# — Ineffable Cacaphony —

## A Tribute to Jack Edmonds, hanging in there at 75

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# Outline

# How old?

JACK EDMONDS

$2^6$

JACK EDMONDS

# 65

JACK EDMONDS

66

JACK EDMONDS

67

JACK EDMONDS

68

JACK EDMONDS

# 69

JACK EDMONDS

70

JACK EDMONDS

# 71

JACK EDMONDS

# 72

JACK EDMONDS

## 73

JACK EDMONDS

# 74

JACK EDMONDS

# 75 years + 3 days

JACK EDMONDS

# You can't teach an old dog . . .

Being perhaps the oldest friend of Jack here present,
I happily accept a certain responsibility,
being the repository of a number of old tales,
and having a privileged perspective
on certain sensitive topics.

# You can't teach an old dog . . .

They say "you can't teach an old dog new tricks",
and I think we can all agree, from experience,
any given mathematician
only knows a handful of things.
I'm sure Jack will agree.

# You can't teach an old dog . . .

The upside is:

An old dog doesn't tend to forget his old tricks.

# You can't teach an old dog . . .

Since the organizers specifically requested:
*We are hoping that the talks will be introductory expositions.*
*Jack says he likes easy talks on stuff he has forgotten or never got to.*

So I have elected to talk only about old tricks.

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Since the organizers specifically requested:
*We are hoping that the talks will be introductory expositions.*
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So I have elected to talk only about old tricks.

I'd better assume that you know what a matroid is.

# You can't teach an old dog . . .

In case you don't, just think of a matroid as:

- P a set of points in a projective space
- V a set of vectors in a vector space
- G the set of edges in a graph

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With closure operator, respectively

- P projective flat spanned by
- V linear subspace spanned by
- G completion of broken circuits

## You can't teach an old dog . . .

Many thanks to the organizers at JPOC, the IHP who let us use
*cette venerable salle du Général Bourbaki*,
with special thanks to Kathie Cameron, who has done wonders,
and also to Pierre Fouilleux and Sylvie Bruno,
who have made sure all the machines are compatible,
and that the lectures can begin in relative calm and serenity,
and finally to all involved, for the interesting lectures
and fine company.

# You can't teach an old dog . . .

And second, an apology:
Kathie pointed out to me that
cacaphony, in the title, is misspelled,
and asked whether I wanted to correct it.
She is right.
We wrote it that way in 1970.

# Ineffable Cacaphony

On the Foundations of Combinatorial Theory:
Combinatorial Geometries

# Ineffable Cacaphony

On the Foundations of Combinatorial Theory:
Combinatorial Geometries

*" For simplicity, we also assume that every point in a geometry is a closed set. Without this additional assumption, the resulting structure is often described by the ineffably cacaphonic term matroid, which we prefer to avoid in favor of the term 'pregeometry'."*

# Ineffable Cacophony

On the Foundations of Combinatorial Theory:
Combinatorial Geometries

$\kappa\alpha\kappa\omega$, in Greek, means *"bad"*.
So *cacophony*, *"has a bad ring to it"*.
*ineffable cacophony*, *"unspeakably bad sounding"*.

# Ineffable Cacophony

On the Foundations of Combinatorial Theory:
Combinatorial Geometries

cacaphony,
a misspelling, and unintentional exaggeration.

## Ineffable Cacophony

On the Foundations of Combinatorial Theory:
Combinatorial Geometries

Lord of the Rings:,
*Like the sound a toddler produces with his elder sisters violin.*
*Just such an ineffable cacophony afflicted Imladris.*

# Ineffable Cacophony

On the Foundations of Combinatorial Theory:
Combinatorial Geometries


Figure: From a group called Pure Volume.

# NBS

Jack organized the first-ever meeting on matroid theory in 1964 at the National Bureau of Standards, near Washington, D.C.

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- single-element extensions
- the Tutte polynomial
- Today I'll tuck in:
- Dilworth completion.
- A question about rigidity.

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It's fine, when you're just starting out,
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and someone is willing to read your thesis!
Thanks, Jack.

# Edge-colored Boolean Algebras

I had basically one trick in my bag:
the idea that matroids were naturally describable
in terms of the set of Yes–No answers
to the questions (for all subsets $B$ and all points $a \notin B$):
Is the point $a$ in the closure of the set $B$?

# Edge-colored Boolean Algebras

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to the questions (for all subsets $B$ and all points $a \notin B$):
Is the point $a$ in the closure of the set $B$?

This relation is easily displayed by coloring
the edges between covering pairs of subsets
in the Hasse diagram of the Boolean algebra:

# Edge-colored Boolean Algebras

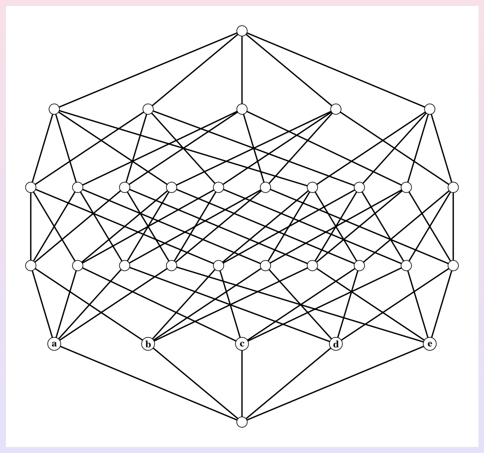



Figure: The Boolean algebra for 5-element set.

# Edge-colored Boolean Algebras

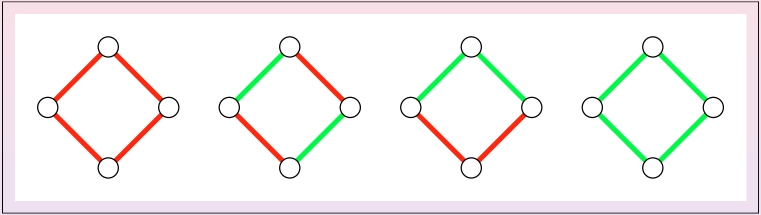


Figure: Every diamond is one of these four types.

This is yet another *cryptomorphic* axiomatization of matroids.

# Edge-colored Boolean Algebras

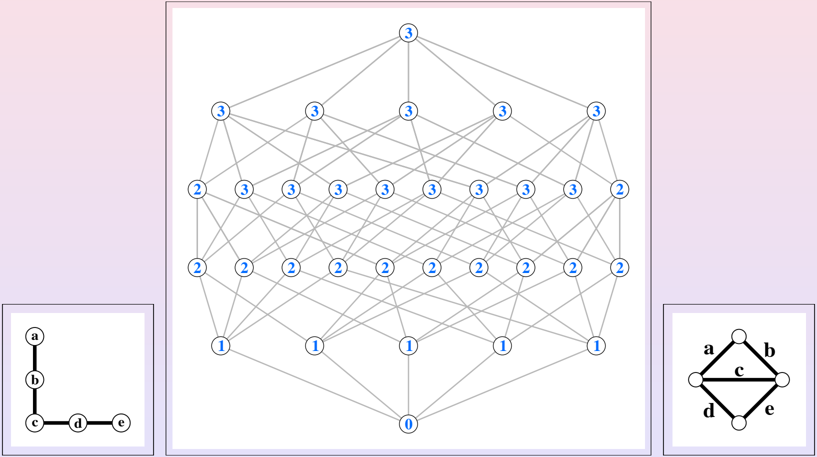



Figure: The (semimodular) rank function of the $L$-matroid.

# Edge-colored Boolean Algebras

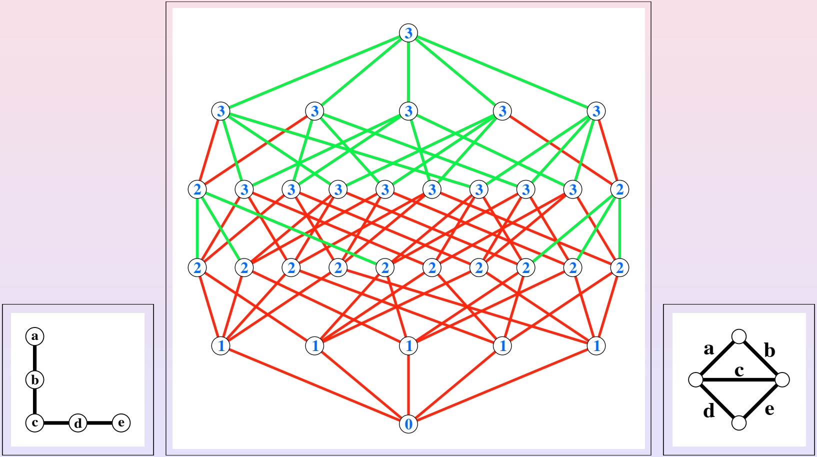



Figure: The edge-colored derived from that rank function.

# Edge-colored Boolean Algebras

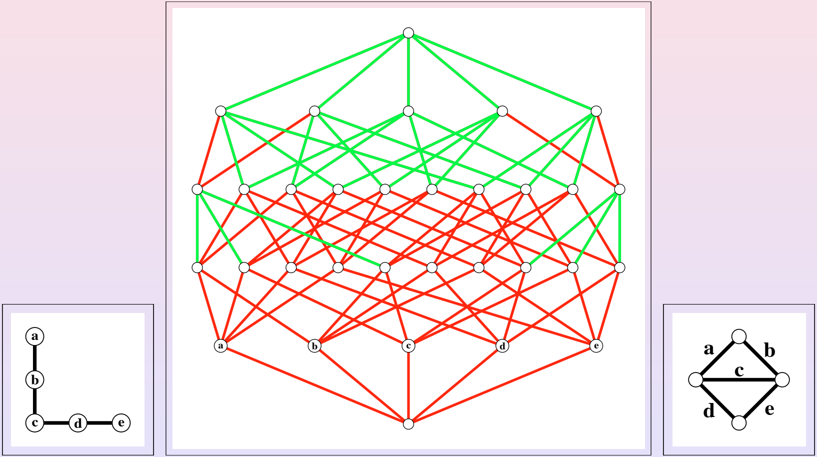



Figure: The edge-colored BA for the $L$-matroid.

# Edge-colored Boolean Algebras

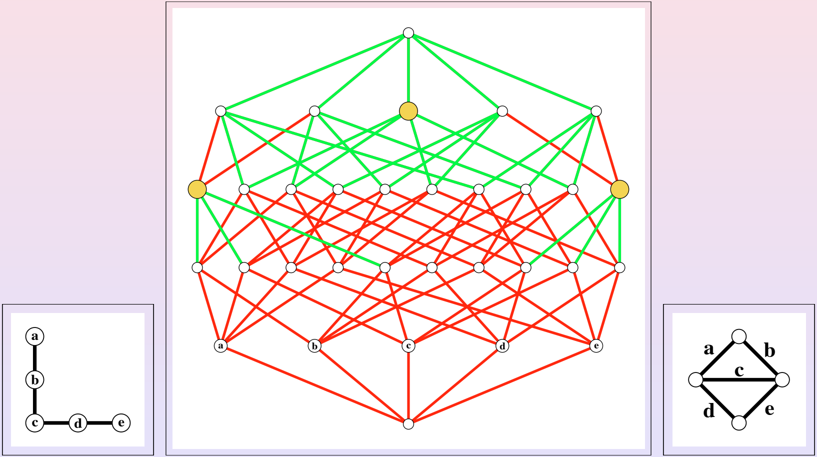



Figure: The circuits of the $L$-matroid.

# Edge-colored Boolean Algebras

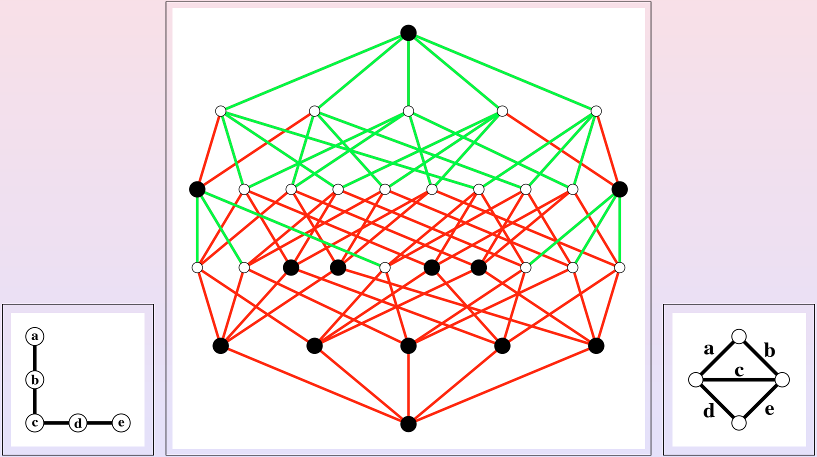



Figure: The flats of the $L$-matroid.

# Edge-colored Boolean Algebras

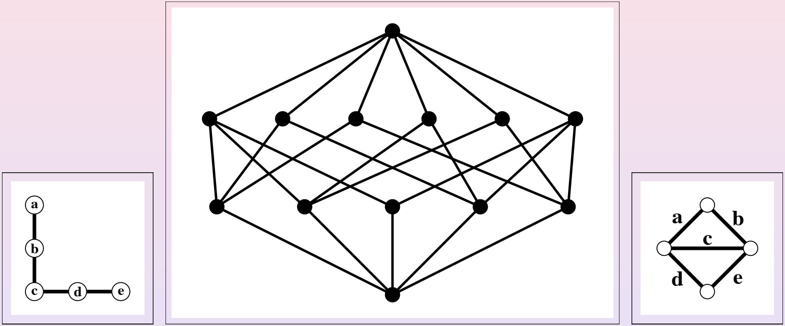



Figure: And its associated geometric lattice.

# Edge-colored Boolean Algebras

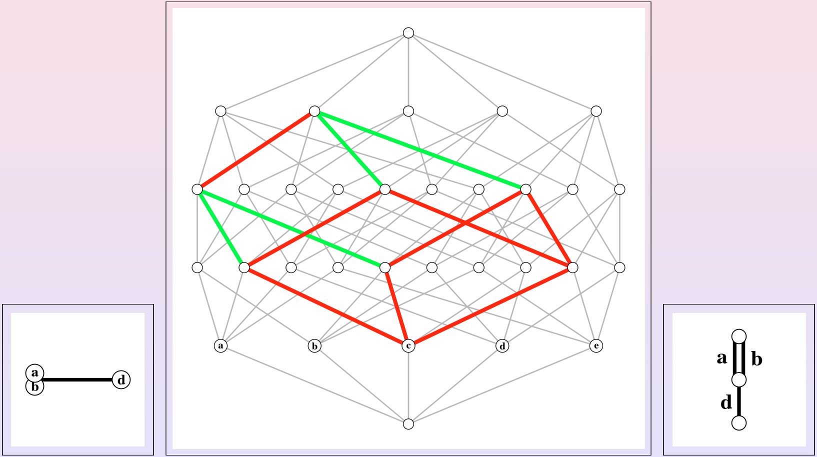



Figure: The minor $(L\backslash c) - e$, simply an interval.

# Edge-colored Boolean Algebras

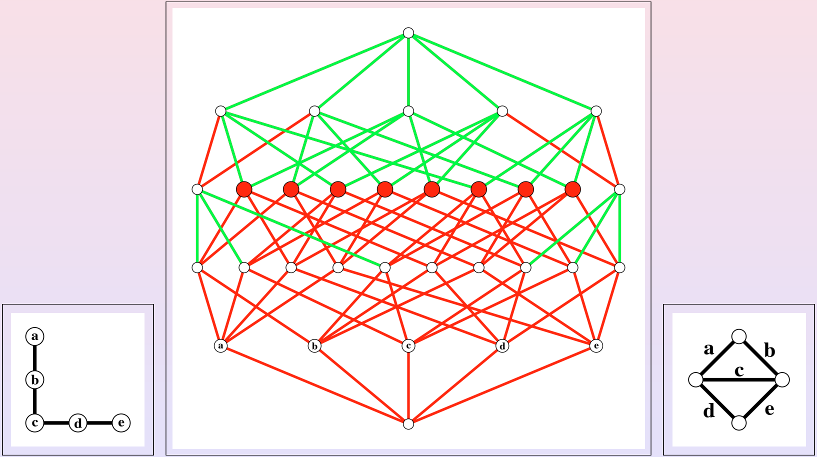
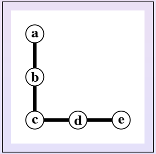
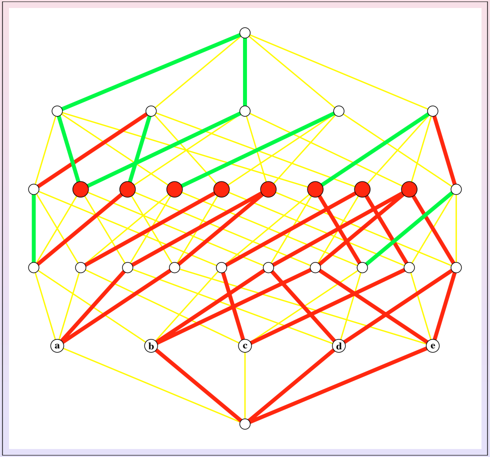



Figure: The bases for the $L$-matroid.

# The Tutte Polynomial



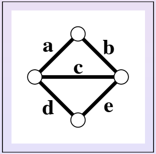
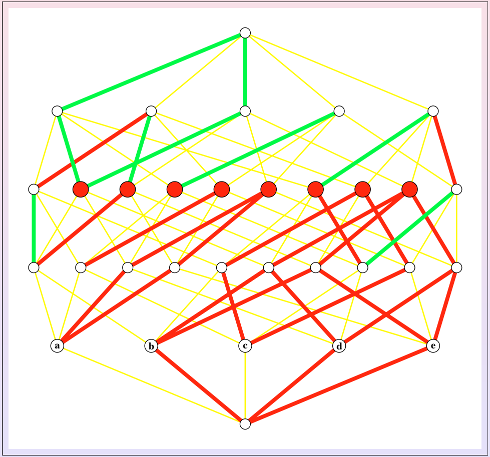




Figure: The Tutte polynomial: int and ext activities.

# The Tutte Polynomial

| | | | |
|---|---|---|---|
| 1 | | | |
| 5 | 2 | | |
| 8 | 10 | 5 | 1 |




| | | | |
|---|---|---|---|
| 1 | | | |
| 1 | 2 | | |
| • | 1 | 2 | 1 |

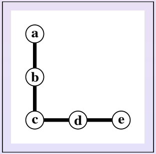
# The Tutte Polynomial

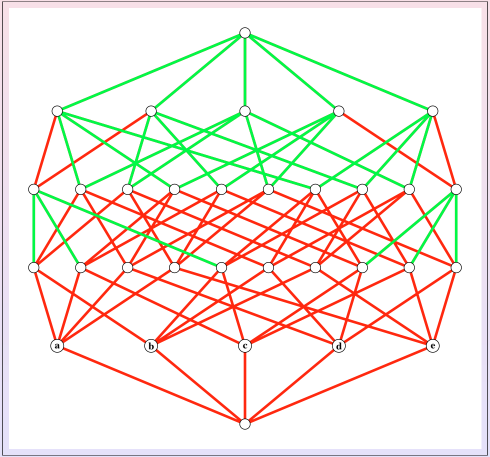
$$\rho \qquad \begin{array}{llll} 1 & & & \\ 5 & 2 & & \\ 8 & 10 & 5 & 1 \end{array} \qquad\qquad \begin{array}{llll} 1 & & & \\ 1 & 2 & & \\ \bullet & 1 & 2 & 1 \end{array} \qquad \tau$$

Rank generating function $\leftrightarrow$ Tutte Polynomial

$$\rho(x, y) = \tau(x + 1, y + 1)$$

# Duality



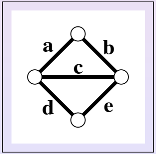




Figure: Back to the edge-colored BA for the $L$-matroid.

# Duality

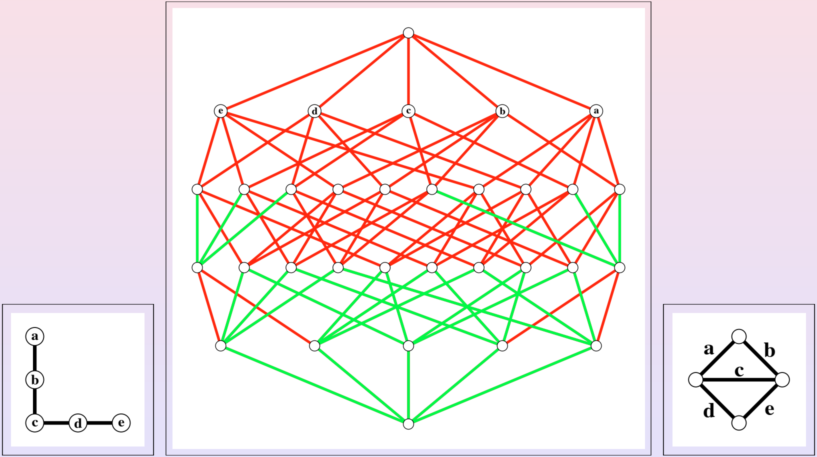



Figure: Duality, via the opposite Boolean algebra.

# Duality

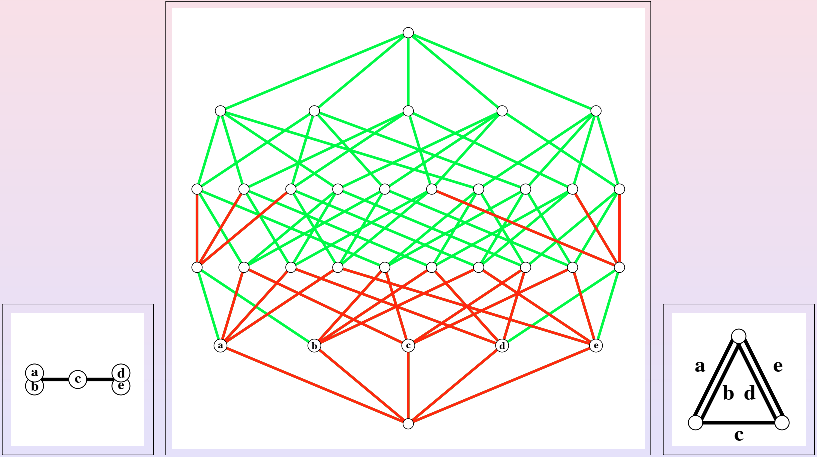



Figure: The dual $L^*$ of the matroid $L$.

# Duality

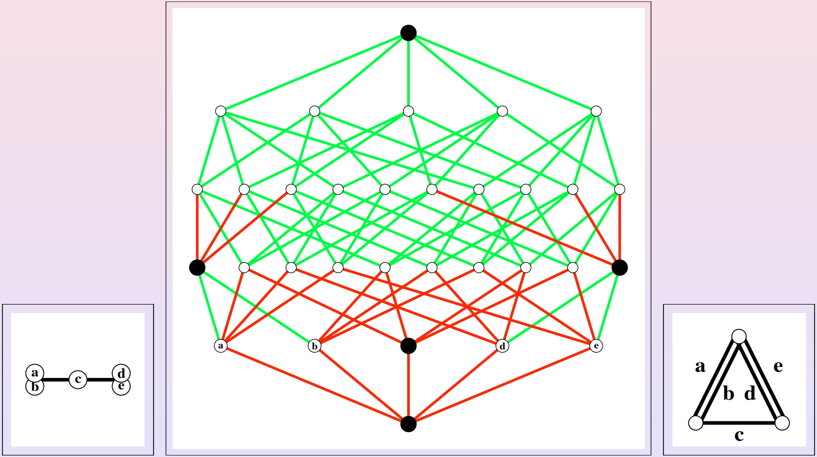



Figure: The flats of the matroid $L^*$.

# Edge-colored Lattices, Then and Now

In my thesis I extended these lattice-coloring methods
from Boolean algebras
to distributive lattices,
then to
complemented modular lattices.

Possible $Q$-*analogues* of matroids.

## Edge-colored Lattices, Then and Now

Lattice colorings and edge-labelings also give rise to combinatorial coalgebras (Rota, Joni, Schmitt), via the minor coproduct:

$$\partial M \;=\; \sum_{A \subseteq S} M|_A \,\otimes M \setminus A$$

The dual Hopf product produced what Bill Schmitt and I called the free product $M(S) \,\square\, N(T)$ of matroids the freest matroid $F(S+T)$ (in the weak order) having $F_{[\emptyset,S]} \simeq M$ and $F_{[S,S+T]} \simeq N$.

# Edge-colored Lattices, Then and Now

Bill and I also completed work on a Hopf algebra project initiated with Gian Carlo Rota, the Whitney algebra of a matroid $M$.

This algebra is formed from the free exterior algebra of points of $M$, taking tensor powers, then dividing out by the ideal generated by coproducts of dependent sets.

The Whitney algebra is a universal coordinatizing algebra for matroids.

## Edge-colored Lattices, Then and Now

Here's the sort of thing you can do with the Whitney algebra.

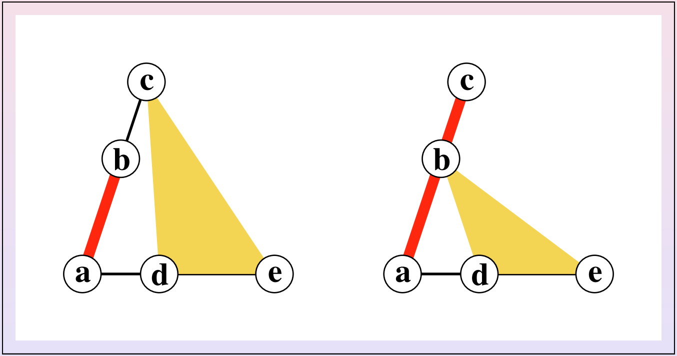



Figure: $ab \otimes cde = ac \otimes bde$.

# An Example of the Free Product

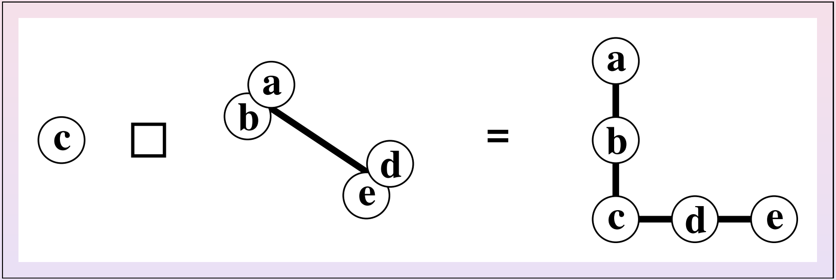



Figure: A point, times a line of two double points.

# Construction of the Free Product

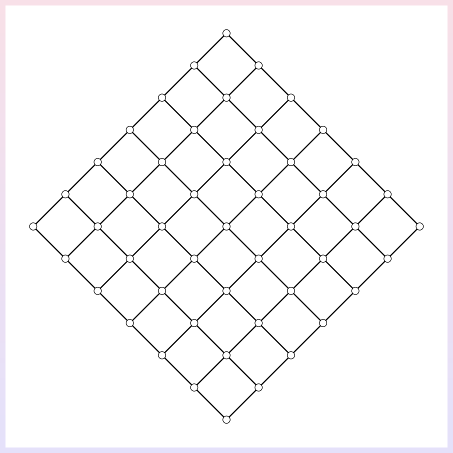


Figure: A $2$-dimensional section of the Boolean algebra.

# Construction of the Free Product

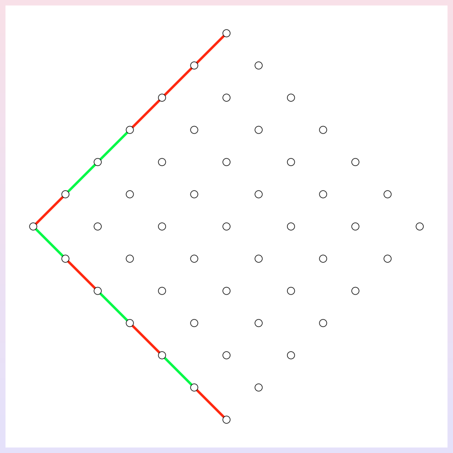


Figure: The two free factors. Start with action on 2nd factor.

# Construction of the Free Product

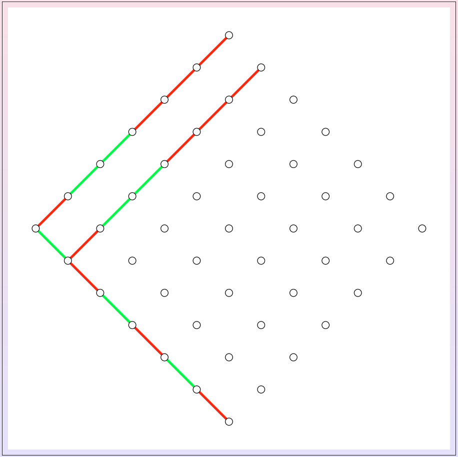


Figure: Copy descending across green.

# Construction of the Free Product

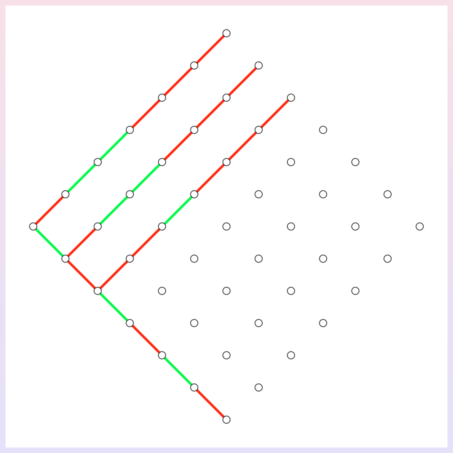
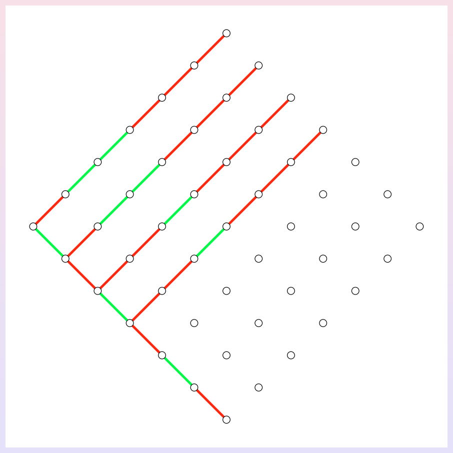


Figure: Lift descending across red.

# Construction of the Free Product



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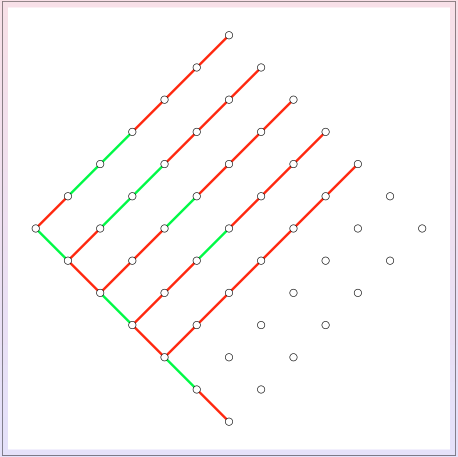

Figure: Lift descending across red.

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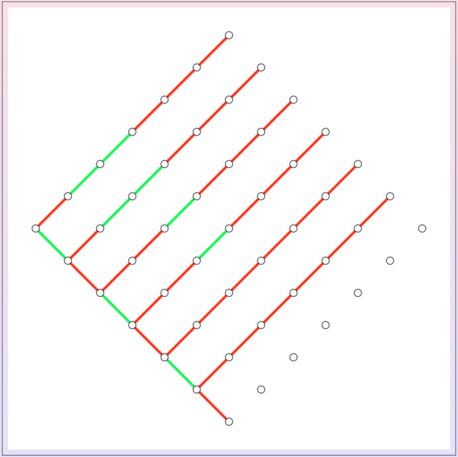


Figure: Copy descending across green.

# Construction of the Free Product

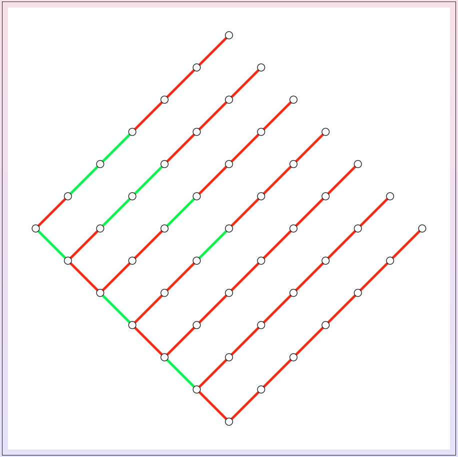


Figure: Lift descending across red (already at max).

# Construction of the Free Product

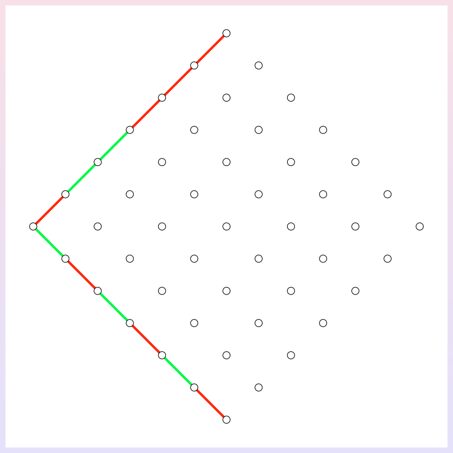


Figure: Acting on the first factor.

# Construction of the Free Product

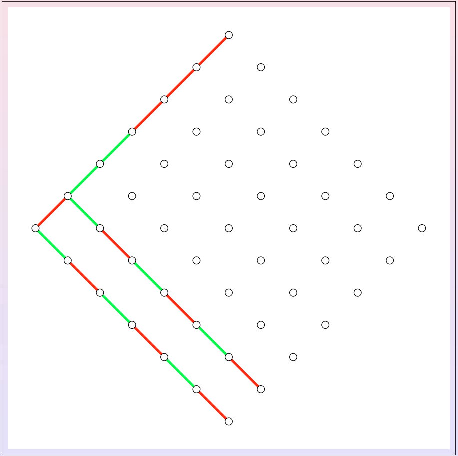


Figure: Copy passing upward under red.

# Construction of the Free Product

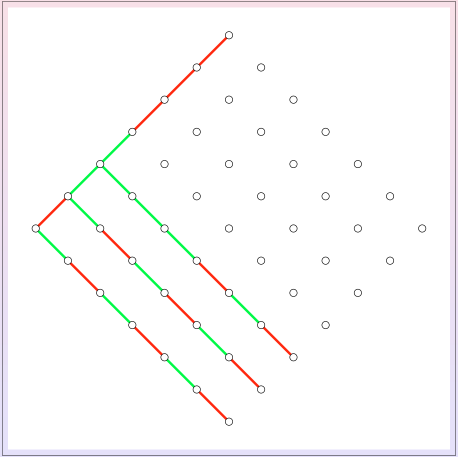
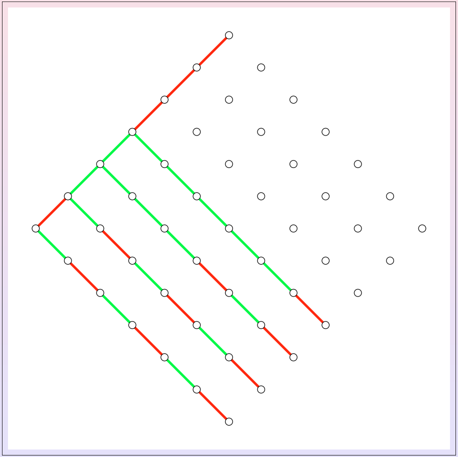


Figure: Truncate passing upward under green.

# Construction of the Free Product



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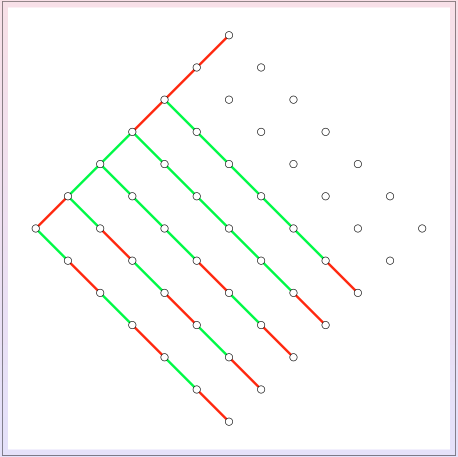

Figure: Copy passing upward under red.

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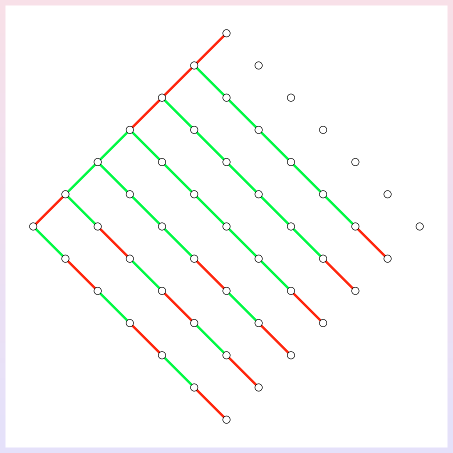

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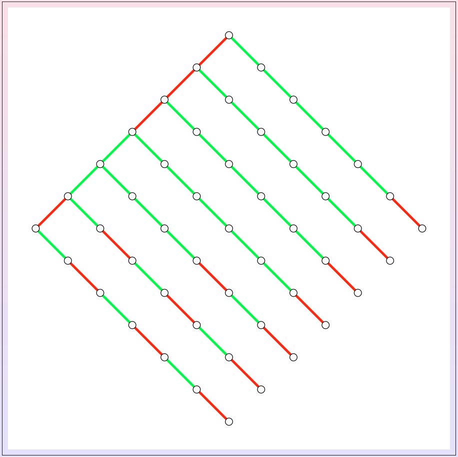

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# Construction of the Free Product

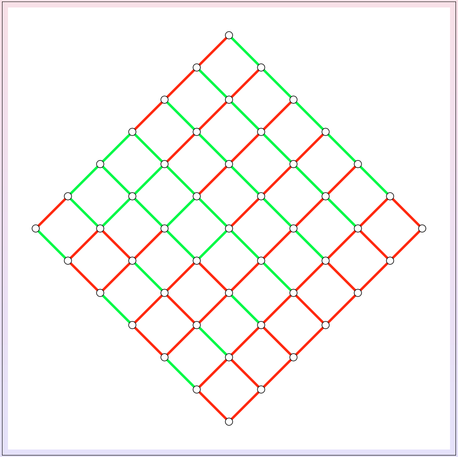


Figure: The free product on this section of the Boolean algebra.

# Construction of the Free Product

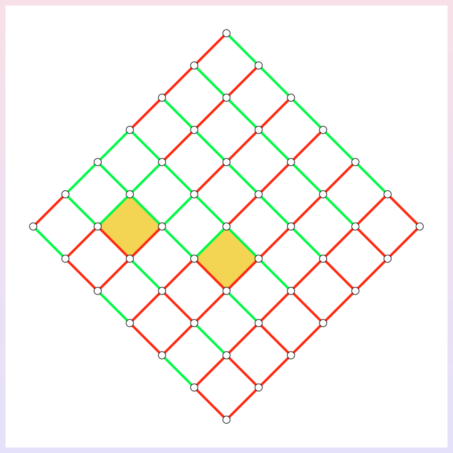


Figure: And here's where the exchanges occur !

# Single-Element Extension

Extensions of a matroid $M$ by a single point $p$ can be achieved in a variety of ways, each such extension $M'$ being determined by a substructure of $M$, either

- a modular filter in the lattice of flats of $M$, or
- a linear class of copoints of $M$

Systematic use of this concept provides inductive proofs of many theorems of matroid theory.

# Single-Element Extension

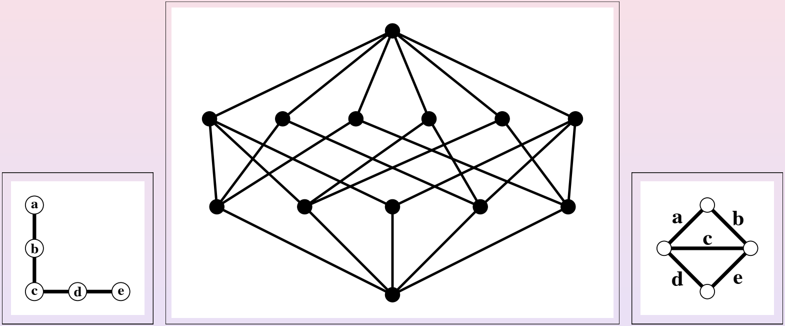
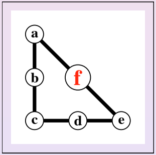
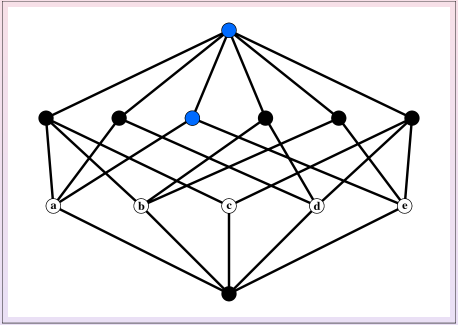



Figure: The flats of the matroid $L^*$.

# Single-Element Extension




**not binary**

Figure: A modular filter.

# Single-Element Extension

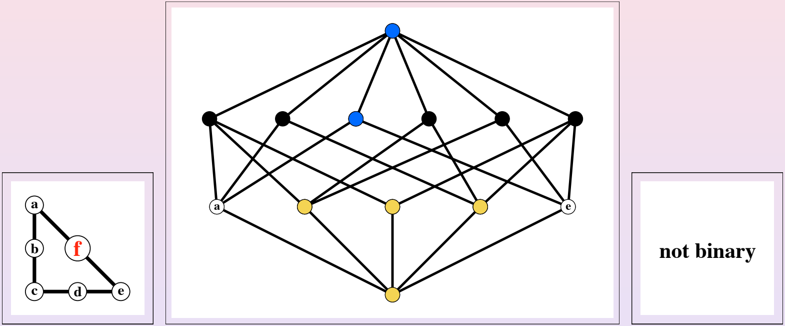
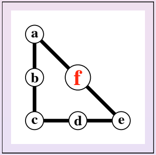
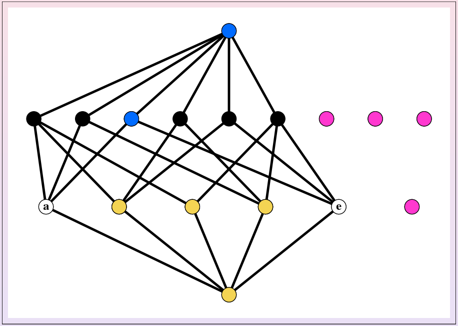



Figure: Elements to be duplicated, in yellow.

# Single-Element Extension




**not binary**

Figure: The new elements in pink.

# Single-Element Extension

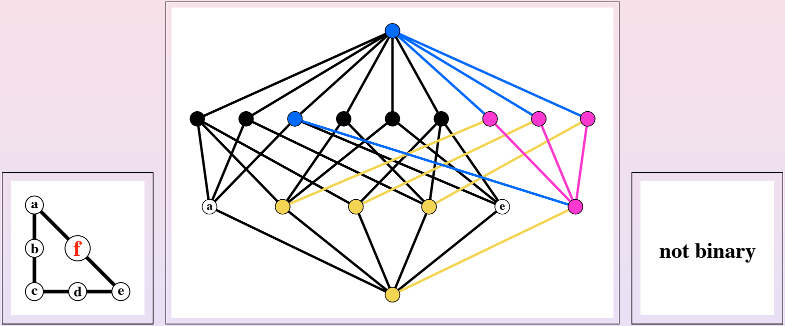



Figure: Connecting the new elements.

# Single-Element Extension

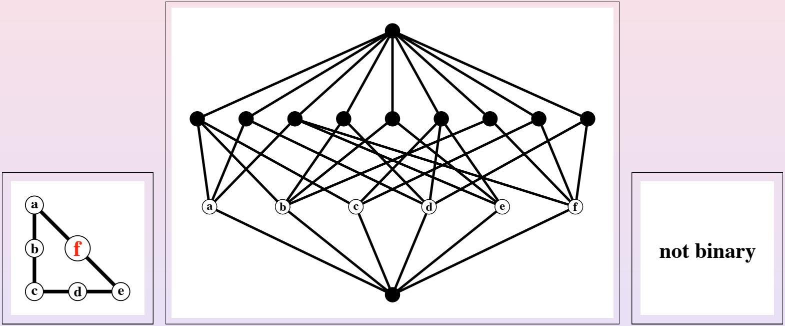



Figure: The single-element extension $L_{+f}$.

# Another Single-Element Extension

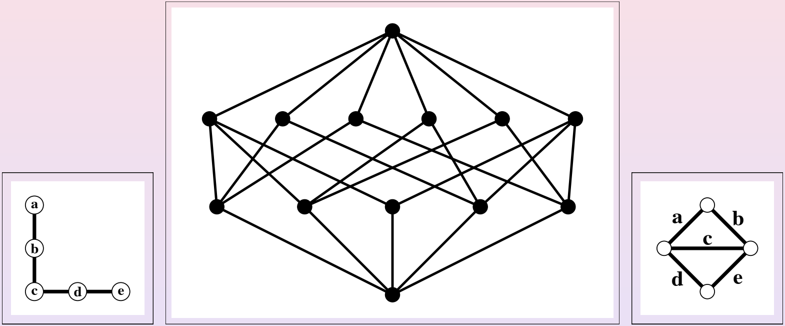



Figure: The flats of the matroid $L^*$.

# Another Single-Element Extension

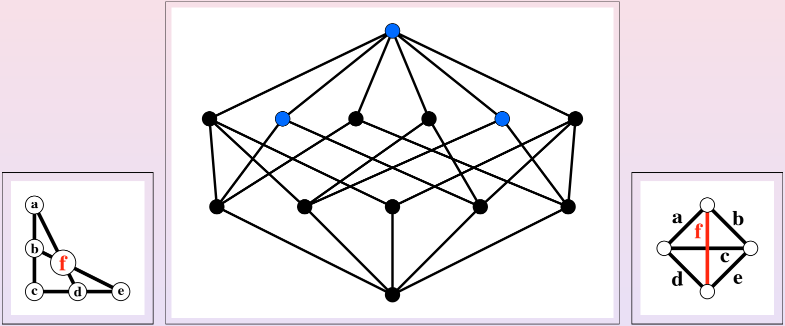



Figure: A modular filter.

# Another Single-Element Extension

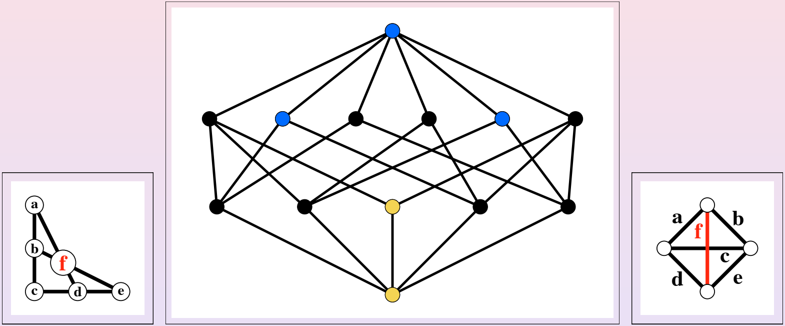



Figure: Elements to be duplicated, in yellow.

# Another Single-Element Extension

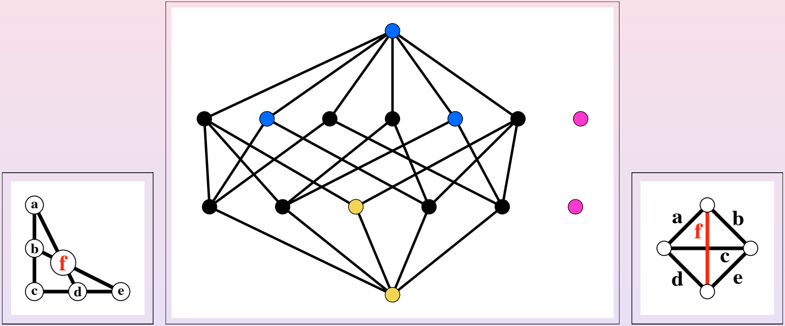
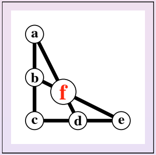
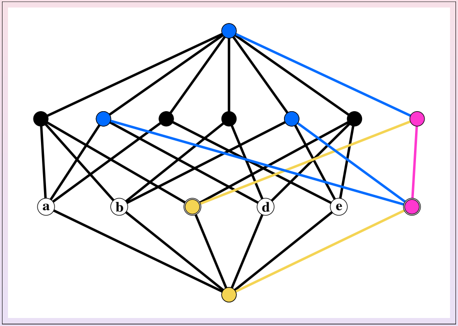



Figure: The new elements in pink.

# Another Single-Element Extension



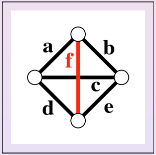




Figure: Connecting the new elements.

# Another Single-Element Extension

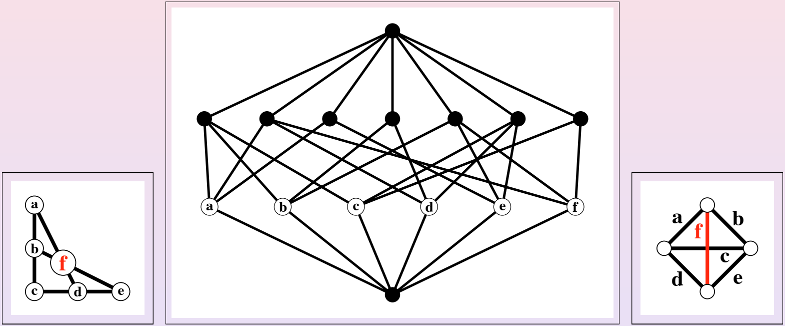



Figure: The single-element extension $L_{+f}$.

## 1969, an interesting year, Jack arrives in Waterloo

The University of Waterloo was energetic in developing strategies for the privatization of state-funded public universities,

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The University of Waterloo was energetic
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of state-funded public universities,

a process showing signs of making headway in France.

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The University of Waterloo was energetic
in developing strategies for the privatization
of state-funded public universities,

(Canada was about to begin to dismantle
its national railway system,
and was privatizing the Post Office.
The telephone system was already private.)

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- University chancellor = Mr. Pollock, president of *Electrohome*, a local electonics manufacturing firm.
- Statistics Department linked to *Mutual Life Assurance Co*, headquarters in Waterloo.
- The Faculty of Mathematics rakes in funds from profitable alliances with computer manufacturers (*IBM, Honeywell*).
- Deans of the Mathematics chosen for loyalty to this system.
- Students were pushed toward *cooperative education*, half of their university career at jobs in these businesses.
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The C&O department could justify its existence simply by providing an office for the remarkable Bill Tutte. Since then:

- Denis Higgs, Ramchandra Murty, Jack Edmonds, Adrian Bondy.
- The MapleSoft spin-off.
- The Digital Oxford dictionary.
- Jim Geelen

so at long last the matroid minors project,
Geoff Whittle, Jim Geelen, Bert Gerards,
sequel to Robertson/Seymour/Thomas on graphs.

## 1969, an interesting year, Jack arrives in Waterloo

4/4/9

Greetings from Jim Geelen and the matroid minors project:

Unfortunately, I will not be attending Jack's meeting.

The project is going well.
Binary matroids are well-quasi-ordered
and minor-testing is poly-time, as expected.
All of the interesting ideas are in the structure theorem;
the applications follow as for graphs.

# 1969, an interesting year, Jack arrives in Waterloo

Figure: The Waterloo campus, with its Math Bldg.

# 1969, an interesting year, Jack arrives in Waterloo

Figure: Affectionately called Fort Stanton.

# 1969, an interesting year, Jack arrives in Waterloo

Figure: Paraphrasing Claudio Lucchesi:

A structure is Edmonds $\Leftrightarrow$ one of its bricks is Edmonds.

# 1969, an interesting year, Jack arrives in Waterloo

Combinatorial mathematics was less competitive in those days. Mathematics research groups had not yet taken to calling themselves *institutes of operations research*.

I always wondered at the time why the term *optimization*, not *pessimization*,
is employed, since the main task is more often than not that of *minimization* (time, space, cost), not maximization.

LP IP NP
were all being born.

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# 1967-1973, Balls and Boxes

Combinatorial theory was emerging as a discipline of its own, and a number of crucial international meetings were held:

# 1967-1973, Balls and Boxes

- Oberwolfach, 1967.
- M.I.T. Summer Seminar in Combinatorial Theory, 1967.
- Symposium in Combinatorics, A.M.S., Los Angeles, 1968.
- Symposium on honor of Oystein Ore, Yale, 1968.
- Calgary Int'l Conference on Combinatorial Structures, 1969.
- Combinatorial Theory and its Appl's, Balatonfured, 1970.
- Combinatorial Theory, Chapel Hill, 1970.
- Geometry Week, Lakehead University, 1970.
- International Congress of Mathematiciens, Nice, 1970.
- Geometria Combinatorie, Perugia, 1970.
- Geometria Combinatorie, Accademia dei Lincei, Rome, 1973.
- Lattice Theory, Houston, 1973.

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# 1969, in Calgary

Jack attended an important meeting in Calgary, Alberta,
*Combinatorial Structures and their Applications*,
and gave a wonderful $19$-page paper
with $138$ numbered statements,
mostly statements of theorems, without proofs,
and without examples, entitled
*Submodular functions, matroids, and certain polyhedra*,
which introduced the concept of *polymatroid*.

I talked about Dilworth completion of lower-truncated Boolean algebras, also leaving out the essential proof, which was subsequently revealed in Jack's talk.

# 1969, in Calgary

I should give a simple example of a polymatroid, and its lattice of flats:

- Underlying set $E = \{a\}$
- Polymatroid $P = \{(x) \mid 0 \leq x \leq 5\}$ (values at $a$)

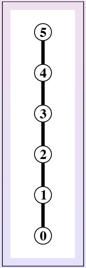


Figure: The lattice of flats of the polymatroid $P$.

# 1969, in Calgary

This example convinces me that it would help if there were a systematic review of connections to the work on matroids on posets:

- Christian Hermann
- Ulrich Faigle
- Marilena Barnabei, Giorgio Nicoletti, Luigi Pezzoli
- Anders Björner, Laszlo Lovasz (greedoids)

# Dilworth Completion

If you truncate a geometric lattice of a matroid $M$ of rank $n$ at the top, the result is still a geometric lattice.

But if you truncate it at the bottom, removing the points, semimodularity is destroyed,

This can be corrected by introducing new flats wherever necessary, a process called Dilworth completion.

Actually, this process was described early on by Juris Hartmanis, once a Ph.D. student of Bob Dilworth,
in a paper it took me a very long time to understand.

# Dilworth Completion, an example.

Start with the Boolean algebra $B_4$,

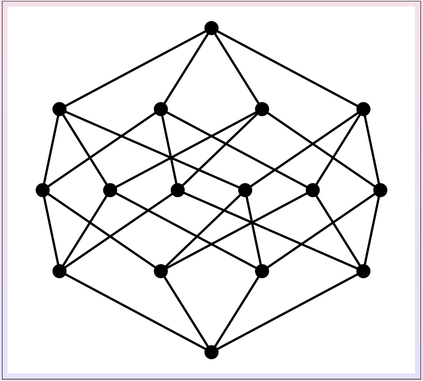

Figure: The Boolean algebra $B_4$.

# Dilworth Completion

and lower truncate, removing the points.

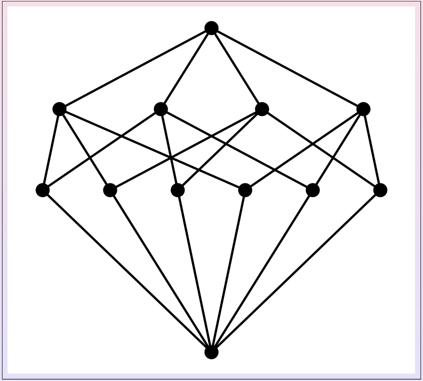

Figure: Its lower truncation.

# Dilworth Completion

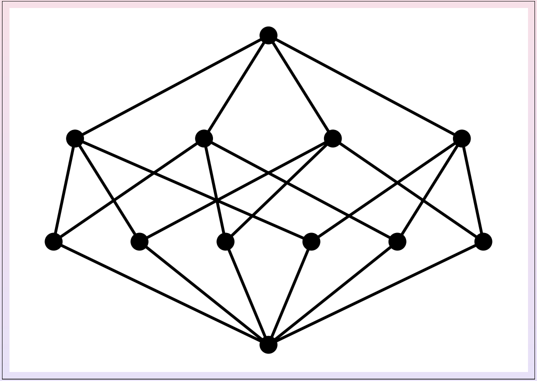

Figure: It's not geometric.

# Dilworth Completion

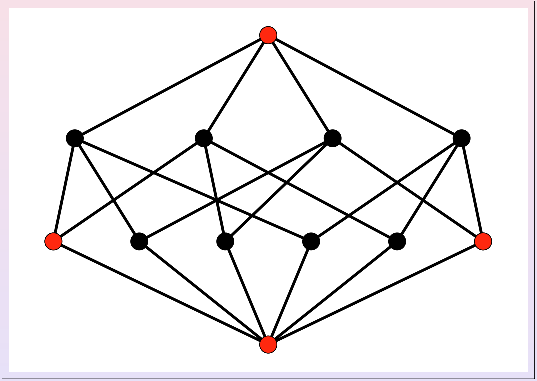

Figure: It's not geometric.

The atoms $ab$ and $cd$ cover $0$ but have join at rank $3$.

# Dilworth Completion

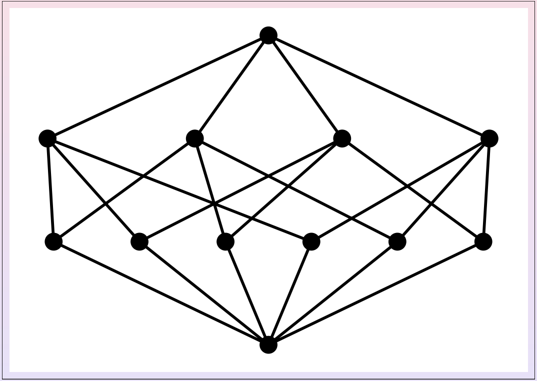

Figure: Make some room for the needed copoints.

# Dilworth Completion

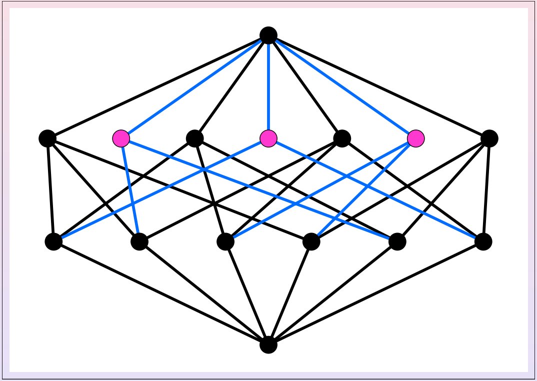

Figure: Insert the new copoints.

# Dilworth Completion

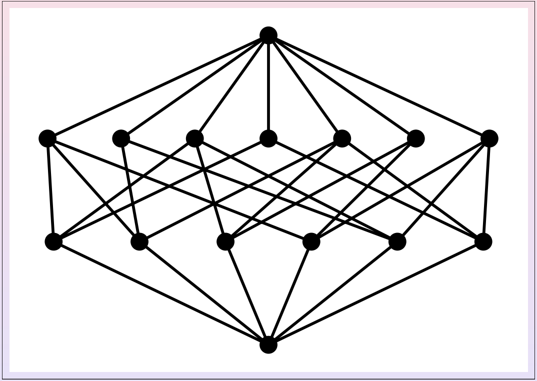

Figure: The Dilworth completion, $H_{1,4}$.

# Dilworth Completion

The Dilworth completion $H_{k,n}$
of the $k$-fold lower truncation of $B_n$
has three principal avatars:

- The lattice of Hartmanis $k$-partitions of an $n$-element set.
- The intersection figure of $n$ general hyperplanes (copoints) in a space of rank $n-k$.
- The matroid of circuits of the uniform matroid $U_{k,n}$.

- Since 1-partitions are simply partitions, $H_{1,n}$ is also the lattice of closed subgraphs of the complete graph $K_n$.
- And the geometry of mirrors in the Coxeter group $A_{n-1} = S_n$.

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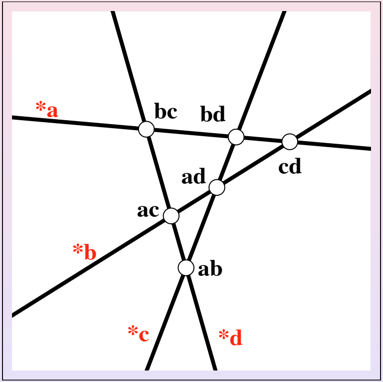


Figure: The Dilworth completion, $H_{1,4}$ of the graphic $K_4$.

# Dilworth Completion

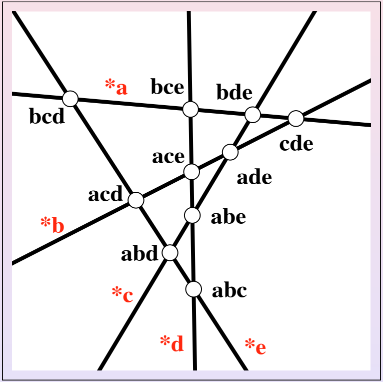


Figure: The Dilworth completion, $H_{2,5}$.

# Dilworth Completion

- The points of $H_{k,n}$, with $S$ a set of cardinality $n$ are the subsets of $S$ of size $k+1$.
- Every $j$-subset $T \subseteq S$, for $k+1 \leq j \leq n$, produces a flat $\binom{T}{k+1}$,
- If a point p (as $k+1$-set) is a subset of $T$, we write $p \lhd T$ .
- A set $Q \subseteq \binom{n}{k+1}$ of points is closed if and only if $Q$ contains, along with any $j$ distinct points $p \lhd T$ in any $(k+j)$-set $T$, also all points $p \lhd T$ .

- Statements like this are always hard to read, painful to check, but that's combinatorics.

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## Geometries of Circuits

Any matroid has in principle a number of distinct possibilities for a geometry of circuits, of which these Dilworth completions are the most general.

This phenomenon is intrinsic to the concept of representation of matroids.

Any representation produces an infinite sequence of
derived matroids,
matroids of circuits of circuits of . . . ,
matroids of *higher order syzygies.*

Circuit geometries were considered by Edouardo Amaldi this morning.

# Geometries of Circuits

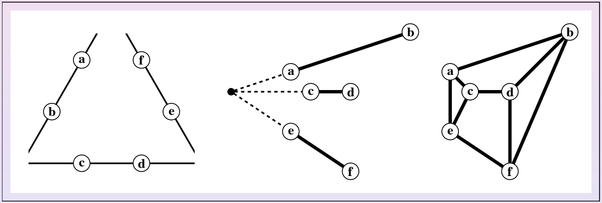


Figure: Possible representations of $U_{3,6}$.

# Geometries of Circuits

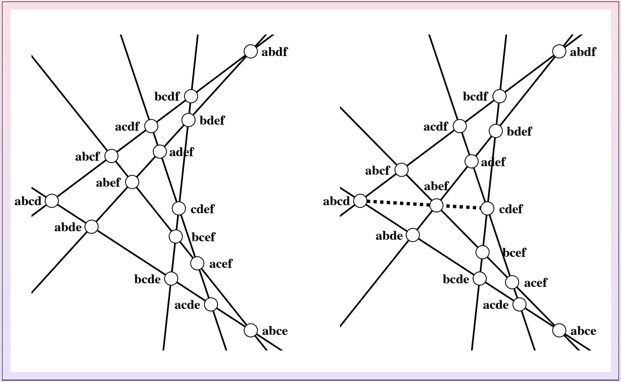


Figure: The Dilworth completion, $H_{3,6} = U'_{3,6}$ .

# Dilworth Completion, the series with $k = 1$

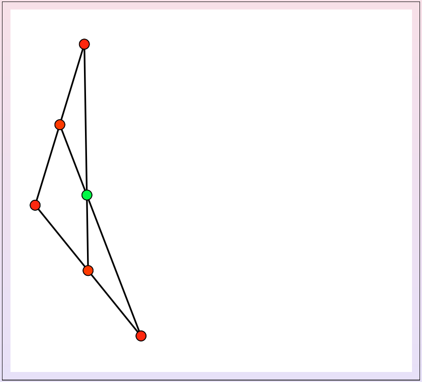

Figure: The complete quadrilateral, $H_{1,4}$.

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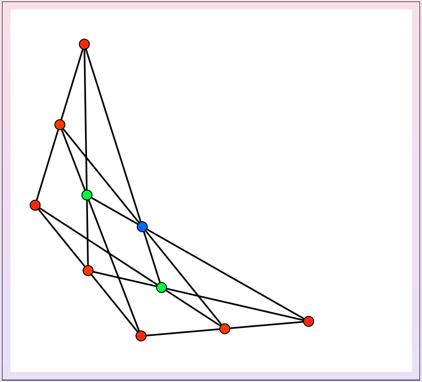

Figure: Desargues, rank $4$, $H_{1,5}$.

# Dilworth Completion, the series with $k = 1$

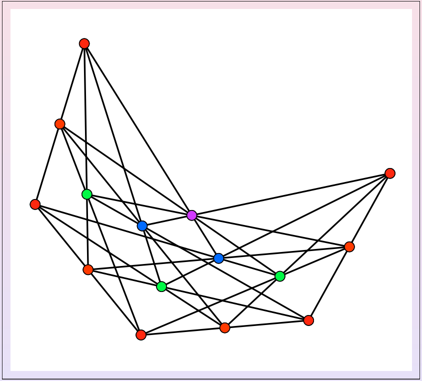

Figure: super Desargues, rank $5$, $H_{1,6}$.

## Sublattice embeddings of Hartmanis Lattices $H_{k,n}$

Write $\mathcal{L} \Longrightarrow \mathcal{M}$ if every lattice in class $\mathcal{L}$ can be represented as a sublattice of a lattice in class $\mathcal{M}$.

- $\mathcal{L}$ = finite lattices
- $\mathcal{G}$ = finite geometric lattices
- $\mathcal{H}_k$ = lattices $H_{k,n}$ for all $n \geq k$.

- Philip Whitman (1941) asked whether $\mathcal{L} \Longrightarrow \mathcal{H}_1$.
- Bob Dilworth (1950s) proved $\mathcal{L} \Longrightarrow \mathcal{G}$.
- Juris Hartmanis proved $\mathcal{H}_j \Longrightarrow \mathcal{H}_k$ for all $j \geq 2, k \geq 2$.
- P. Pudlàk & Tůma (1977) proved $(\mathcal{L}) \Longrightarrow \mathcal{H}_1$.
- but a reasonable proof with better bounds would be a fine gift to mankind (*l'Humanité*).
- Especially, $\mathcal{H}_2 \Longrightarrow \mathcal{H}_1$ !!!

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## 2009

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There are lots of vital applications!

# 2001


Figure: 911 was an inside job

This *was* a rigid structure, brought down by controlled demolition.

# Structural Rigidity

The matroid $R_{d,n}$ of generic $d$-rigidity
is the matroid on the set of edges of a graph on $n$ vertices
having as bases those subgraphs that are
isostatic (just rigid) in general position in $d$-dimensional space:

- $d = 1$ Spanning trees (ie: graphic matroids)
- $d = 2$ $2v - 3$ edges on $v$ vertices, no $2v' - 2$ on any $v' < v$. (typical construction from a submodular function.)
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# Structural Rigidity

Prove exchange:

- bases, generically isostatic, $B, C$
- Removal of an edge $e$ of $B$ permits a motion, with one infinitesimal degree of freedom, which changes the distance between certain pairs of vertices.
- Some one of those pairs $\{a, b\}$ must be an edge $f$ in $C$.
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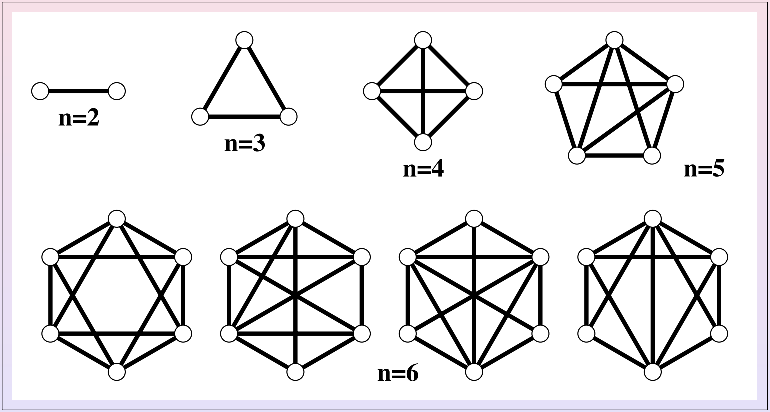


Figure: 3-isostatic graphs, bases of $R_{3,n}$.

## Structural Rigidity

Rank is crudely determined by extending to all edge sets the function having values

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | ... | $n$ |
| 0 | 0 | 1 | 3 | 6 | 9 | ... | $3n-6$ |

on edge sets of complete graphs $K_n$.

# Structural Rigidity

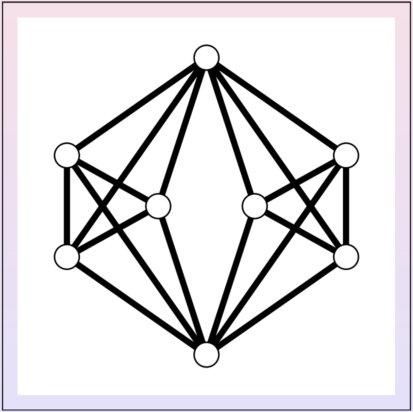

Figure: Two bananas: satisfies "$3v - 6$".

# Structural Rigidity

Rigid components (as vertex sets)
don't have to have any edges!
(replace them by bananas.)

## Structural Rigidity

The improved count, including hinges, is even wrong:
Jackson & Jordán, biplanes example.

## Structural Rigidity

What are the circuits, or bases?

There's a problem for all you fans of semimodularity!
Tamás Király, Stephan Thomassé, Jack Edmonds
This is a job for Batman!

## 2009

> And if my friends and colleagues will please stop having sesqui-decimal birthdays, or worse, memorial services, maybe I'll get my book going on Geometric Homology.

Thank you for your attention.

# from Vincent Duquenne

Abstraction misses Concreteness

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Concreteness misses Abstraction

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Concreteness
and
Abstraction
— nothing less !—