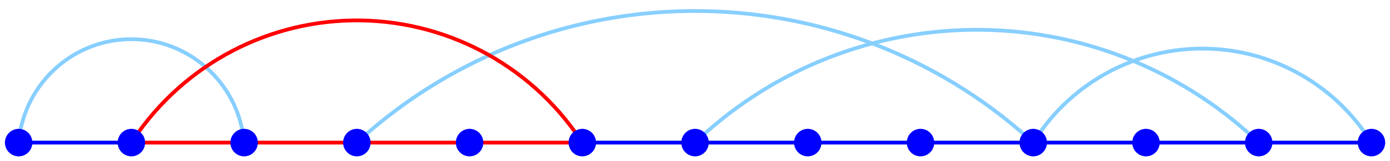

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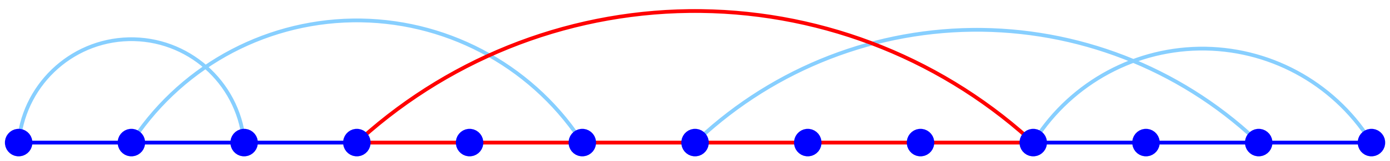

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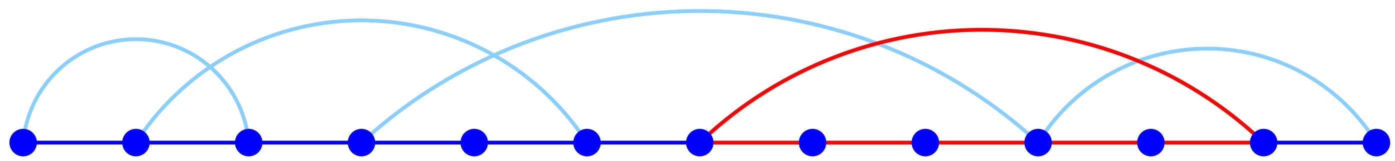

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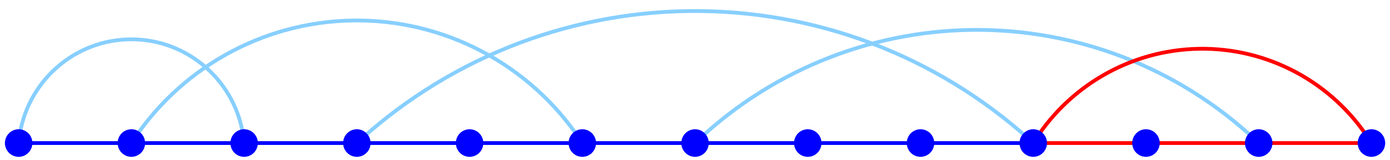
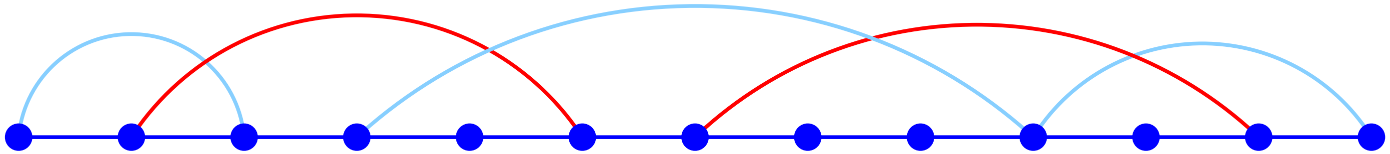


CYCLES IN VINES

Each pair of ears defines a cycle:



CYCLES IN VINES

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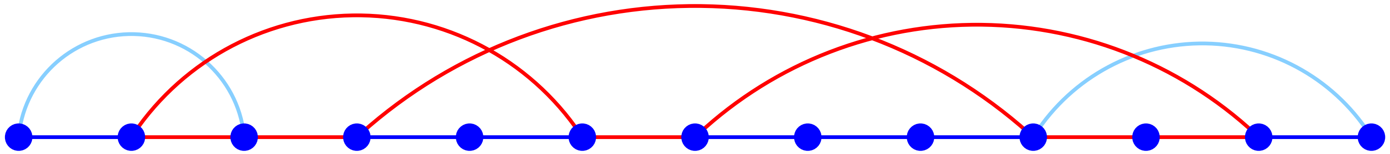
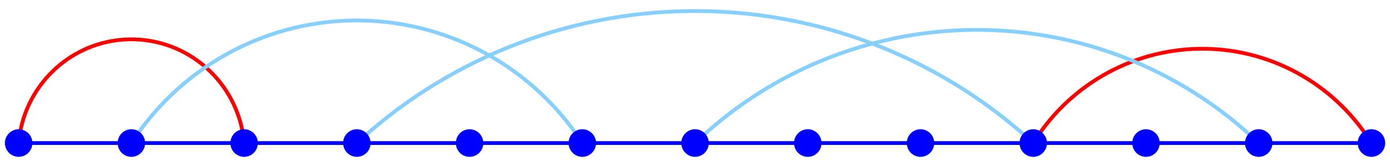


CYCLES IN VINES

In particular, the first and last ears define a cycle C:



CYCLES IN VINES

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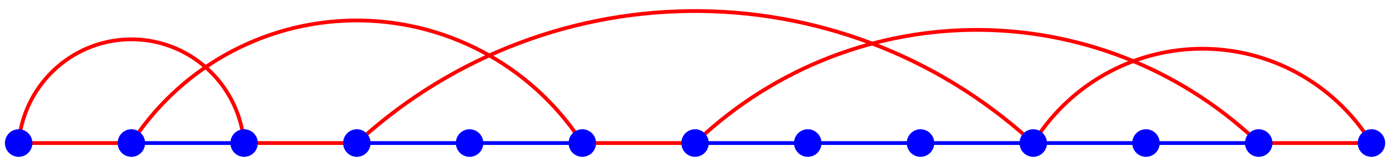


LONG CYCLES

Dirac *A 2-connected graph with minimum degree* d *contains either a cycle of length at least* $2d$ *or a Hamilton cycle.*

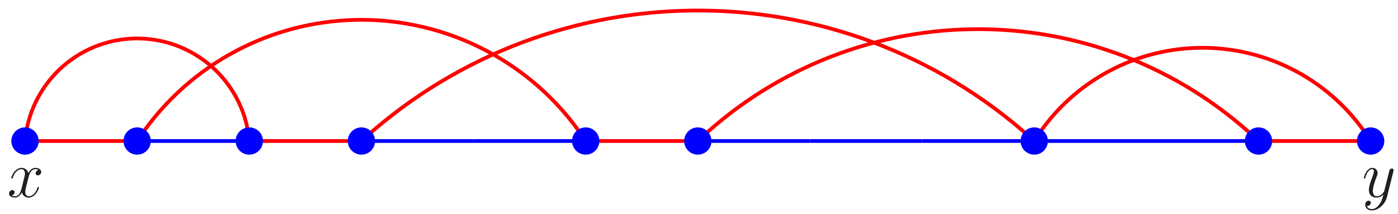
Proof

P a longest path $\mathcal{Q}$ a vine on P such that:

- $|\mathcal{Q}|$ is as small as possible
- subject to this condition, $|V(C) \cap V(P)|$ is as large as possible

LONG CYCLES

Where are the neighbours of x?



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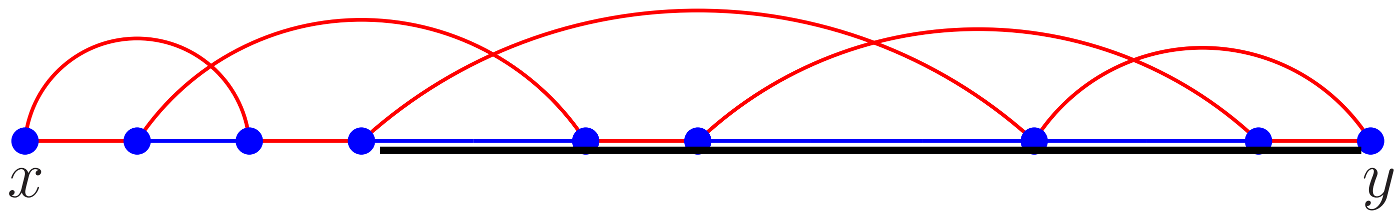

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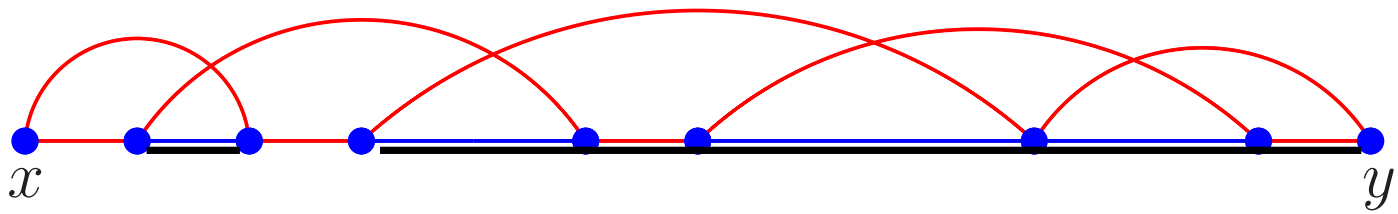

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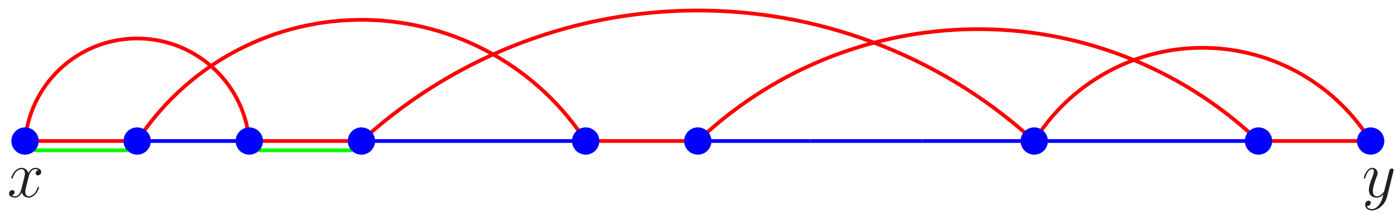

Both x and all its neighbours lie on C.

LONG CYCLES

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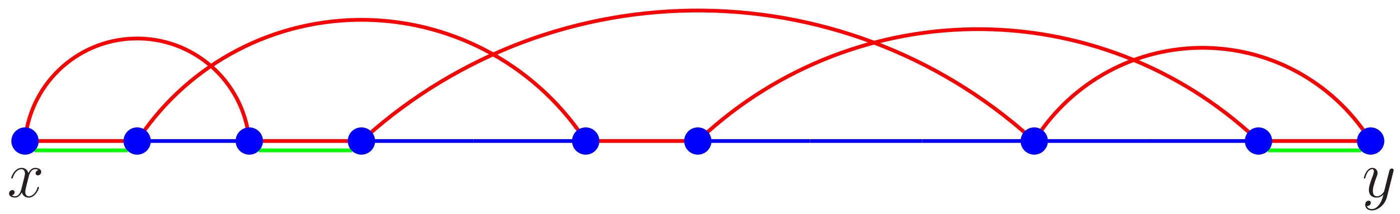

Both x and all its neighbours lie on C. Likewise for y.

LONG CYCLES

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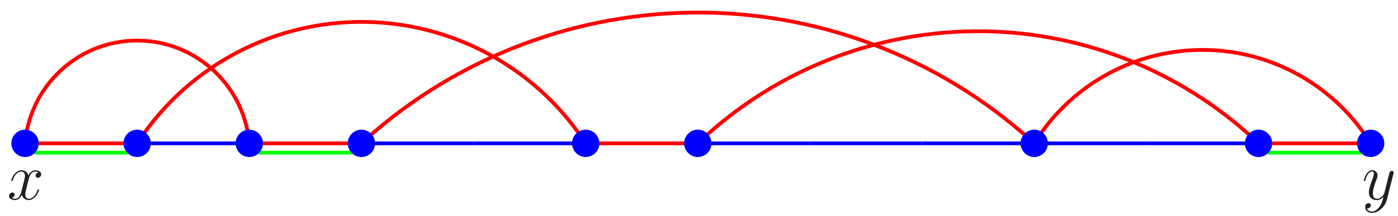


Both x and all its neighbours lie on C. Likewise for y.

This implies that C has length at least $2d$ or is a Hamilton cycle.

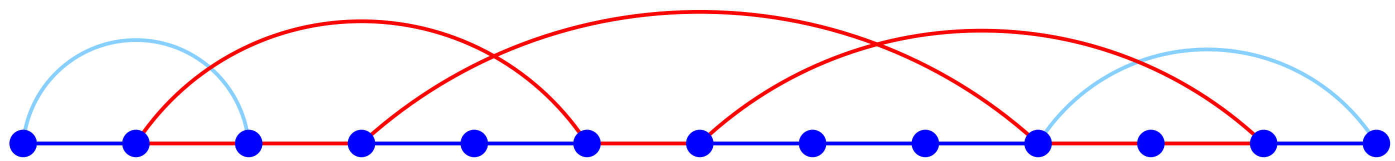
LONG CYCLES

Dirac *A 2-connected graph which contains a path of length l contains a cycle of length at least $2\sqrt{l}$.*

Proof

P a longest path $\mathcal{Q}$ a vine on P

Recall that each pair of ears in $\mathcal{Q}$ defines a cycle:



LONG CYCLES

Suppose (for simplicity) that $|\mathcal{Q}| = 2t - 1$ is odd.

There are t^2 such cycles which include the central ear.

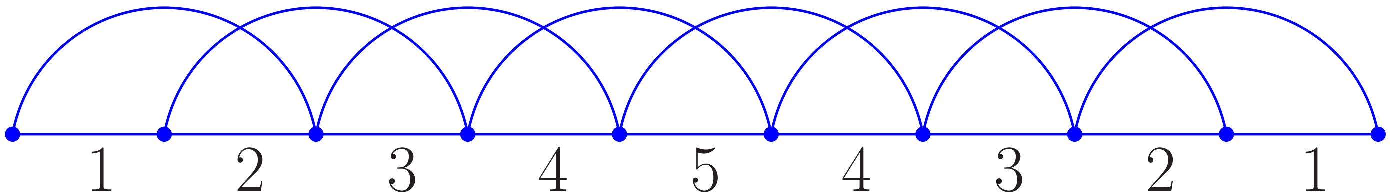
These cycles cover P t times and the ears a total of t^3 times.

So their average length is

$$\frac{lt + t^3}{t^2} = \frac{l}{t} + t \geq 2\sqrt{l}$$

LONG CYCLES

Best possible:



LONG CYCLES

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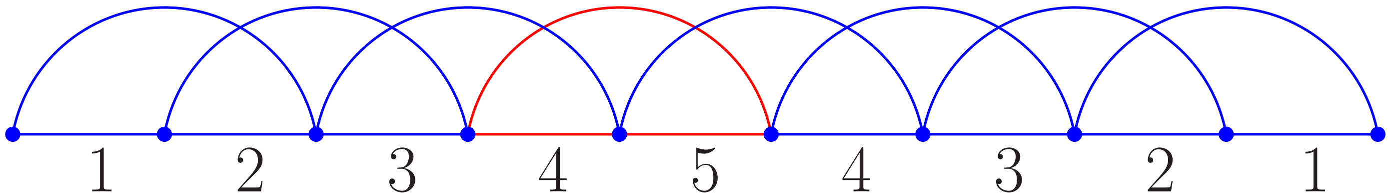

LONG CYCLES

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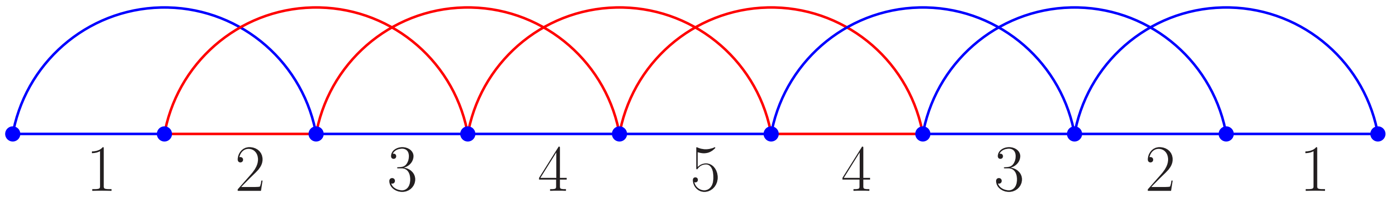
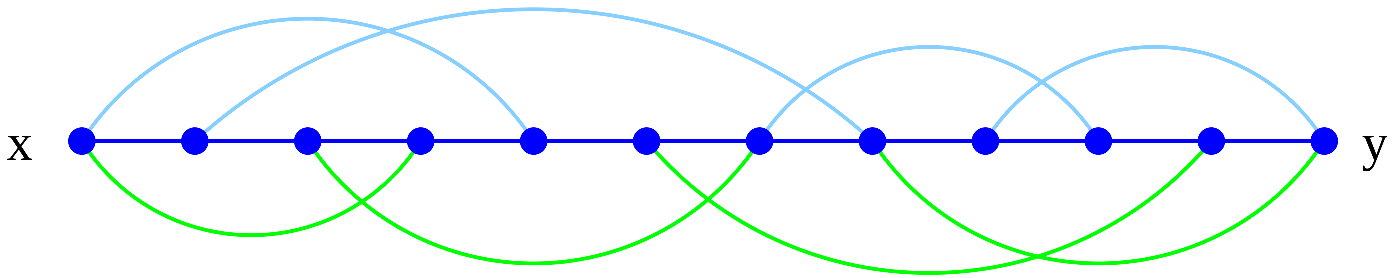


DISJOINT VINES

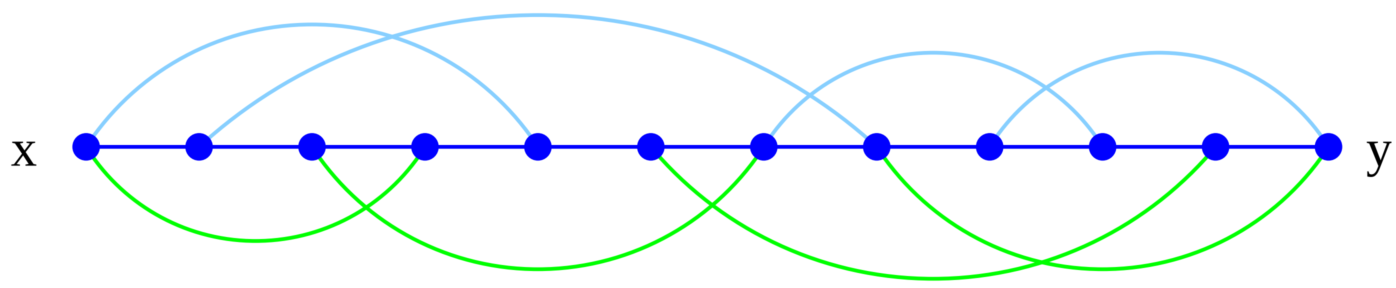
Vines $\mathcal{Q} := (Q_1, Q_2, \ldots, Q_r)$ and $\mathcal{R} := (R_1, R_2, \ldots, R_s)$ on a path xPy are disjoint if:

- their ears meet only on P
- only Q_1 and R_1 have a common first vertex (x)
- only Q_r and R_s have a common last vertex (y)



DISJOINT VINES

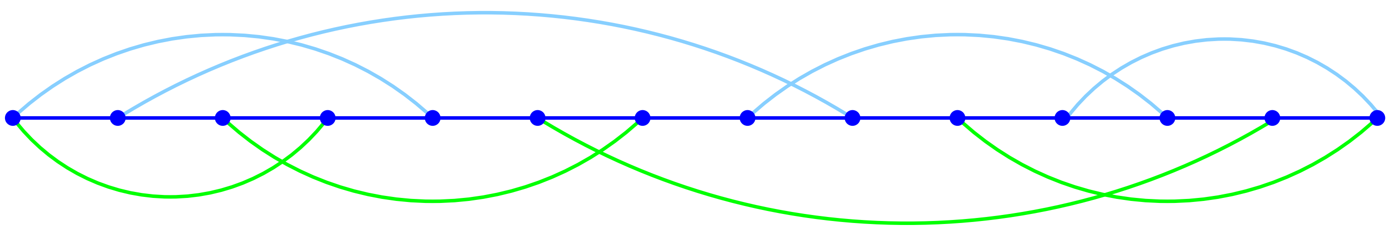
B+Locke *If P is a path in a 3-connected graph, there are two disjoint vines on P.*



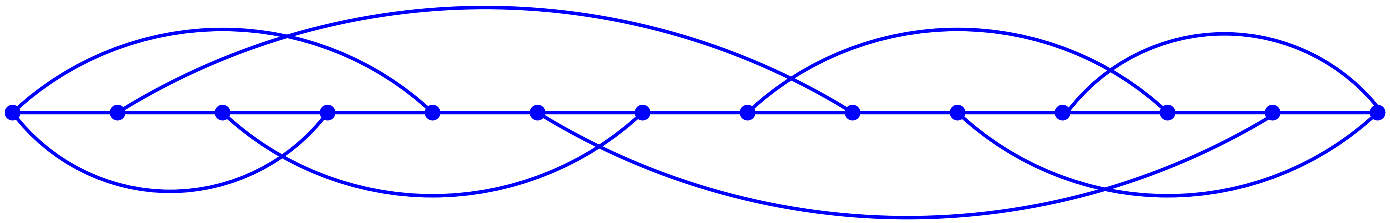
Proof. Menger's Theorem

3-CONNECTED CUBIC GRAPHS

If P has length l, how long a cycle must there be?

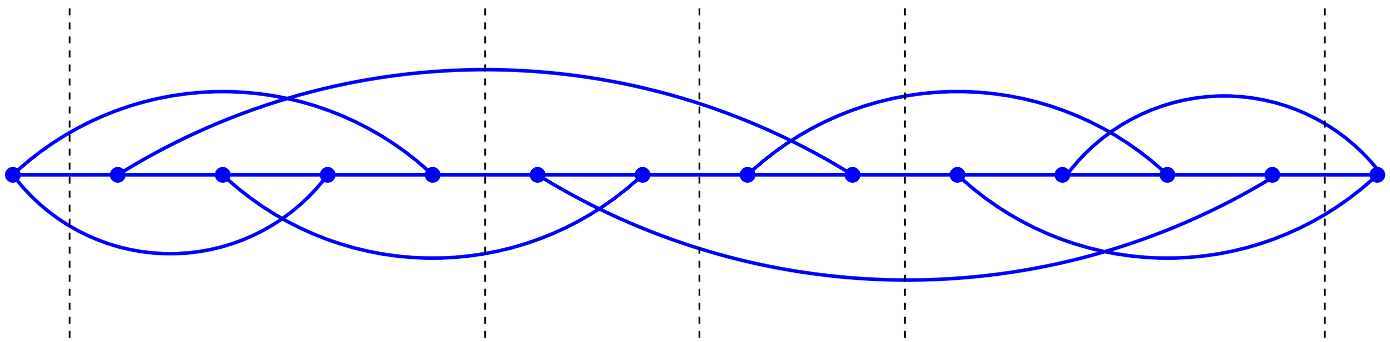


3-CONNECTED CUBIC GRAPHS

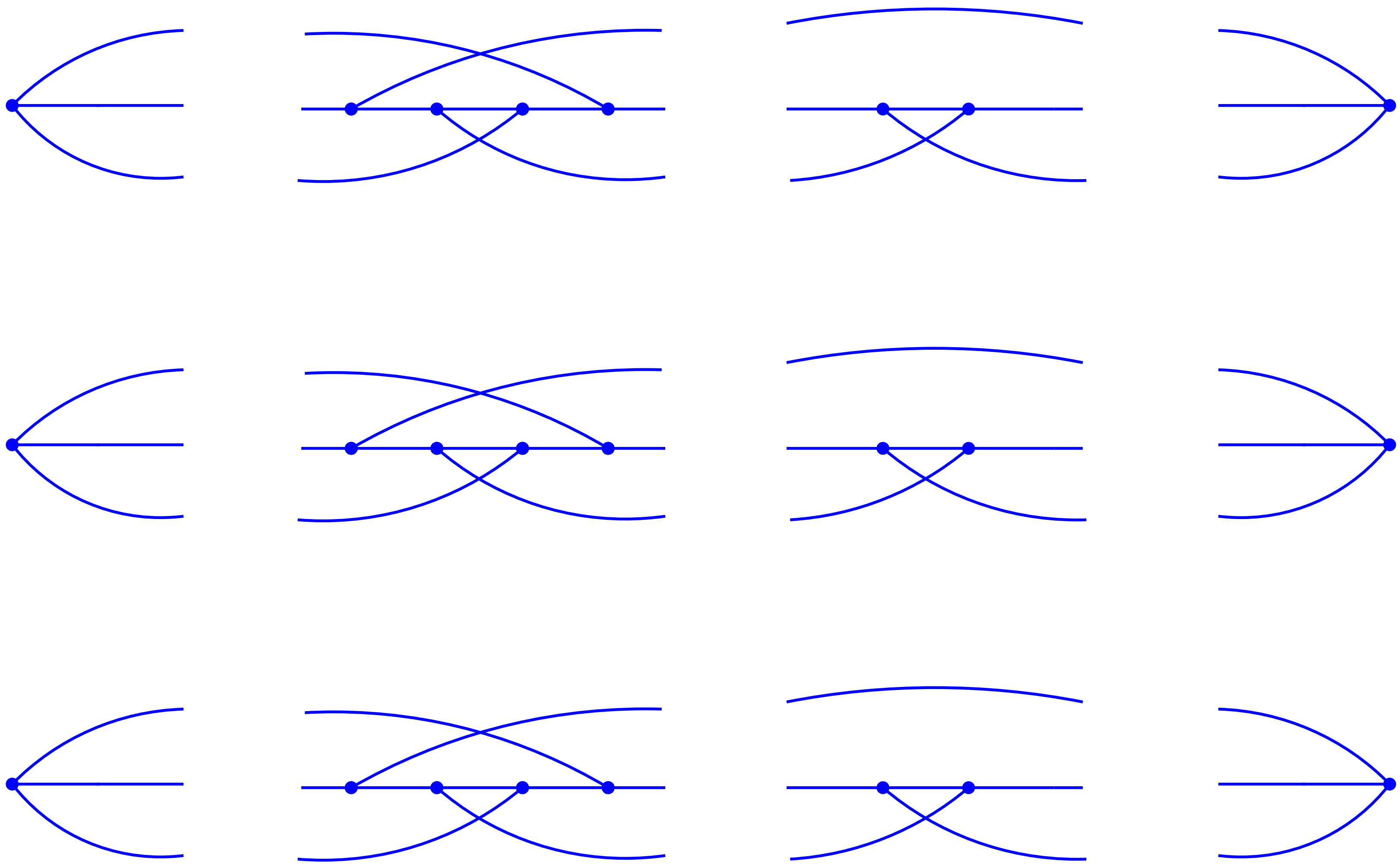


3-CONNECTED CUBIC GRAPHS

Split the subgraph into 'modules':



3-CONNECTED CUBIC GRAPHS



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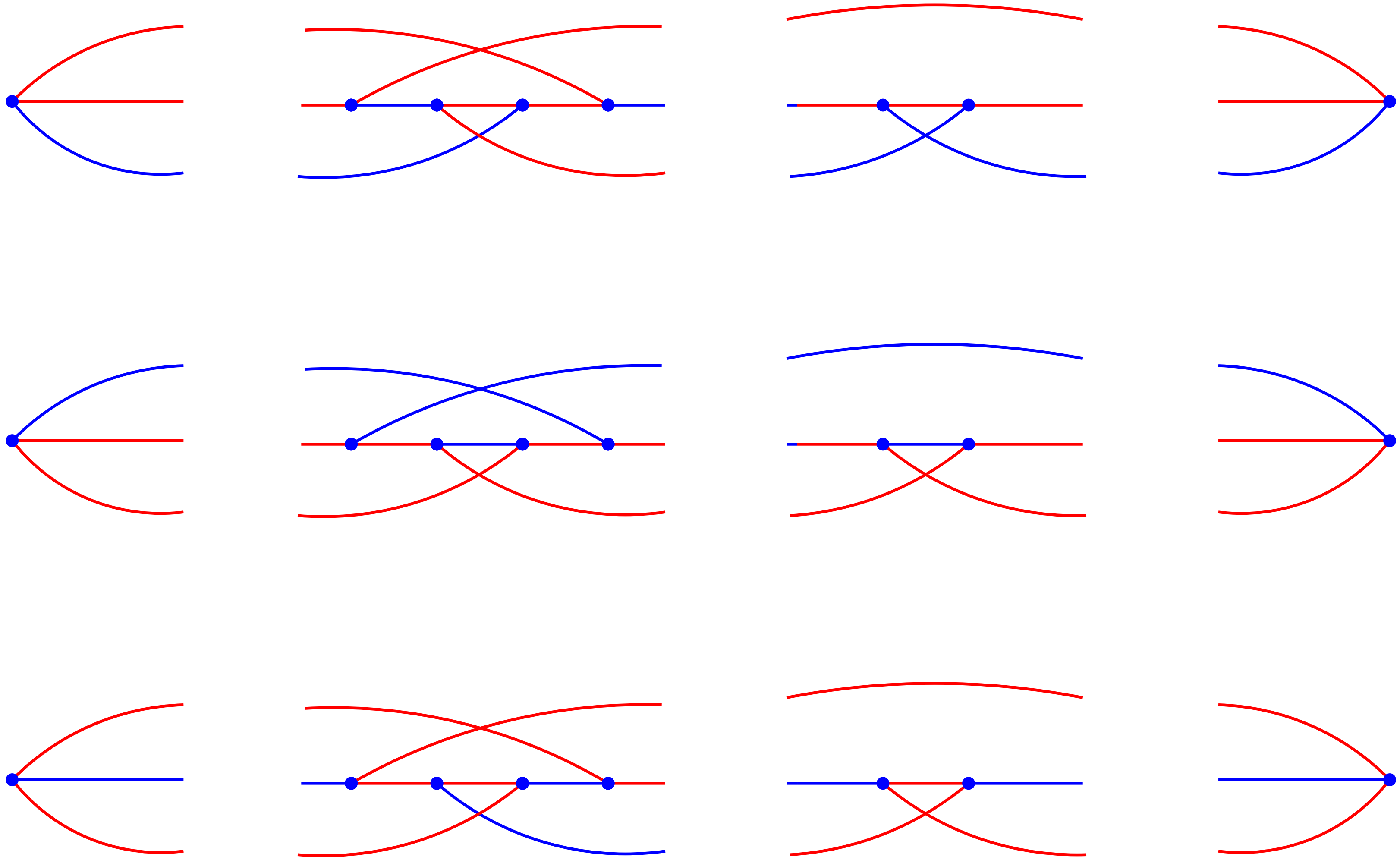

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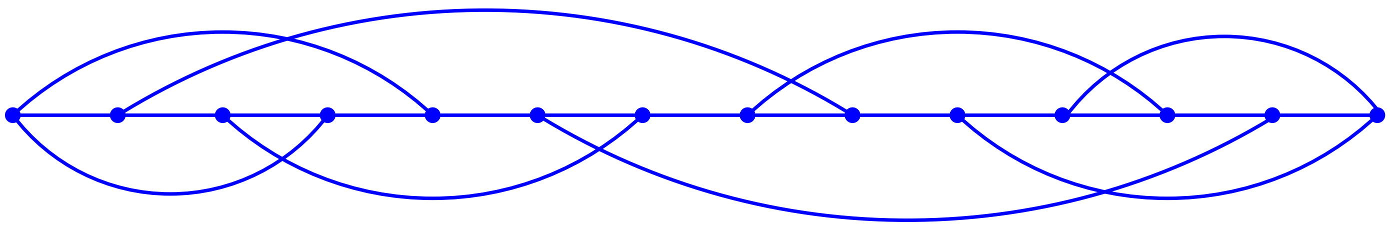

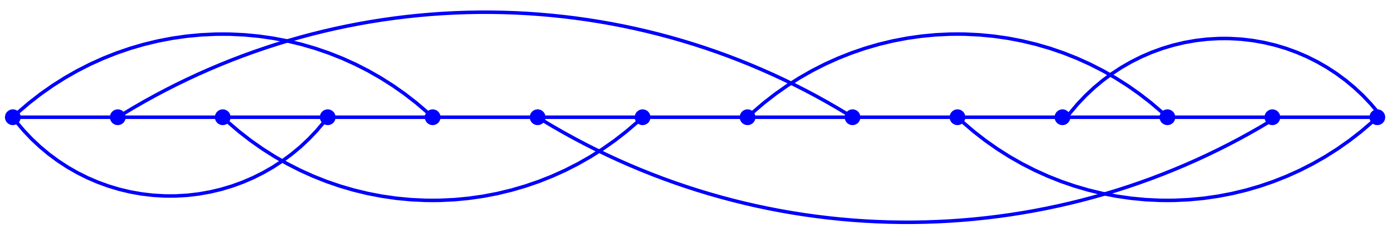

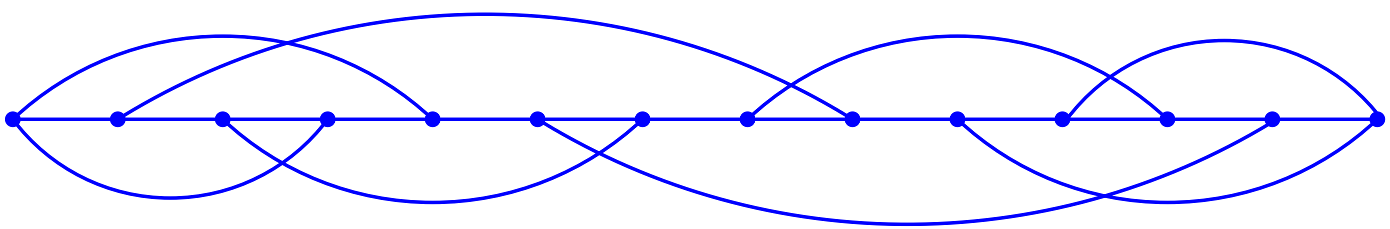

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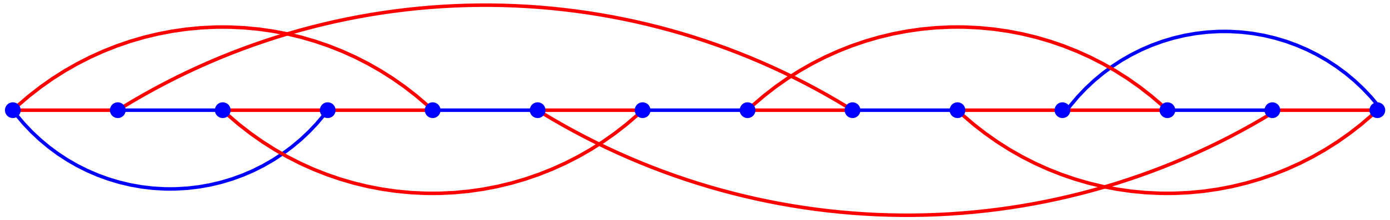

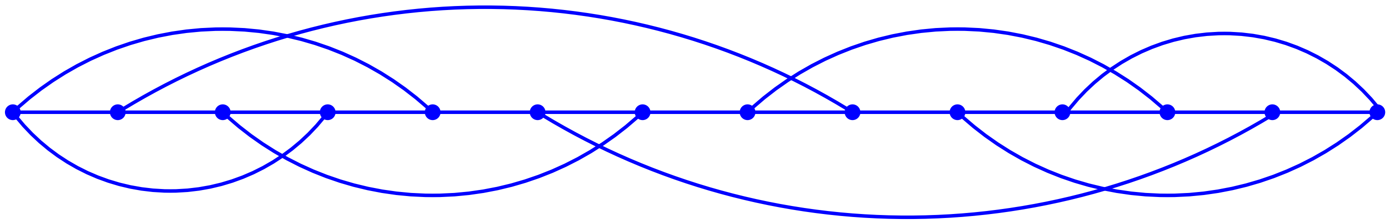

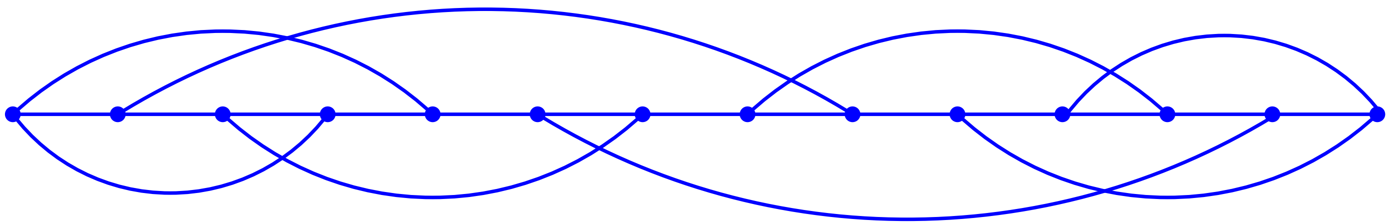

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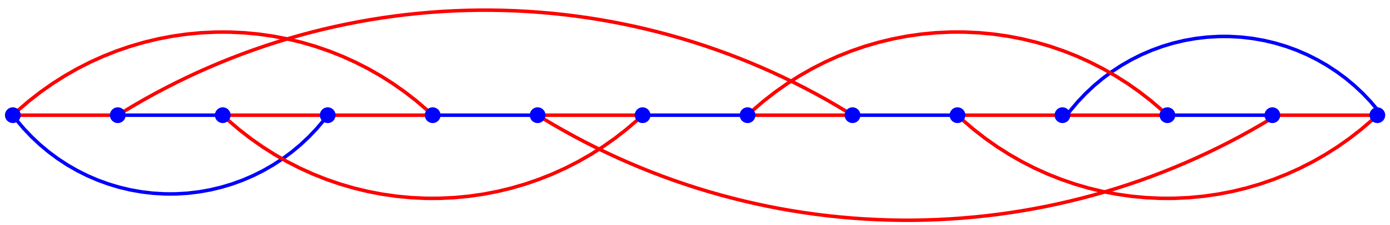

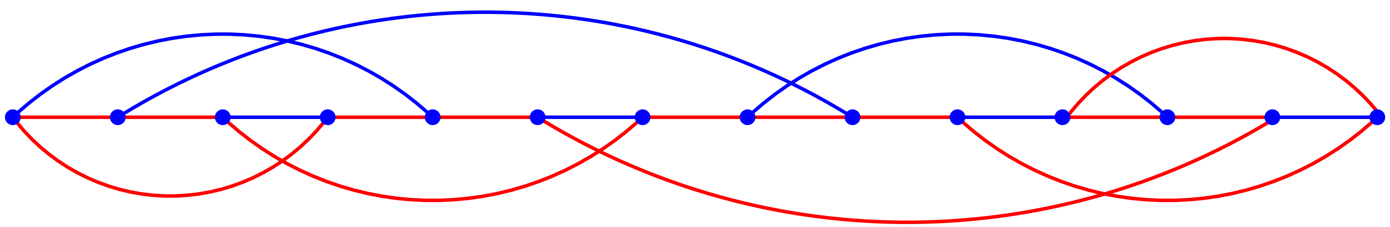

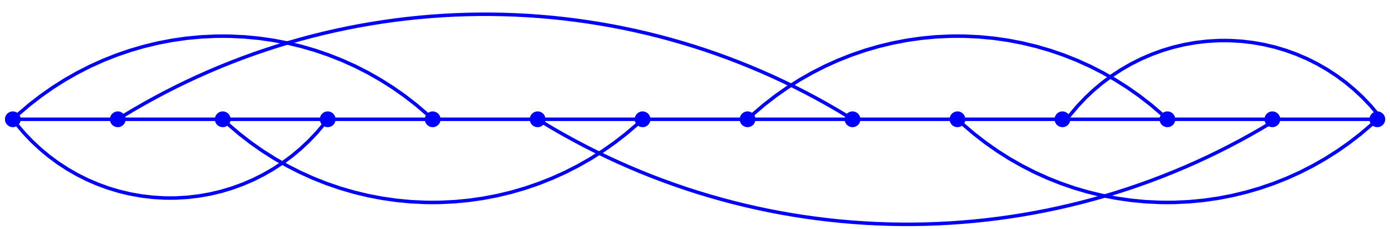

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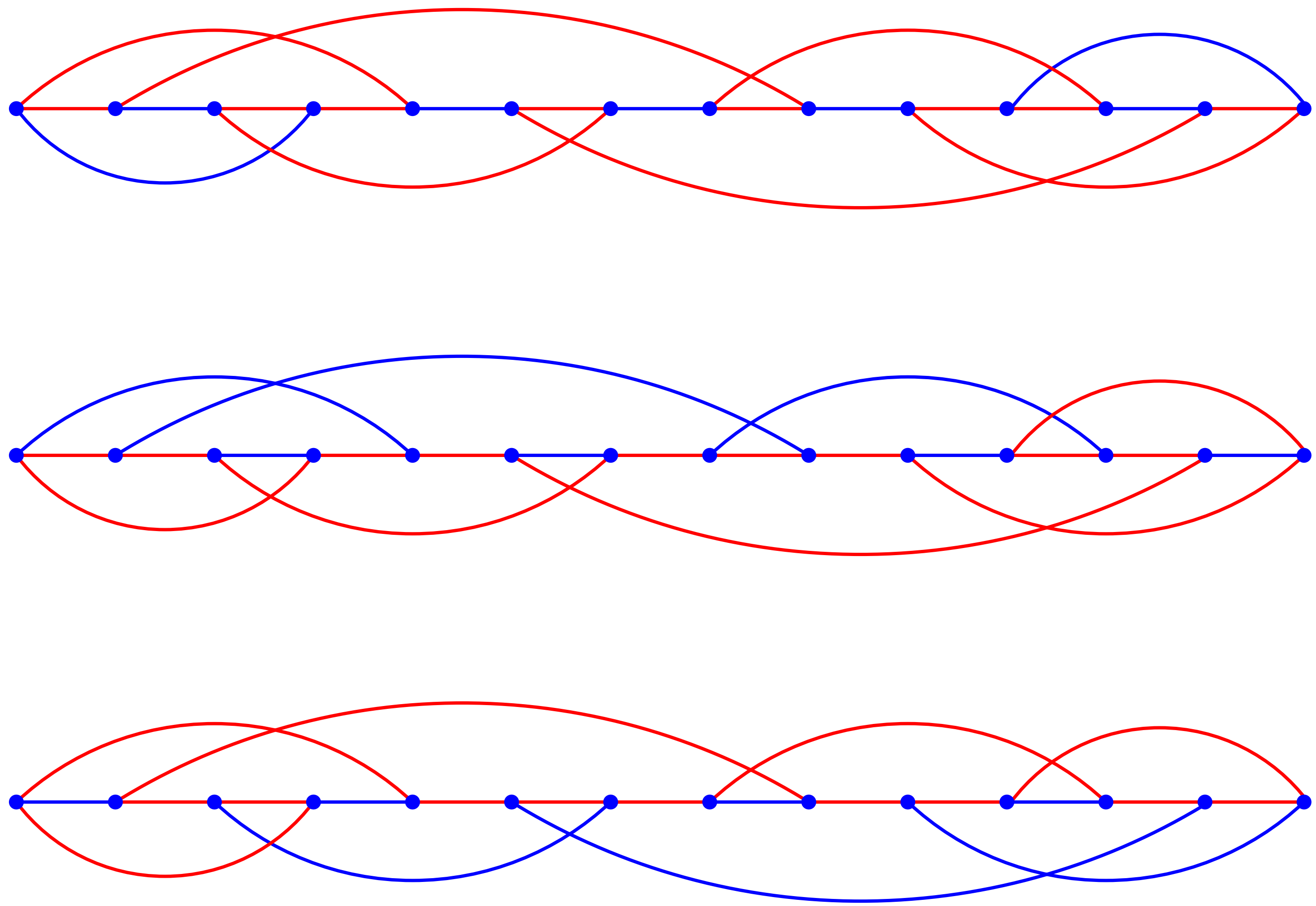
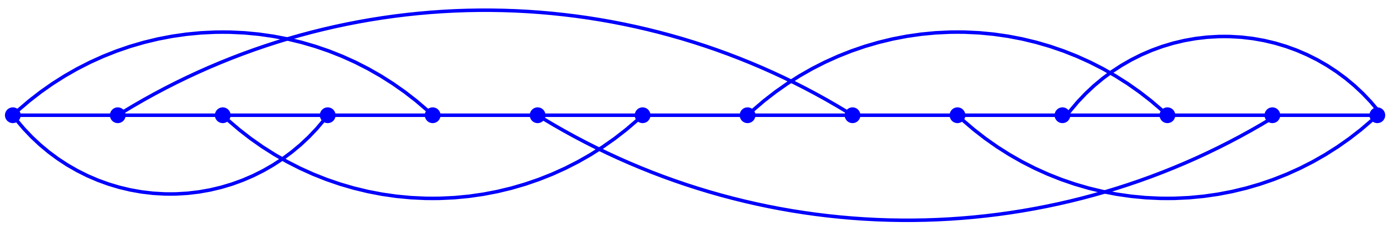


3-CONNECTED CUBIC GRAPHS

Three cycles covering each edge of this subgraph exactly twice.



3-CONNECTED CUBIC GRAPHS

B+Locke *A 3-connected cubic graph which contains a path of length* l *contains a cycle of length at least* $\frac{2}{3}l$.

Upper bound (based on Petersen graph): $\frac{7}{8}l$

3-CONNECTED GRAPHS

Dirac *A 2-connected graph which contains a path of length* l *contains a cycle of length at least* $2\sqrt{l}$.

B+Locke *A 3-connected graph which contains a path of length* l *contains a cycle of length at least* $\frac{2}{5}l$.

Thomassen Upper bound: $\frac{1}{2}l$

k-CONNECTED GRAPHS

Locke *A k-connected graph which contains a path of length* l *contains a cycle of length at least* $\left(\frac{2k-4}{3k-4}\right) l$.

Thomassen Upper bound: $\left(\frac{k-2}{k-1}\right) l$

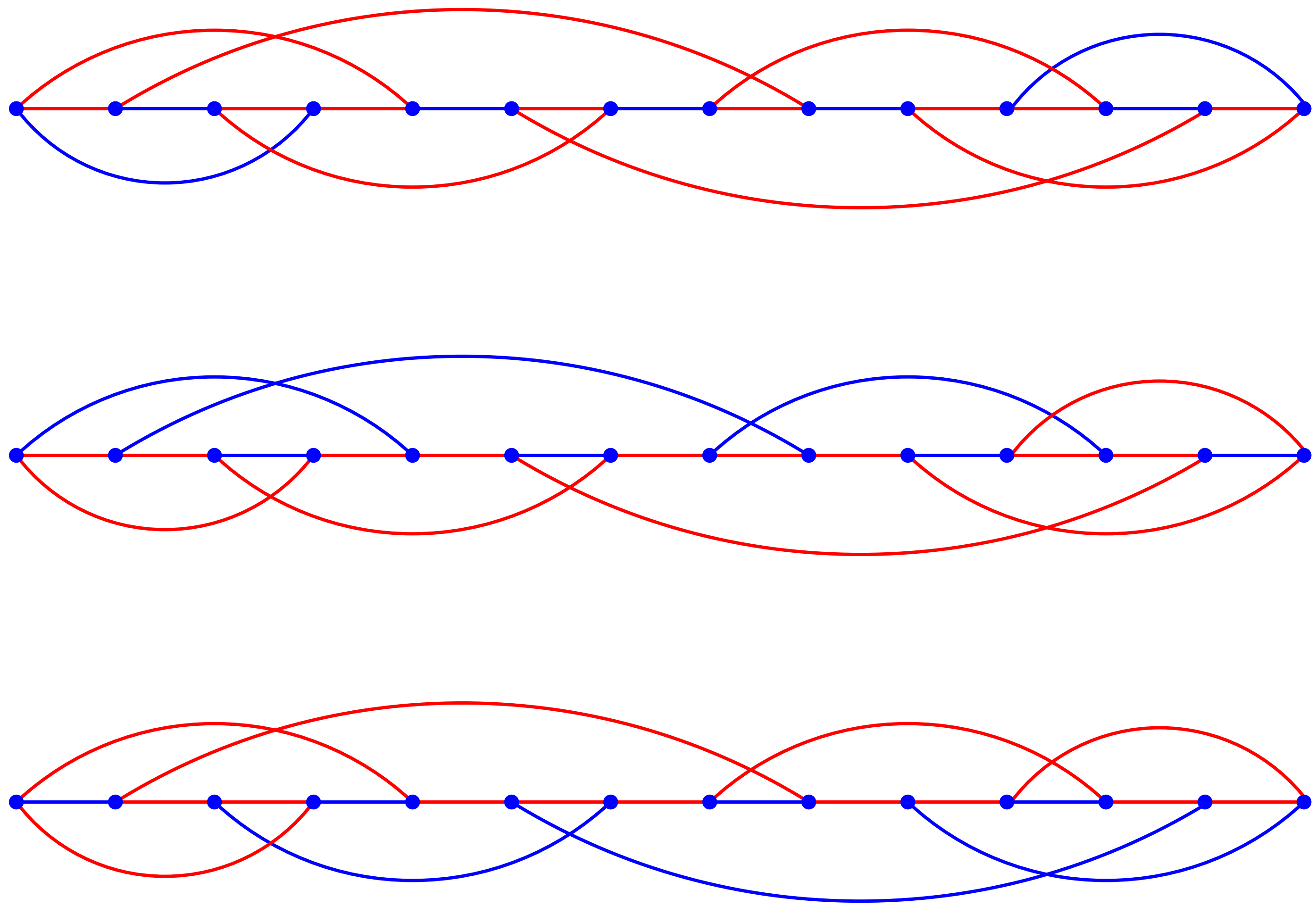
CYCLE DOUBLE COVERS

Tarsi *A 2-edge-connected graph which contains a Hamilton path admits a double cover by six even subgraphs.*

Proof by Goddyn

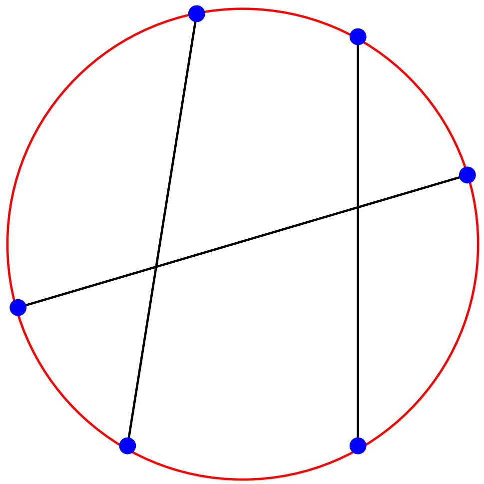
- reduce (by standard arguments) to 3-connected cubic graphs
- consider two disjoint vines on the Hamilton path
- the union of the vines and the path is a spanning subgraph H
- there are three cycles C_1, C_2, C_3 which cover each edge of H exactly twice

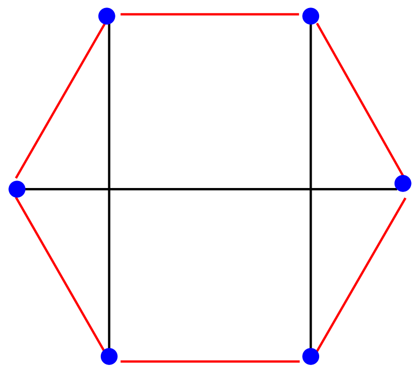
CYCLE DOUBLE COVERS



CYCLE DOUBLE COVERS

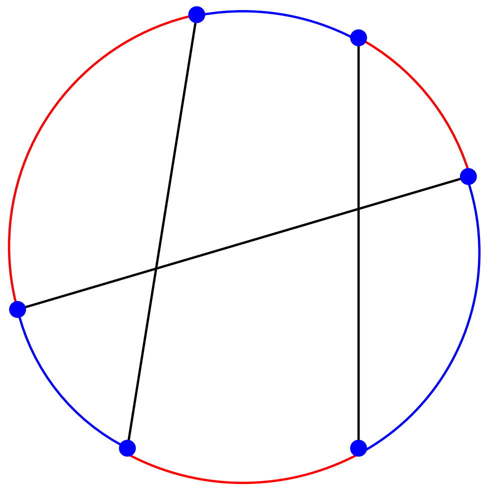
- the remaining set of edges F admits a partition into three subsets F_1, F_2, F_3, where the edges in F_i are chords of C_i, $i = 1, 2, 3$
- $C_i \cup F_i$ is either a cycle or a subdivision of a cubic graph K_i

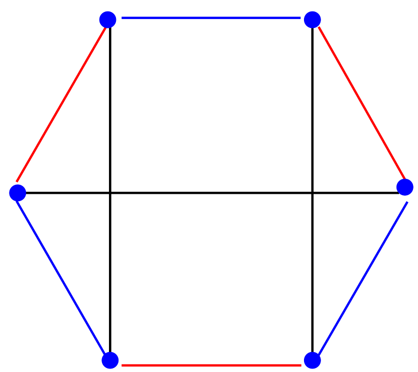




CYCLE DOUBLE COVERS

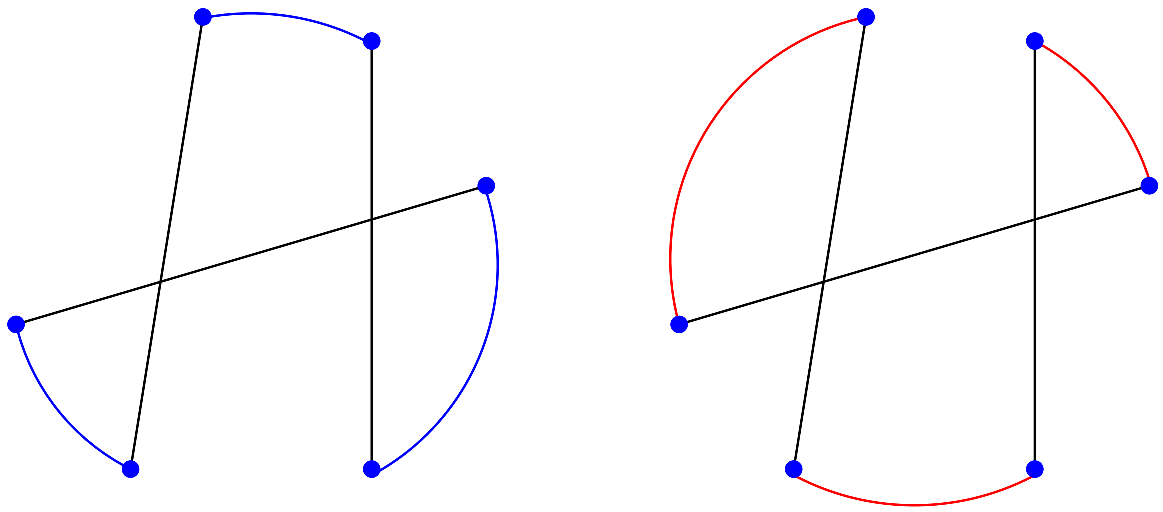
- K_i is hamiltonian, so has a 3-edge-colouring in which the edges of F_i receive one colour and the edges of the Hamilton cycle are coloured alternately with the other two colours





CYCLE DOUBLE COVERS

- the union of F_i with each of the other colours is a 2-factor of K_i
- these two 2-factors correspond to two even subgraphs of $C_i \cup F_i$
- the resulting six even subgraphs constitute a double cover



CYCLE DOUBLE COVERS

Conjecture (Preissmann) *Every 2-edge-connected graph admits a double cover by five even subgraphs.*