An operator for graph parameters between α and $\overline{\chi}$

The sandwich line graph

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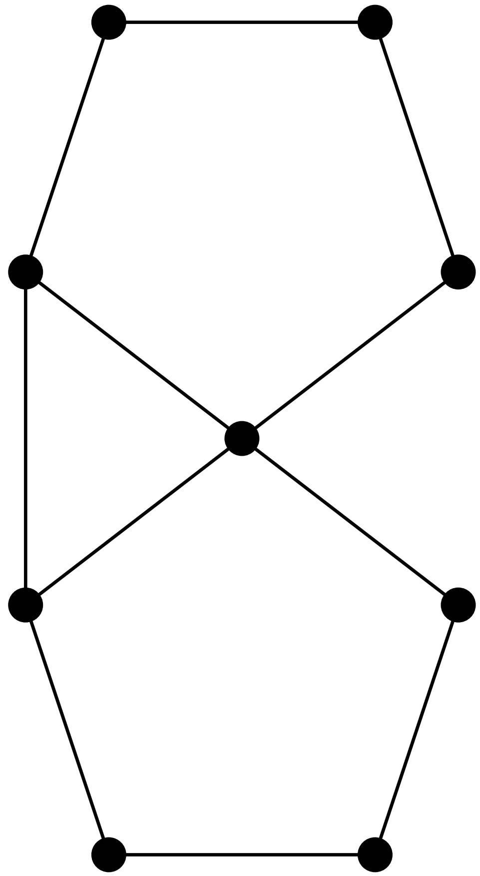
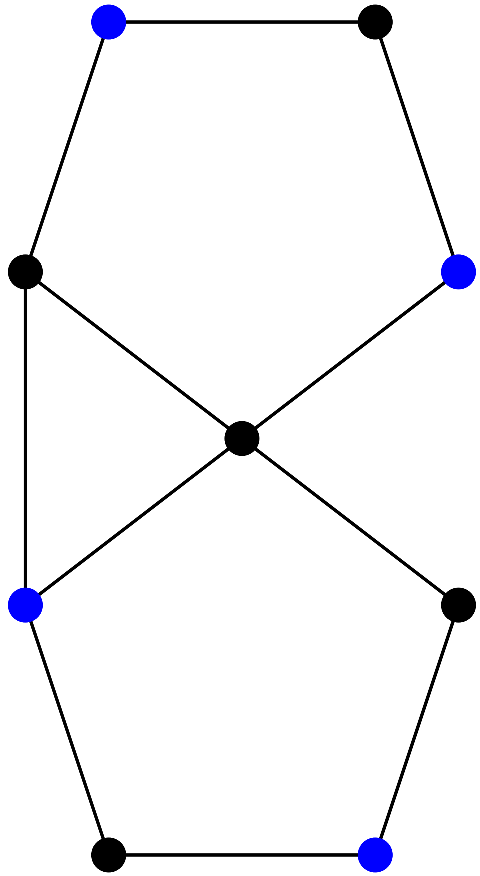
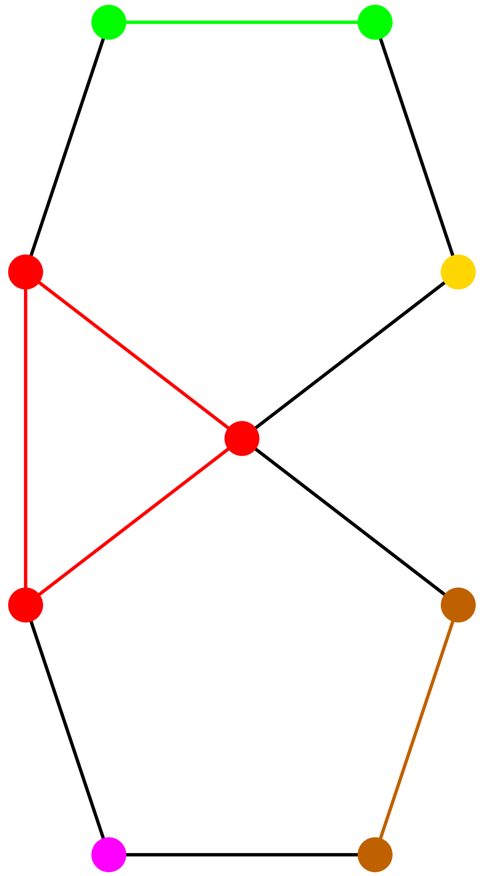
Introduction



Introduction (α)



Introduction ($\overline{\chi}$)



Introduction

We will study the "sandwich property" of the edge graph introduced in:

D.C., V. Jost: A one-to-one correspondence between colorings and stable sets. *Oper. Res. Lett. 36(6) 673-676 (2008)*

The fractional clique covering number $\overline{\chi}_f(G)$

Let $G = (V, E)$ be a graph with clique-set $\mathcal{K}$.

$$\begin{aligned}
\alpha(G) \quad &= \quad \max\{\sum_{v \in V} x_v : \quad \sum_{v \in K} x_v \leq 1, \ \forall K \in \mathcal{K}; \quad x \in \{0,1\}^V\} \\
&\leq \quad \max\{\sum_{v \in V} x_v : \quad \sum_{v \in K} x_v \leq 1, \ \forall K \in \mathcal{K}; \quad x \geq 0\} \\
&= \quad \min\{\sum_{K \in \mathcal{K}} y_K : \quad \sum_{K \in \mathcal{K}} y_K \geq 1, \ \forall v \in V; \quad y \geq 0\} \\
&\leq \quad \min\{\sum_{K \in \mathcal{K}} y_K : \quad \sum_{K \in \mathcal{K}} y_K \geq 1, \ \forall v \in V; \quad y \in \{0,1\}^{\mathcal{K}}\} \\
&= \quad \overline{\chi}(G)
\end{aligned}$$

These three parameters are NP-hard to compute:

$$\alpha(G) \leq \overline{\chi}_f(G) \leq \overline{\chi}(G)$$

The Lovász theta number $\vartheta(G)$

Let $G = (V, E)$ be a graph and let $\mathcal{M}_G$ be the set of symmetric $V \times V$ matrices, the trace of which is 1, with $M_{u,v} = 0$ for distinct adjacent u, v, and which are positive semidefinite.

$$\begin{aligned}
\alpha(G) &\leq \max\{\sum_{u,v \in V} M_{u,v} : \quad M \in \mathcal{M}_G\} \\
&\leq \overline{\chi}_f(G) \\
&\leq \overline{\chi}(G)
\end{aligned}$$

The Lovász theta number $\vartheta(G)$ can be computed in polynomial time (ε approx).

Example : The 5-cycle

If $G = C_5$, then

$$\begin{aligned} \alpha(C_5) &= 2 \\ &< \sqrt{5} \\ &= \vartheta(C_5) \\ &< 5/2 \\ &= \overline{\chi}_f(C_5) \\ &< 3 \\ &= \overline{\chi}(C_5) \end{aligned}$$

($\sqrt{5} \simeq 2.236$)

Other polynomial sandwich functions

$$\vartheta'(G)$$

R.J. McEliece, E.R. Rodemich, H.C. Rumsey Jr.: *The Lovász bound and some generalizations, J. Combin. Inform. System Sci. 3 134-152 (1978)* and

A. Schrijver: *A comparison of the Delsarte and Lovász bounds, IEEE Transactions on Information Theory IT-25 425-429 (1979)*

$$\vartheta^{+}(G)$$

M. Szegedy: *A note on the Theta number of Lovász and the generalized Delsarte bound, FOCS 36-39 (1994)*

$$\vartheta^{+\triangle}(G)$$

P. Meurdesoif: *Strengthening the Lovász Theta(G) bound for graph coloring. Math. Program. 102(3) 577-588 (2005)*

$$\vartheta'^{\triangle}(G)$$

I. Dukanovic, F. Rendl: *Semidefinite programming relaxations for graph coloring and maximal clique problems, Math. Program. 109(2-3) 345-365 (2007)*

The $\overline{\chi}_f(G)$ barrier

The polynomial sandwich functions satisfy:

$$\alpha(G) \leq \vartheta'^{\triangle}(G) \leq \vartheta'(G) \leq \vartheta(G) \leq \vartheta^{+}(G) \leq \vartheta^{+\triangle}(G) \leq \overline{\chi}_f(G) \leq \overline{\chi}(G)$$

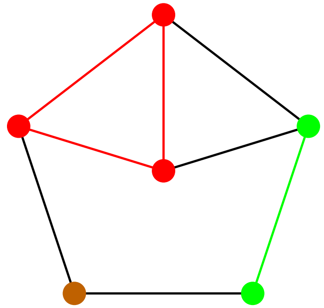
There is no polynomial graph parameter β with $\frac{|V(G)|}{\omega(G)} \leq \beta(G) \leq \overline{\chi}(G)$ unless P=NP.

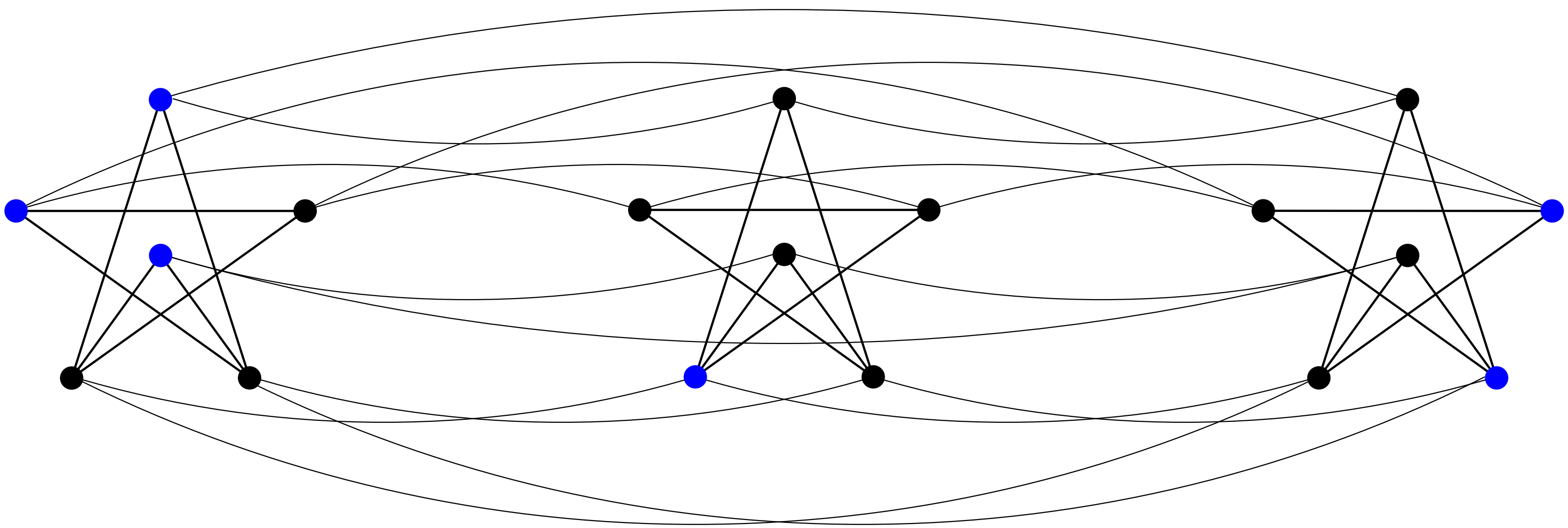
$\Rightarrow$ no poly β with $\overline{\chi}_f \leq \beta \leq \overline{\chi}$

N. Gvozdenović and M. Laurent: *The operator Ψ for the Chromatic Number of a Graph, SIAM J. Optim., 19(2) 592-615 (2008)*

$$\begin{array}{rccc} \Psi: & [\frac{|V|}{\chi}, \overline{\chi}] & \longrightarrow & [\alpha, \overline{\chi}] \\ & [\frac{|V|}{\chi}, \alpha] & \longrightarrow & \{\overline{\chi}\} \\ & [\frac{|V|}{\omega}, \overline{\chi}] & \longrightarrow & \{\alpha\} \\ & \vartheta & \longmapsto & \lceil \vartheta \rceil \\ & \vartheta' & \longmapsto & \lceil \vartheta^{+} \rceil \end{array}$$

$$\Psi_\beta(G) := \min_{t\in\mathbb{N}} t \quad \textbf{\textit{s.t.}}\ \beta(K_t \square \overline{G}) = |V(G)|$$





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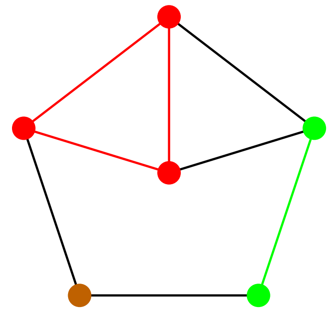

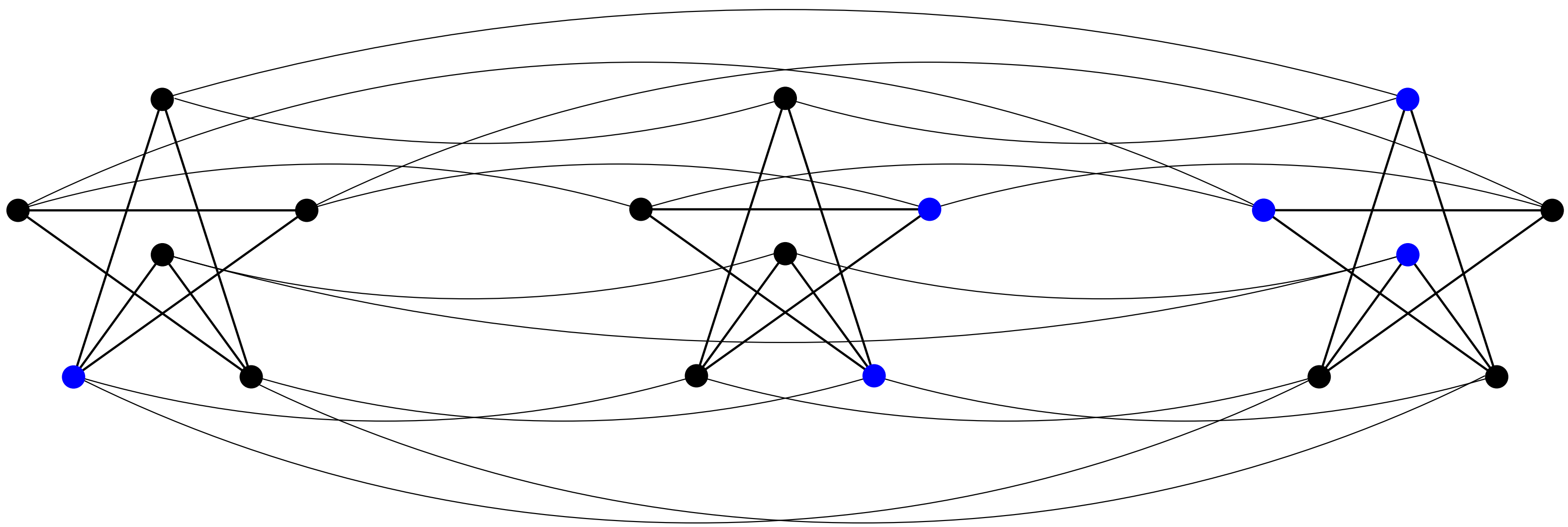

Line graph of triangle free graphs

The line-graph $L(G)$ of G is the graph with node-set the edge-set of G and where two nodes e, f are nonadjacent in $L(G)$ if they correspond to two disjoint edges of G.

If $L(G)$ is the line-graph of a triangle-free G, then :

$$|V(G)| - \alpha(G) = \overline{\chi}(L(G))$$

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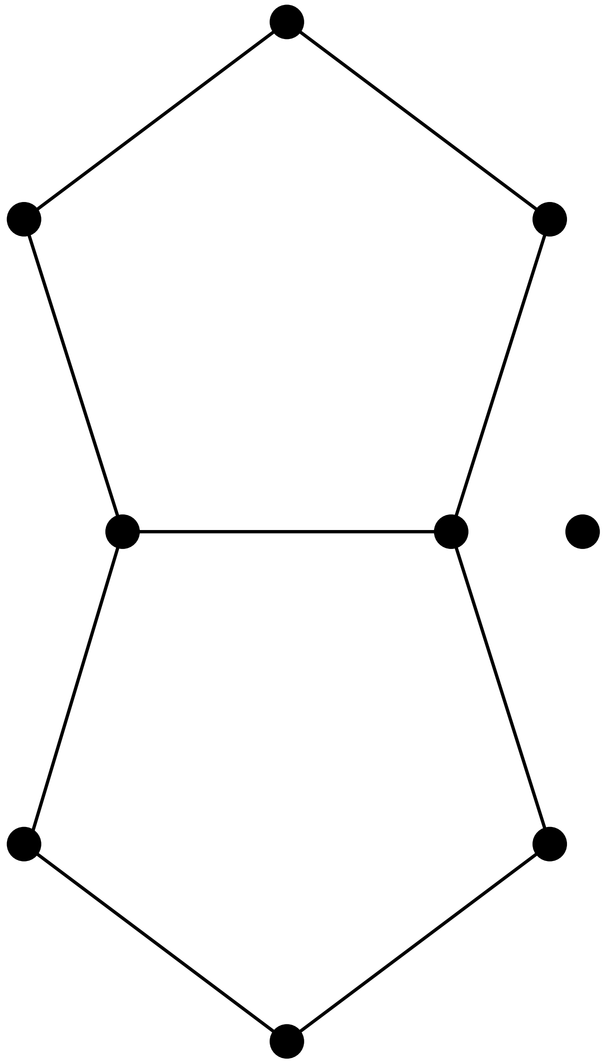
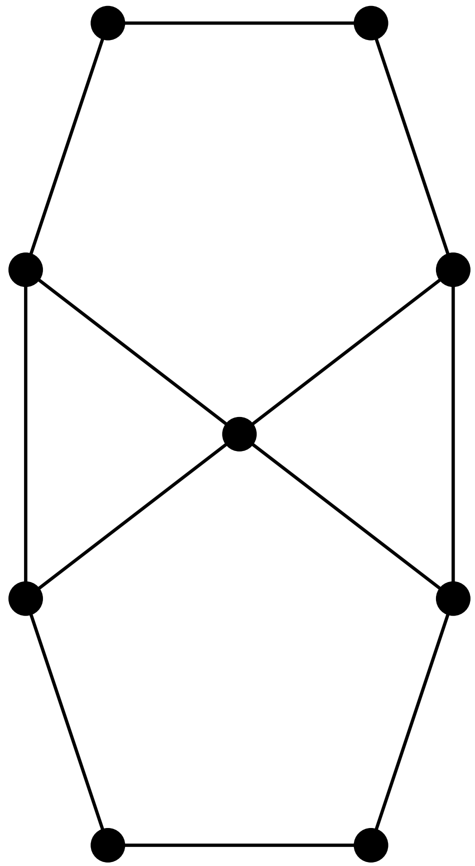
Illustrating $|V(G)| - \alpha(G) = \overline{\chi}(L(G))$





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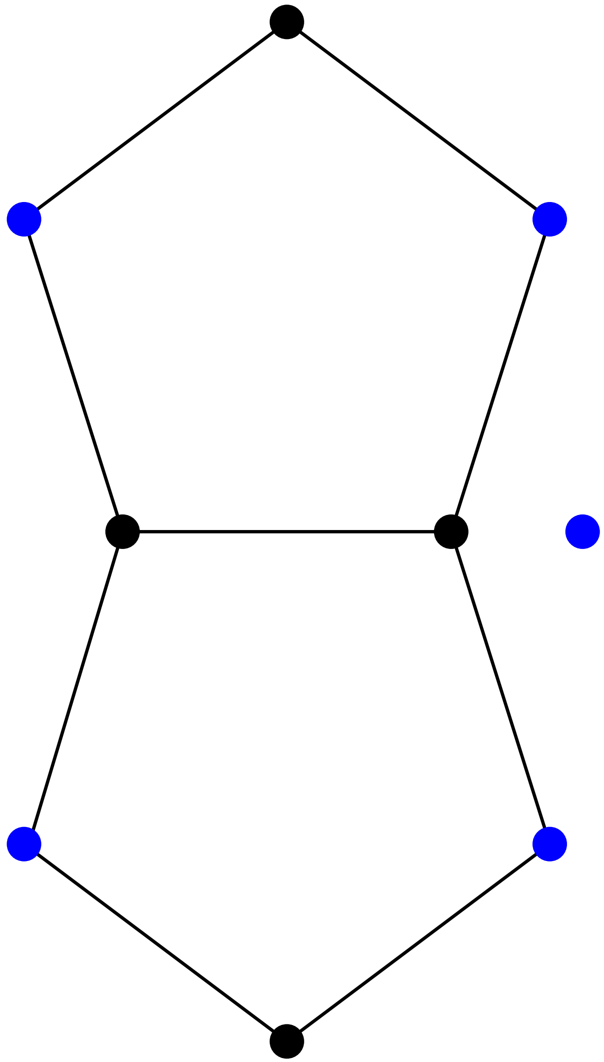

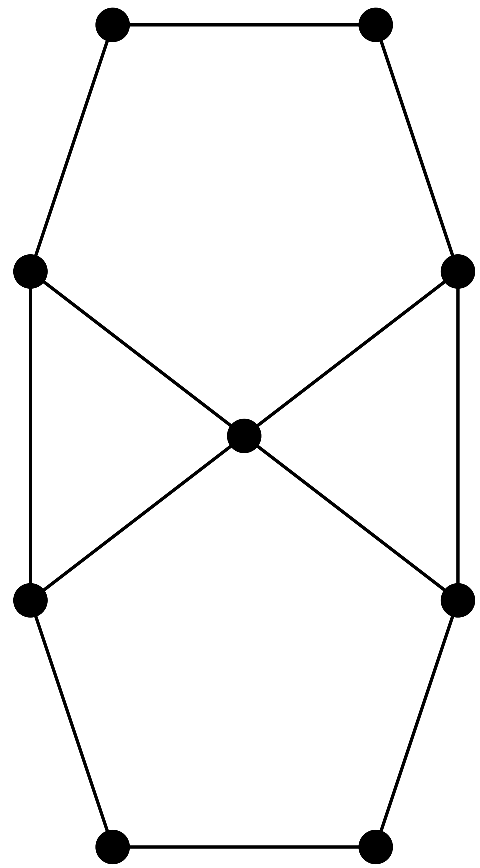

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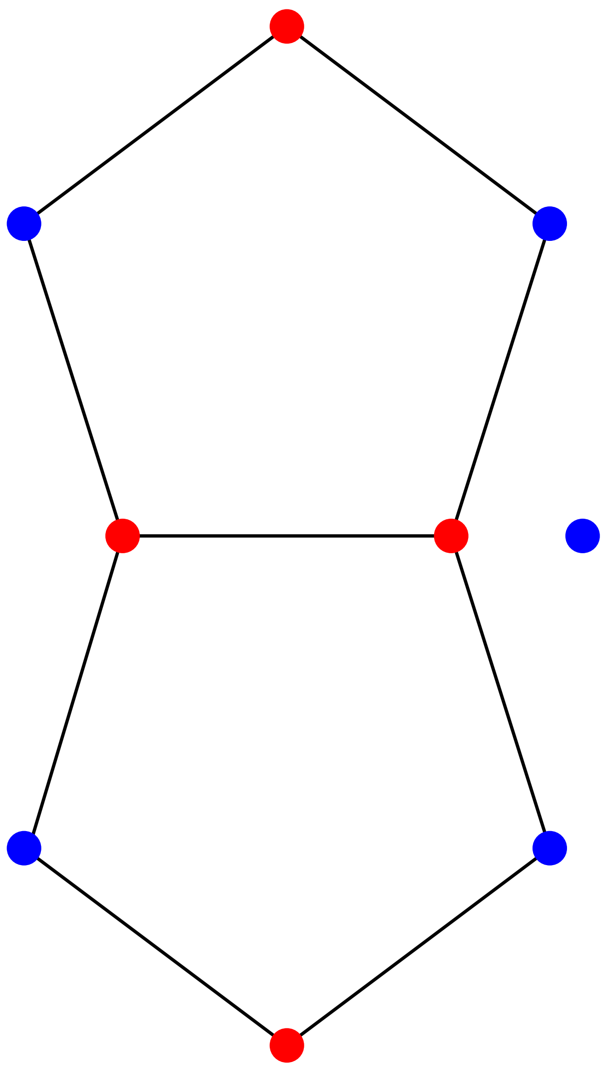

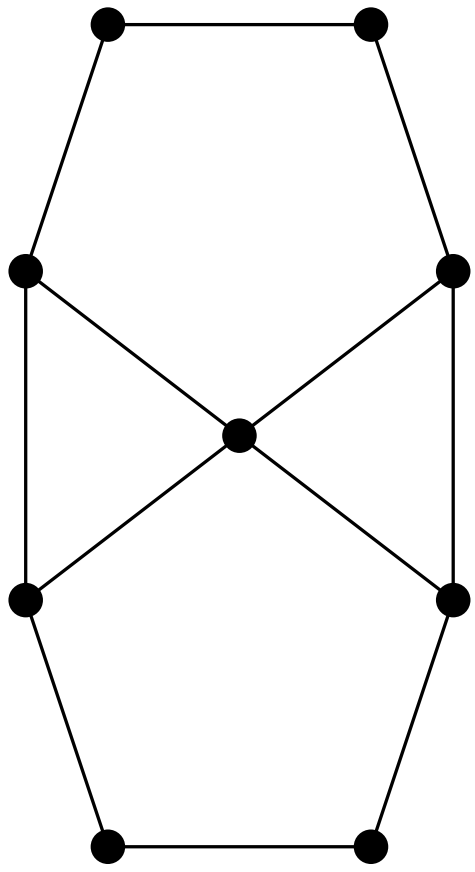

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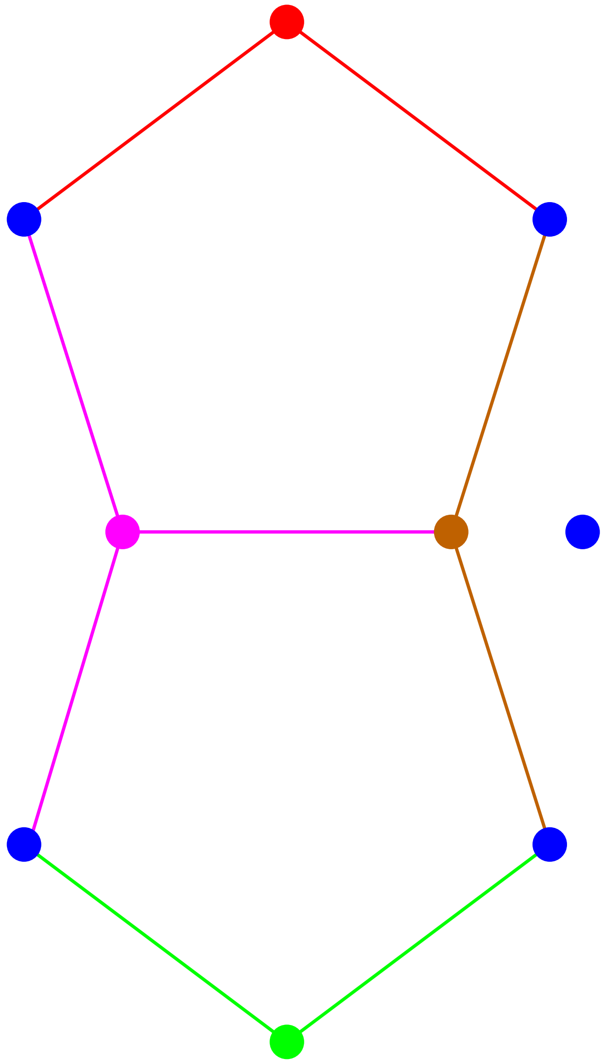

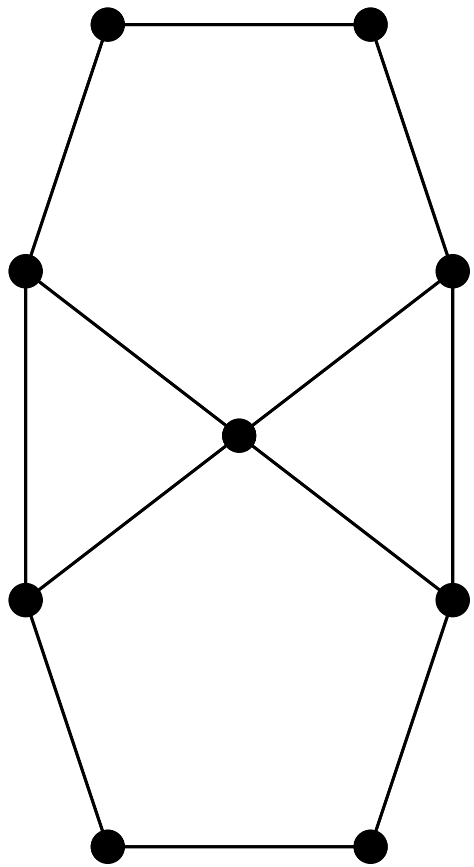

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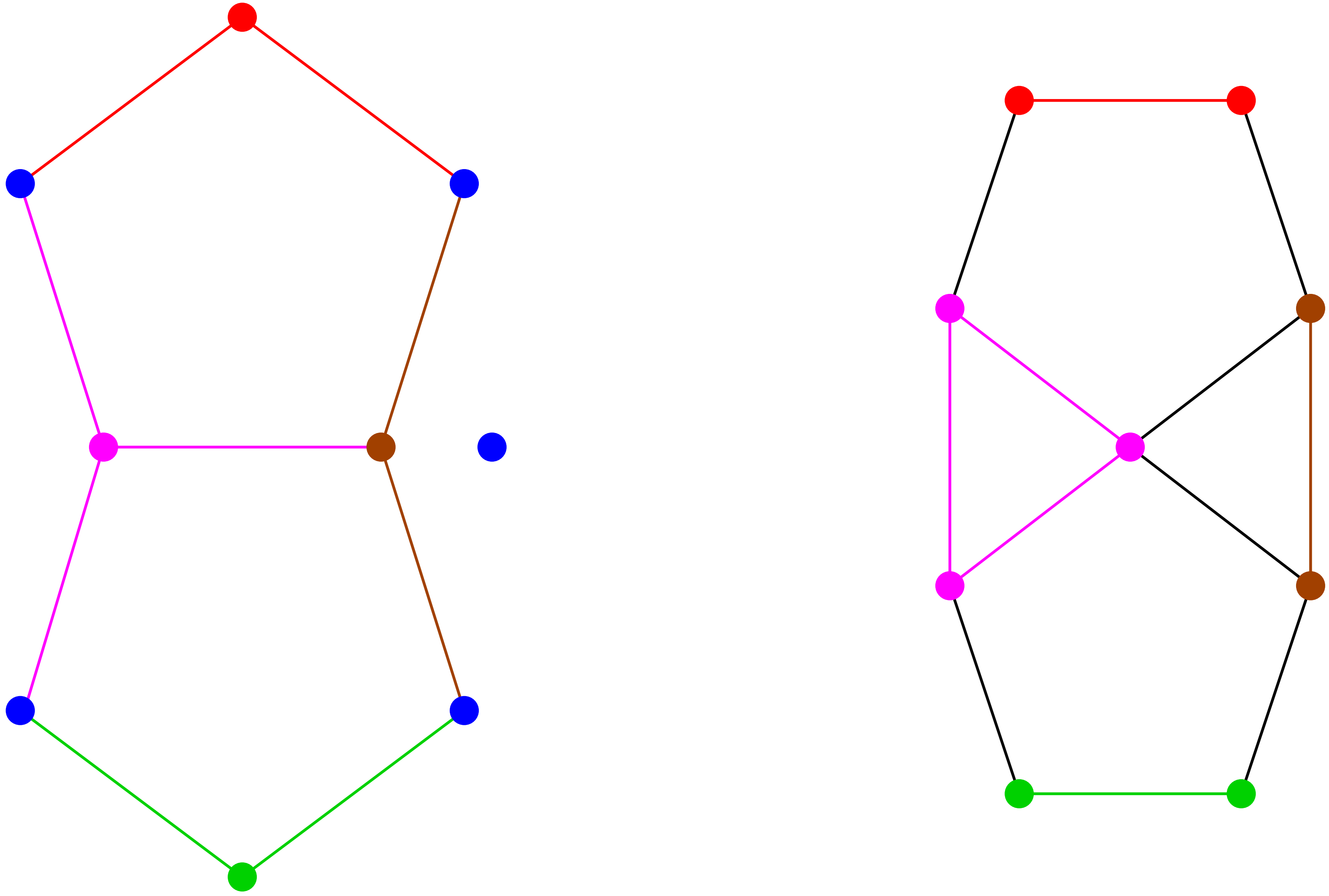

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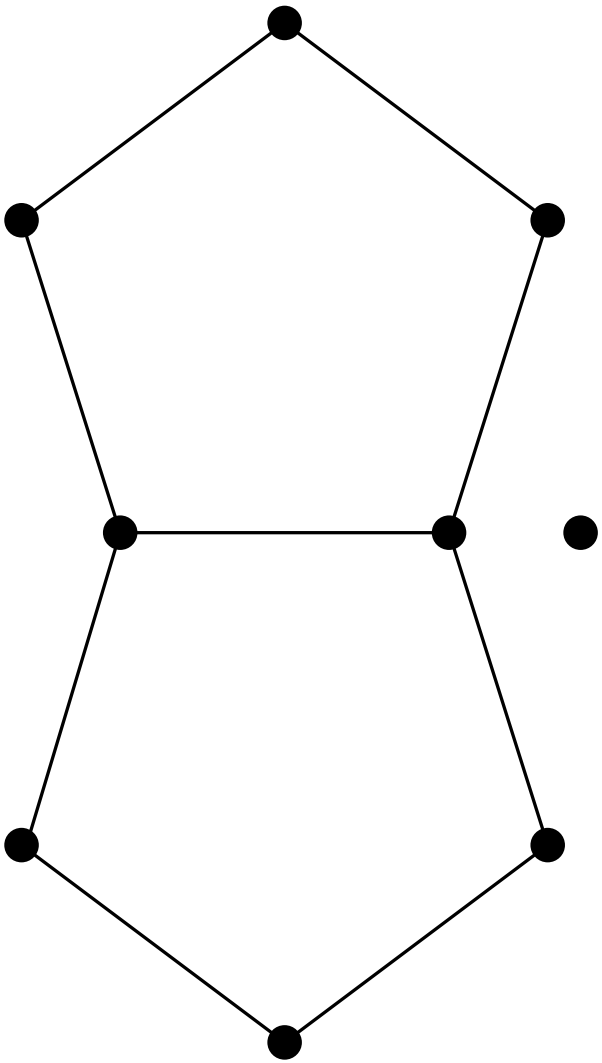

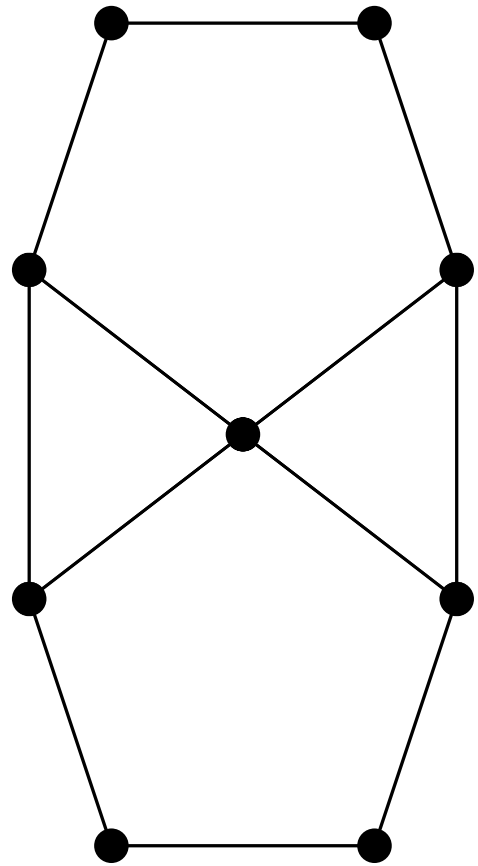

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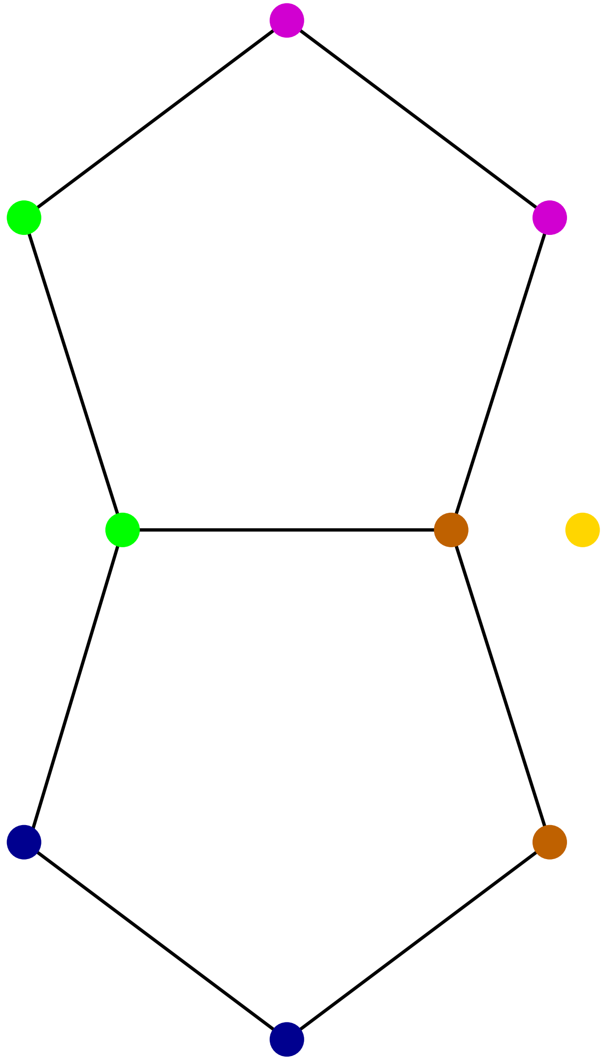

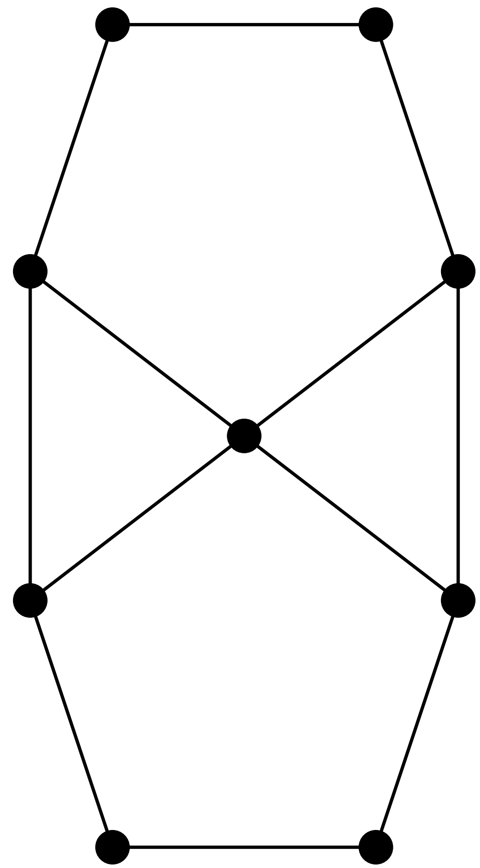

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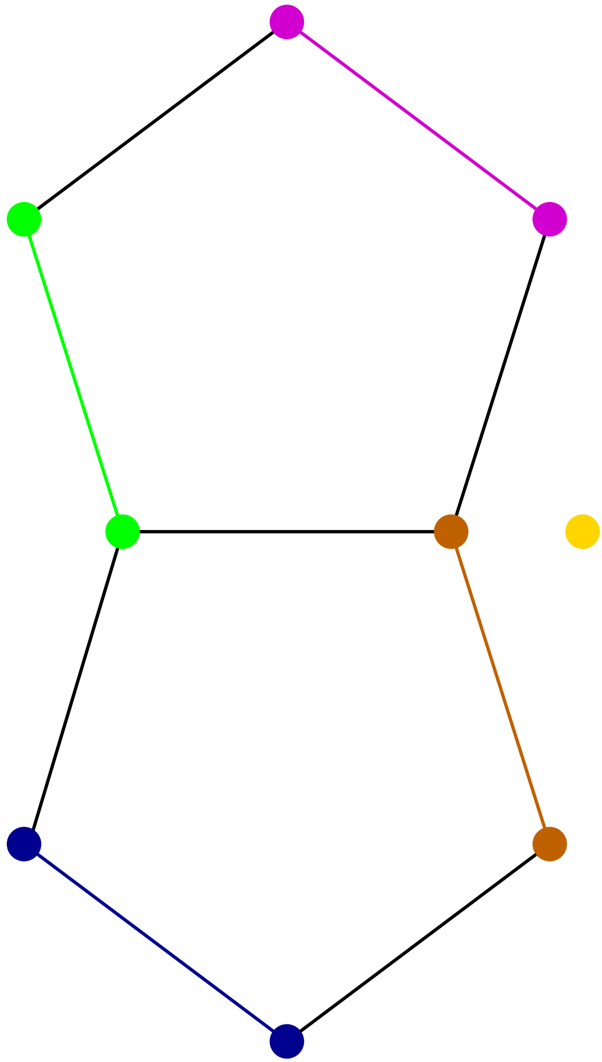

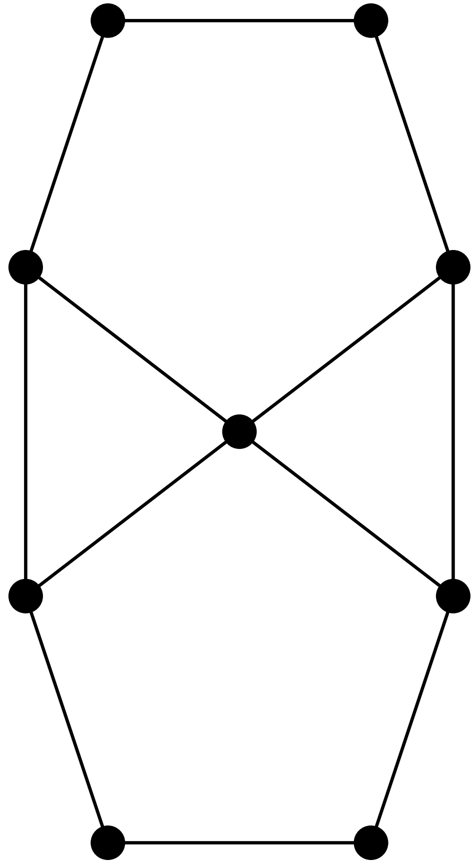

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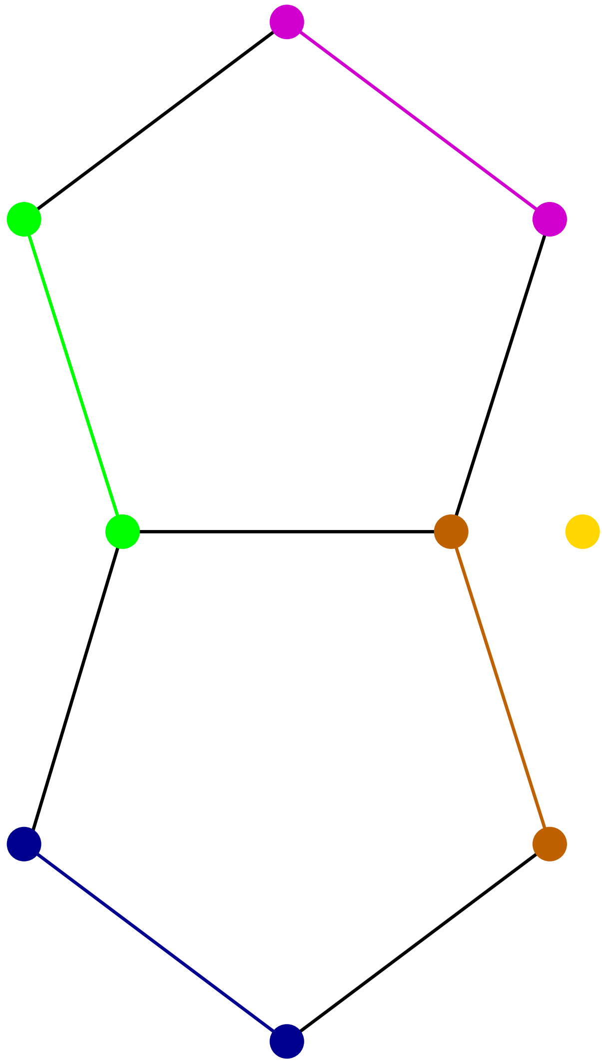

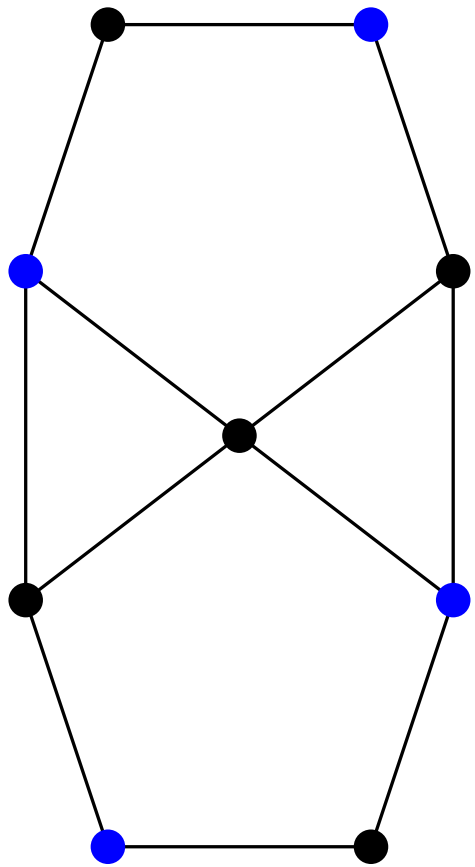

Line graph of triangle free graphs

If $L(G)$ is the line-graph of a triangle-free G, then :

(P1) $|V(G)| - \alpha(G) = \overline{\chi}(L(G))$;

(P2) $|V(G)| - \overline{\chi}(G) = \alpha(L(G))$;

Line graph of triangle free graphs

If $L(G)$ is the line-graph of a triangle-free G, then :

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(P2) $|V(G)| - \overline{\chi}(G) = \alpha(L(G))$;

and it follows from (P1)-(P2) that

(P3) If $\alpha \leq \beta \leq \overline{\chi}$ $(\forall G)$ and if G is triangle-free then,

$$\alpha(G) \leq |V(G)| - \beta(L(G)) \leq \overline{\chi}(G).$$

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$$\alpha \leq \beta \quad \Rightarrow \quad |V(G)| - \beta(L(G)) \leq |V(G)| - \alpha(L(G)) = \overline{\chi}(G).$$

$$\beta \leq \overline{\chi} \quad \Rightarrow \quad \alpha(G) = |V(G)| - \overline{\chi}(L(G)) \leq |V(G)| - \beta(L(G)).$$

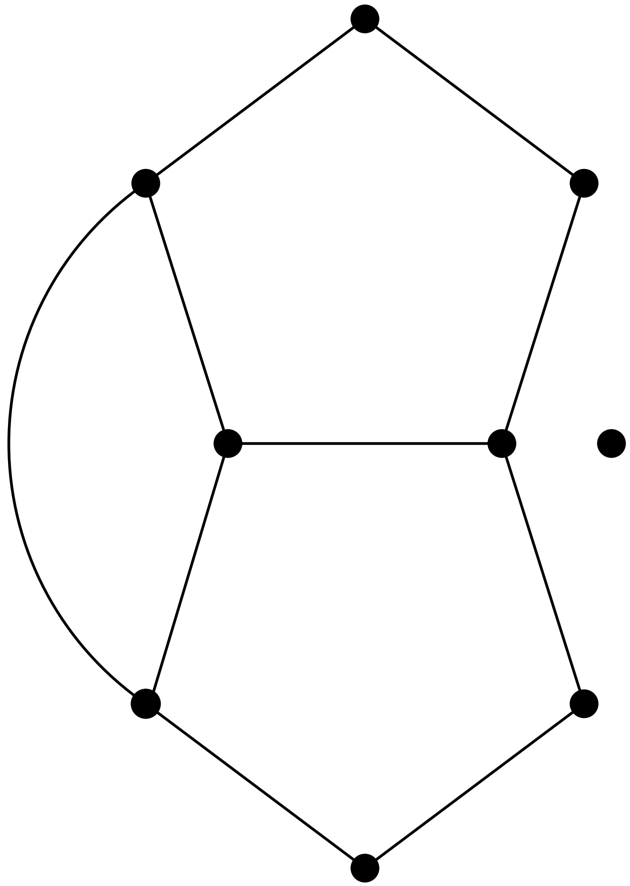
Sandwich line graphs

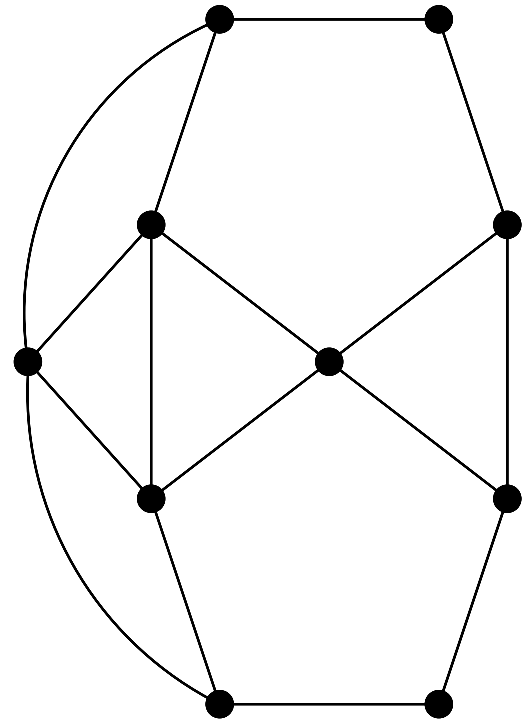
A sandwich line graph $S(G)$ of G is an edge graph of G such that:

(i) $S(G) = L(G)$ if G is triangle free;

(ii) $S(G)$ satisfies (P1), (P2) and (P3) for any graph G.

Now G has triangle





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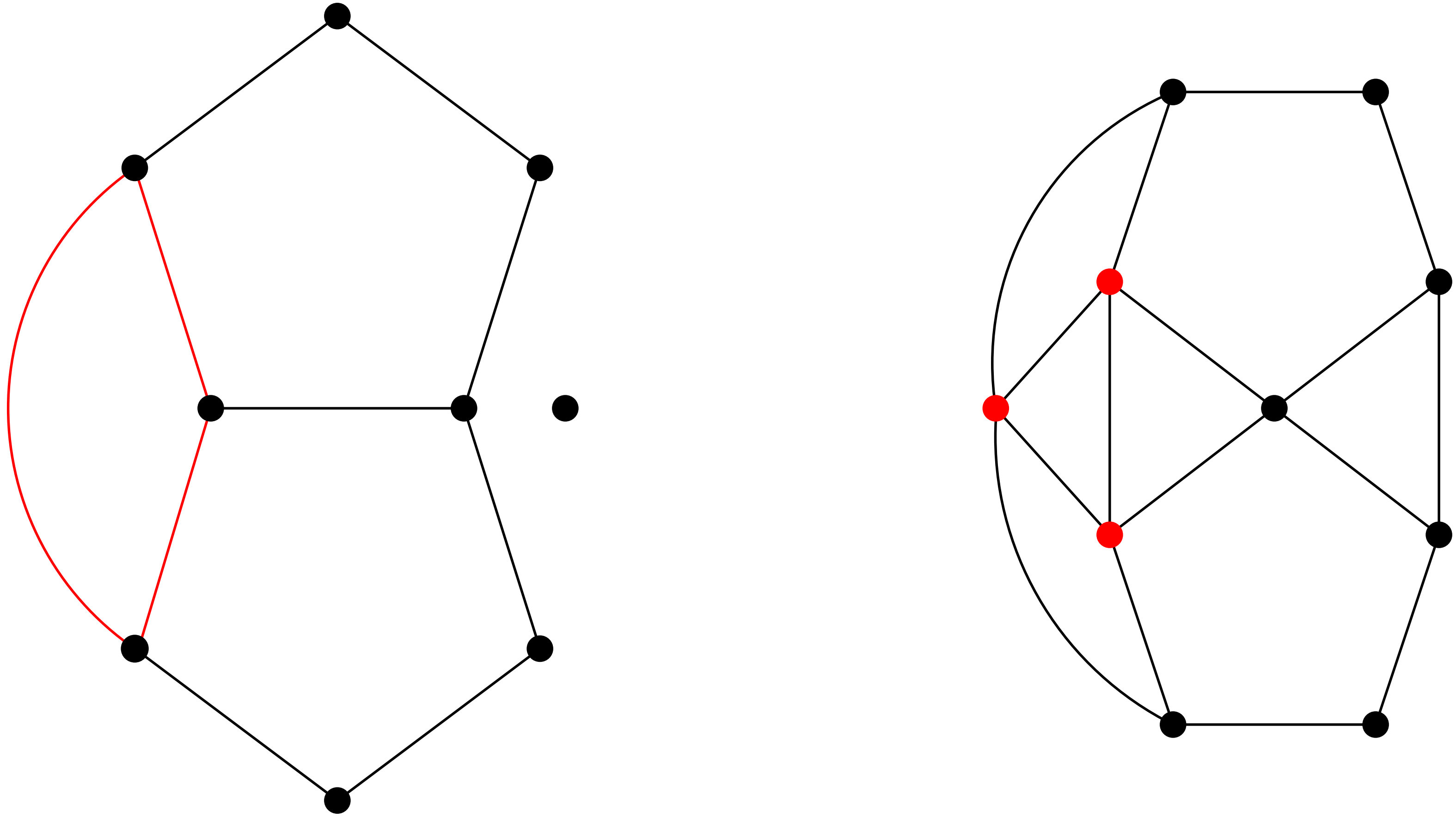
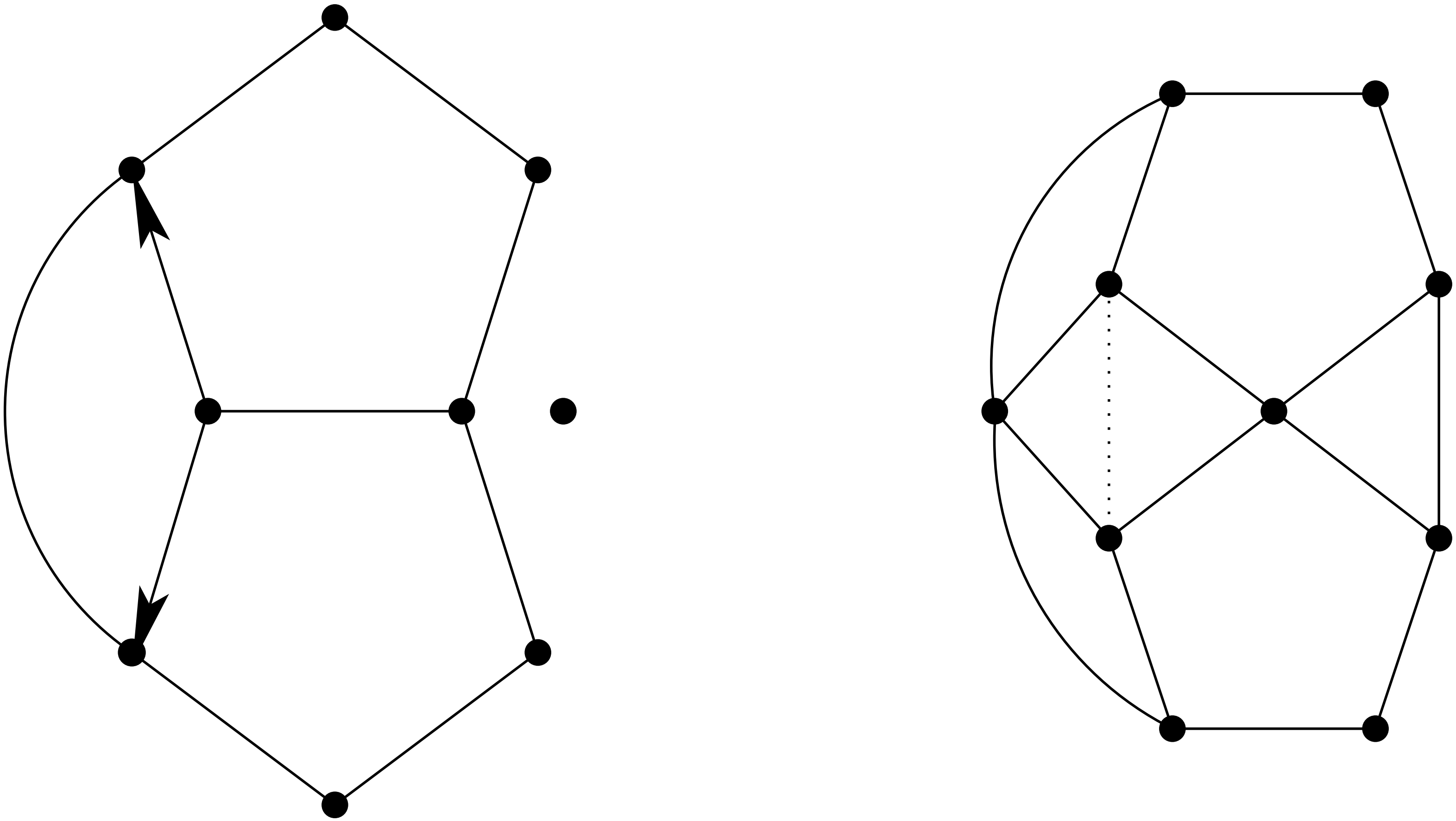


Restoring the sandwich property



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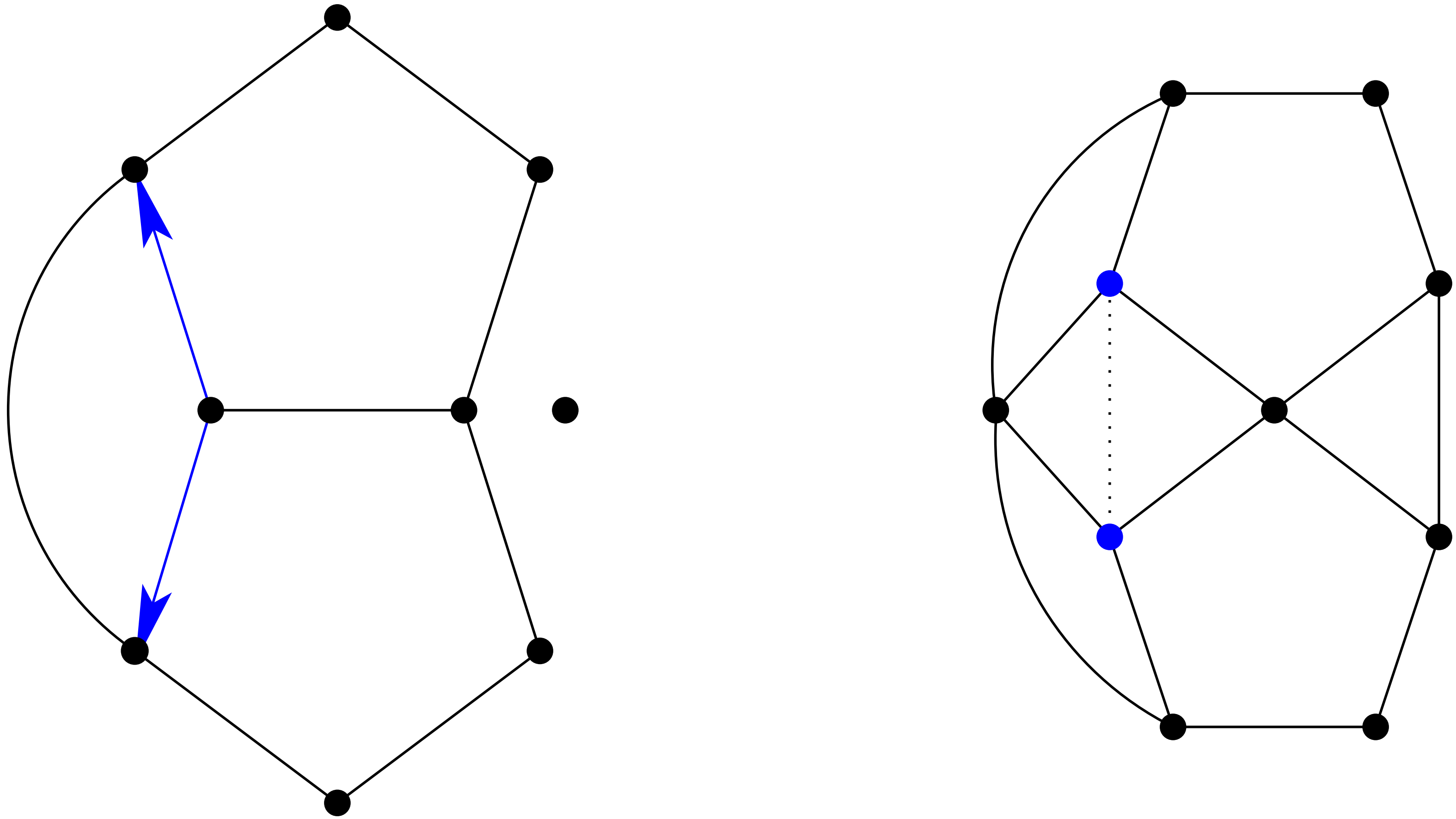

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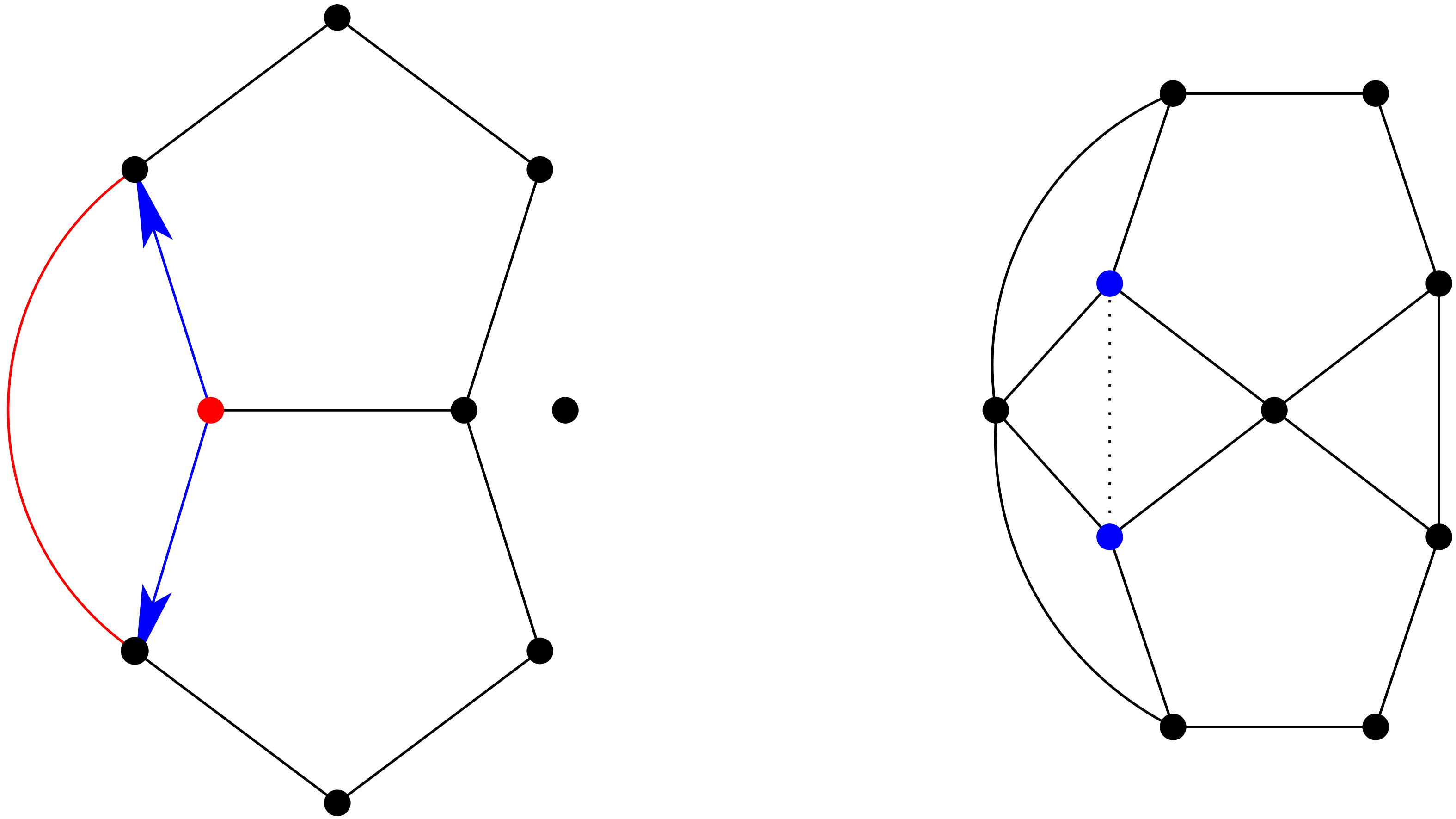
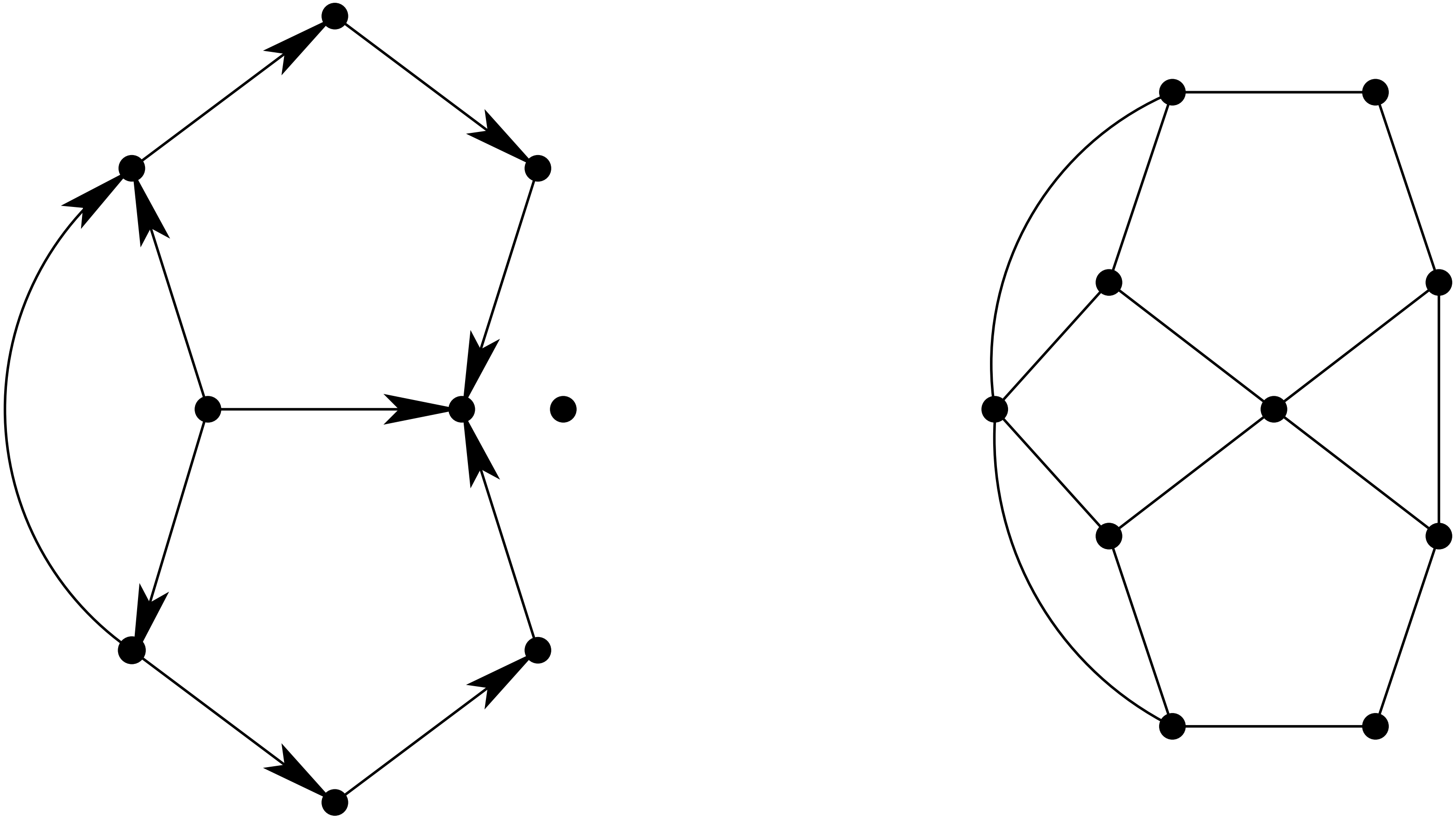
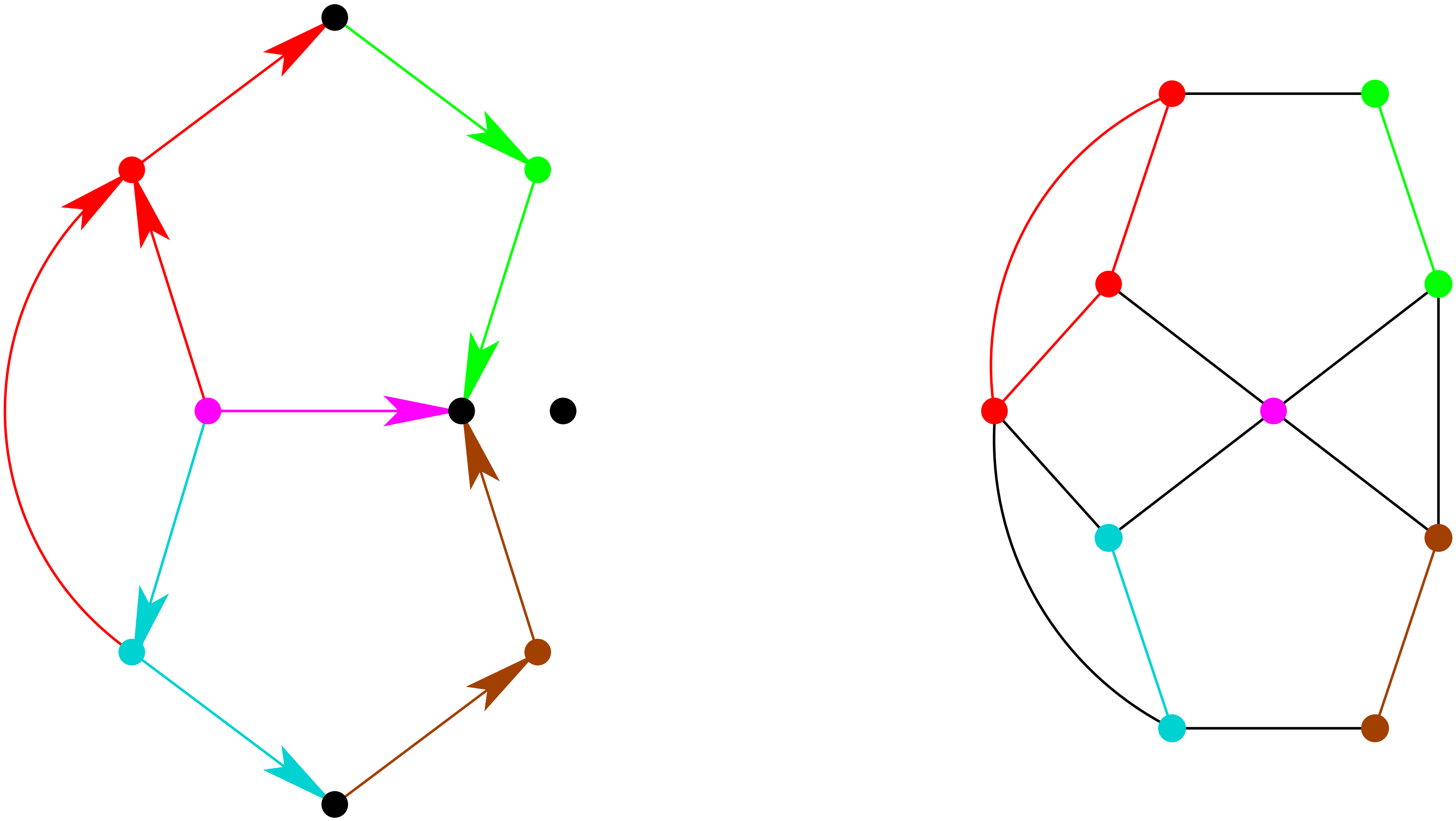


A sandwich line graph $S(G)$ *of* G



Illustrating $|V(G)| - \alpha(G) \geq \overline{\chi}(S(G))$



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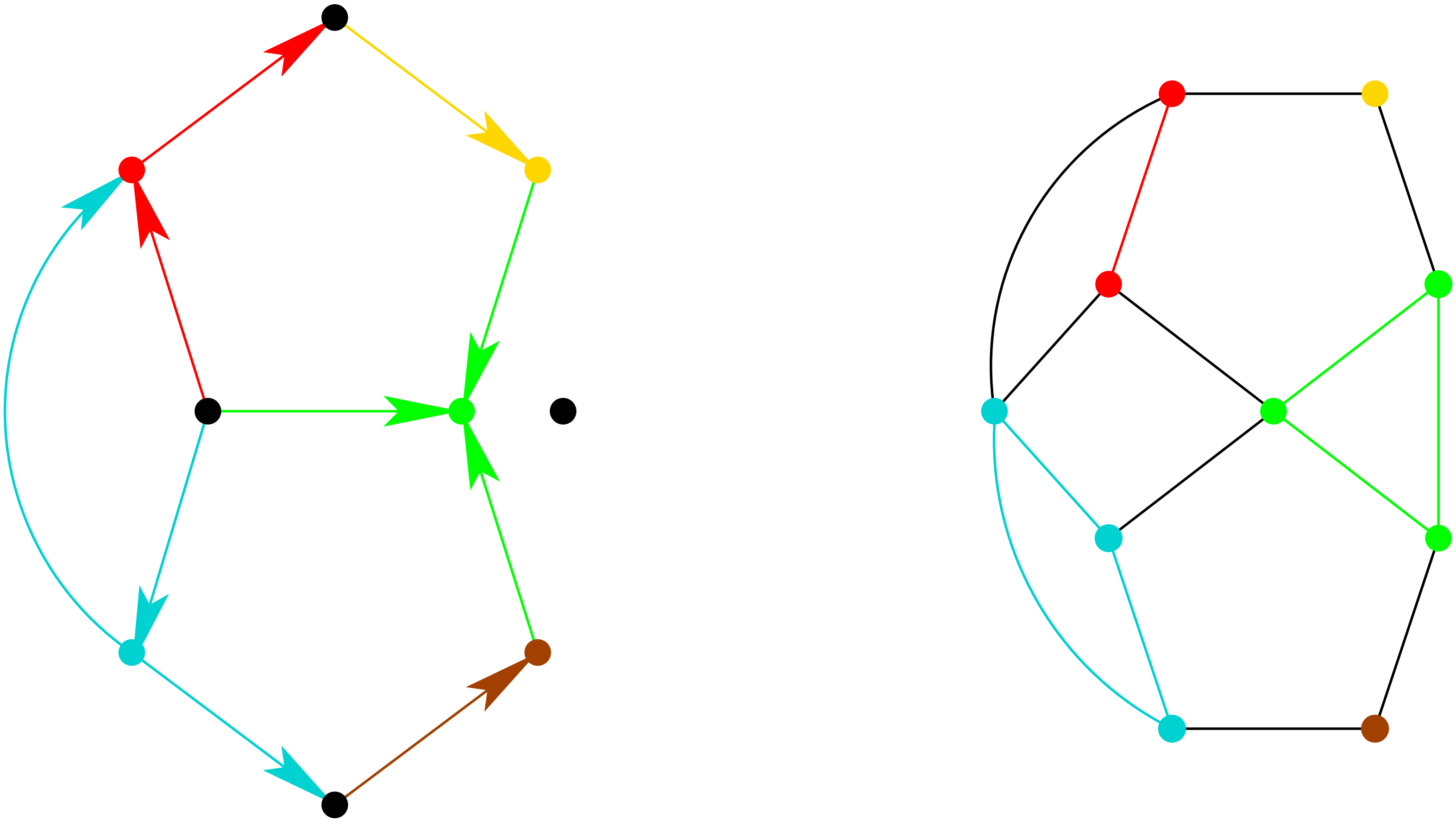

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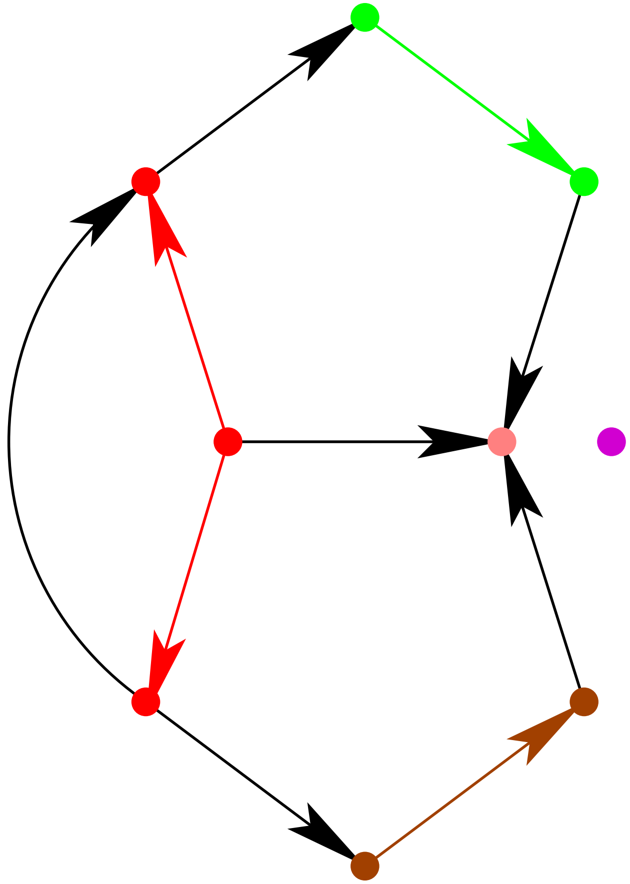

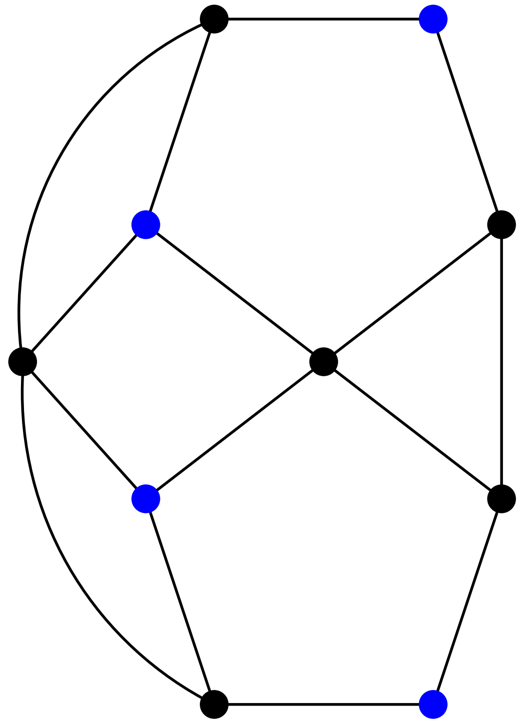

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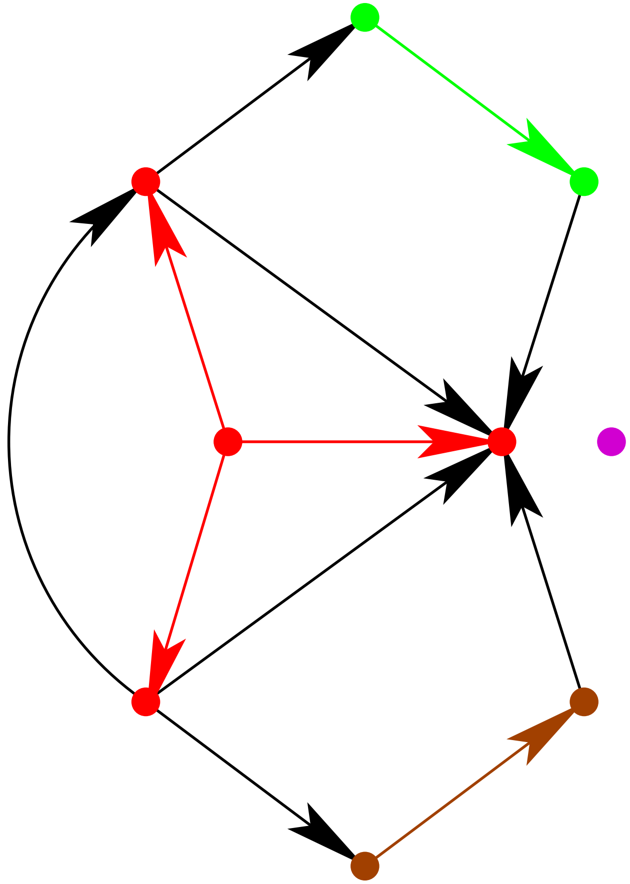

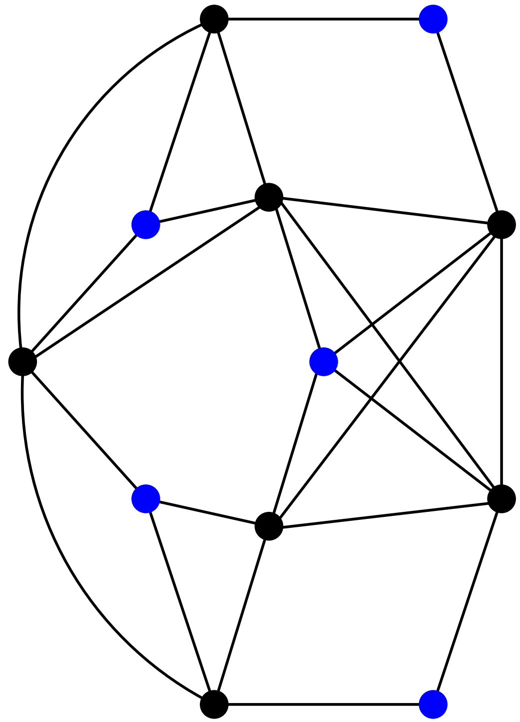

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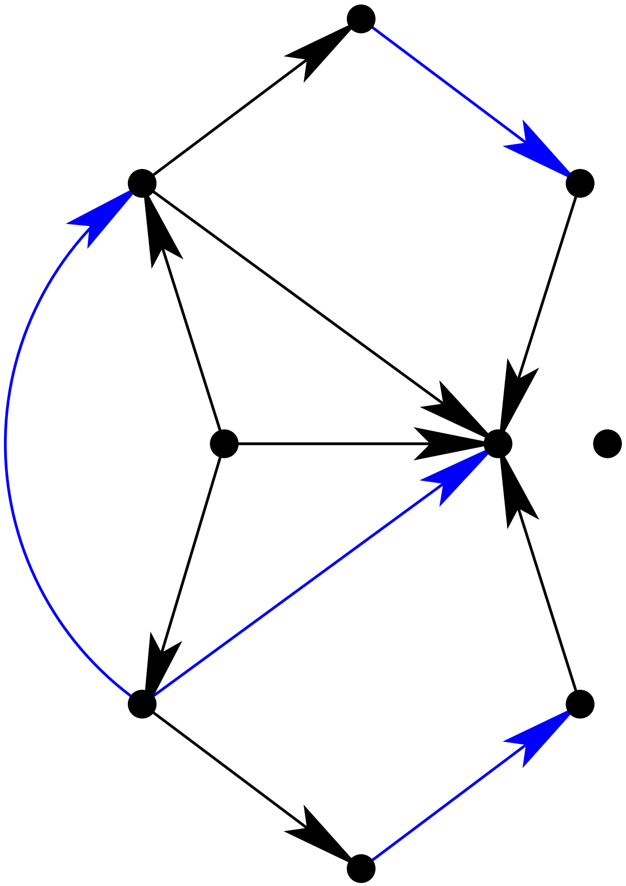

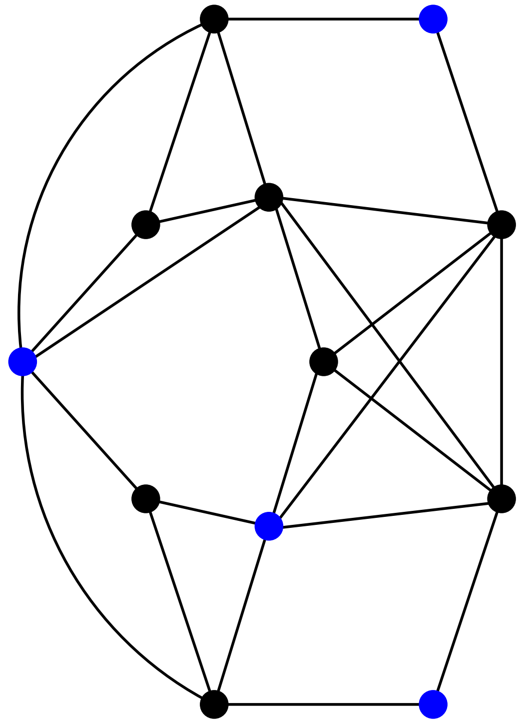

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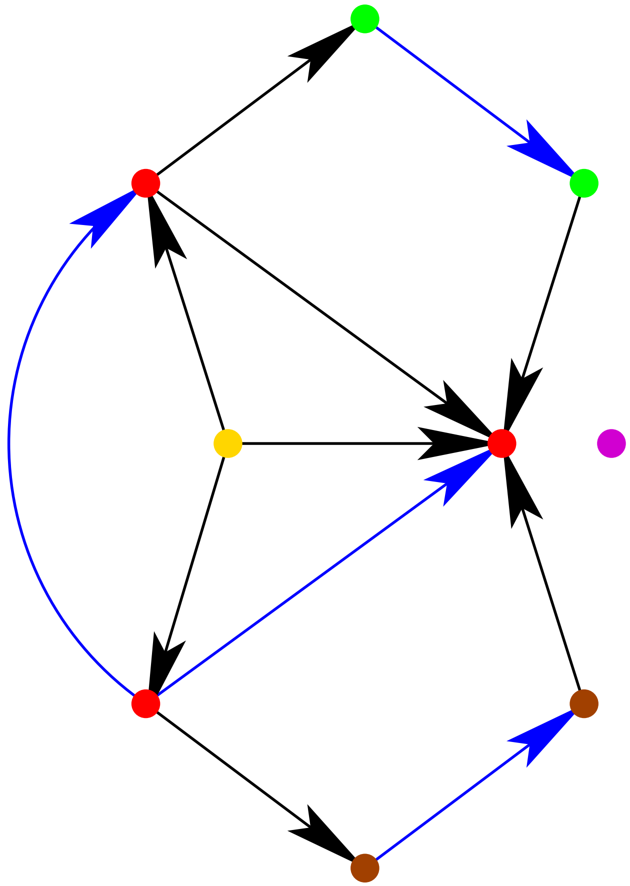

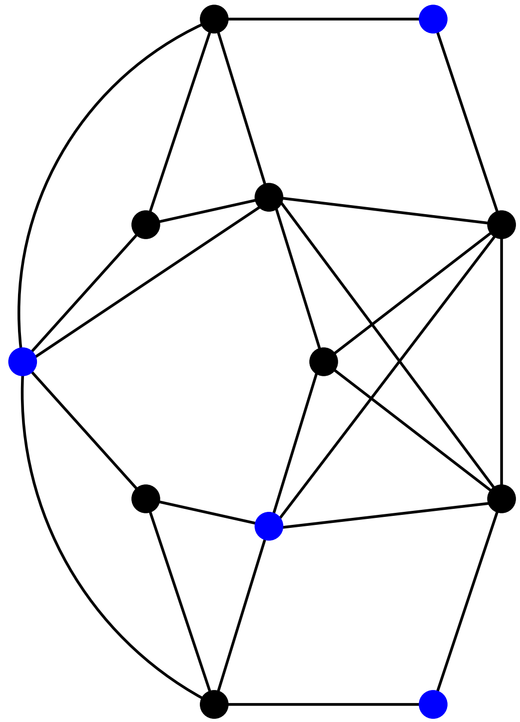

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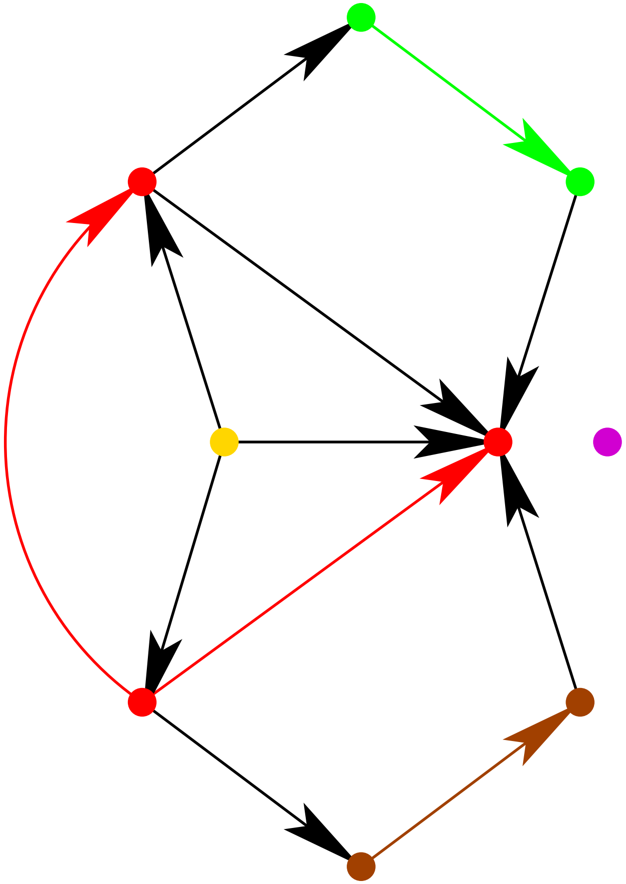

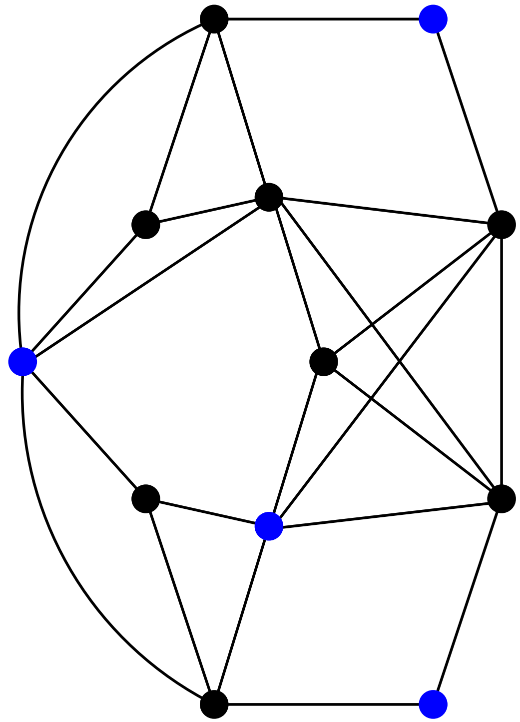

Theorem

Let $\beta(G)$ be a graph parameter with the following sandwich property:

$$\alpha(G) \leq \beta(G) \leq \overline{\chi}(G),$$

for every graph G.

Let $G = (V, E)$ be a simple graph.

Let $\sigma : V \leftrightarrow \{1, 2, \ldots, |V|\}$ and let $S(G_\sigma)$ be the graph with node-set the edge-set of G and where two nodes e, f are nonadjacent in $S(G_\sigma)$ iff they are disjoint edges in G or $e = uv$, $f = uw$ but $vw \in E$ and $\sigma(u) < \min\{\sigma(v), \sigma(w)\}$.

Then

$$\alpha(G) \leq |V(G)| - \beta(S(G_\sigma)) \leq \overline{\chi}(G), \quad \forall \sigma$$

futhermore, if $\beta(G) = \overline{\chi}(G)$, then the left inequality holds with equality; and if $\beta(G) = \alpha(G)$, then the right inequality holds with equality.

A corollary

Denote $S^0(G) = G$ and let $S^{i+1}(G)$ be a sandwich line graph of $S^i(G)$.

Thus, for any sandwich function β and any integer $k \geq 1$, one has

$$\alpha(G) \leq (-1)^k \beta(S^k(G)) - \sum_{i=0}^{i=k-1} (-1)^{i+1} |V(S^i(G))| \leq \overline{\chi}(G).$$

Particular example : numerical values with $G = kC_5$

k	$\alpha = \Psi_{\overline{\chi}_f}$	ϑ	Ψ_ϑ	$\overline{\chi}_f = \Phi_{\overline{\chi}_f}$	Φ_ϑ	$\overline{\chi}$
1	2	2.236	3	2.5	2.764	3
2	4	4.472	5	5	5.528	6
15	30	33.54	34	37.5	41.46	45
100	200	223.6	224	250	276.4	300

where Ψ and Φ are the monotone nonincreasing operators

$$\begin{array}{rccl} \Psi: & [\frac{|V|}{\chi}, \overline{\chi}] & \rightarrow & [\alpha, \overline{\chi}] \\ & \beta & \mapsto & \Psi_\beta(G) = \min_{t \in \mathbb{N}} t \quad \text{s.t. } \beta(K_t \Box \overline{G}) = |V(G)| \\ \Phi: & [\alpha, \overline{\chi}] & \rightarrow & [\alpha, \overline{\chi}] \\ & \beta & \mapsto & \Phi_\beta(G) = |V(G)| - \beta(S(G)) \end{array}$$

The End

Thank You